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A Mathematical Model Water Movement about Bottom-Water-Drive Reservoirs

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ABSTRACT

This paper presents the development and solution of a mathematical model for aquifer water movement about bottom-water-drive reservoirs. Pressure gradients in the vertical direction due to water flow are taken into account. A vertical permeability equal to a fraction of the horizontal permeability is also included in the model. The solution is given in the form of a dimensionless pressure-drop quantity tabulated as a function of dimensionless time. This quantity can be used in given equations to compute reservoir pressure from a known water-influx rate, to predict water-influx rate (or cumulative amount) from a reservoir-pressure schedule or to predict gas reservoir pressure and pore-volume performance from a given gas-in-place schedule. The model is applied in example problems to gas-storage reservoirs, and the difference between reservoir performances predicted by the thick sand model of this paper and the horizontal, radial-flow model is shown to be appreciable.

INTRODUCTION

The calculation of aquifer water movement into or out of oil and gas reservoirs situated on aquifers is important in pressure maintenance studies, material-balance and well-flooding calculations. In gas storage operations, a knowledge of the water movement is especially important in predicting pressure and pore-volume behavior. Throughout this paper the term "pore volume" denotes volume occupied by the reservoir fluid, while the term "flow model" refers to the idealized or mathematical representation of water flow in the reservoir-aquifer system.

The prediction of water movement requires selection of a flow model for the reservoir-aquifer system. A physically reasonable flow model treated in detail to date is the radial-flow model considered by van Everdingen and Hurst.[1] In many cases the reservoir is situated on top of the aquifer with a continuous horizontal interface between reservoir fluid and aquifer water and with a significant depth of aquifer underlying the reservoir. In these cases, bottom-water drive will occur, and a three-dimensional model accounting for the pressure gradient and water flow in the vertical direction should be employed. This paper treats such a model in detail — from the description of the model through formulation of the governing partial differential equation to solution of the equation and preparation of tables giving dimensionless pressure drop as a function of dimensionless time. The model rigorously accounts for the practical case of a vertical permeability equal to some fraction of the horizontal permeability. The pressure-drop values can be used in given equations to predict reservoir pressure from a known water-influx rate or to predict water-influx rate (or cumulative amount) when the reservoir pressure is known.

The inclusion of gravity in this analysis is actually trivial since gravity has virtually no effect on the flow of a homogeneous, slightly compressible fluid in a fixed-boundary system subject to the boundary conditions imposed in this study. Thus, if the acceleration of gravity is set equal to zero in the following equations, the final result is unchanged. The pressure distribution is altered by inclusion of gravity in the analysis, but only by the time-constant hydrostatic head.

The equations developed are applied in an example case study to predict the pressure and pore-volume behavior of a gas storage reservoir. The prediction of reservoir performance based on the bottom-water-drive model is shown to differ significantly from that based on van Everdingen and Hurst's horizontal-flow model.

DESCRIPTION OF FLOW MODEL

The edge-water-drive flow model treated by van Everdingen and Hurst[1] is shown in Fig. 1a. The aquifer thickness b is small in relation to reservoir radius r_b, water invades or recedes from the field at the latter's edges, and only horizontal radial flow is considered as shown in Fig. 1b. The bottom-water-drive reservoir-aquifer system treated herein is sketched in Fig. 2a and 2b. Here the aquifer

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[1]References given at end of paper.

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thickness h is appreciable in relation to r_b, water flows into and out of the reservoir across a roughly horizontal reservoir fluid-water interface, and flow components in the vertical direction exist. The aquifer is considered as a right circular cylinder of height h and exterior radius r_e, with upper and lower faces impermeable except for that portion ($r < r_b$) of the upper face intersected by the reservoir. The aquifer formation is considered to have constant, but unequal, permeabilities in the horizontal and vertical directions. The case of an average vertical permeability equal to a fraction of the average horizontal permeability is a practical one in aquifers riddled with thin, discontinuous shale streaks. This fraction may be taken as 1.0, of course, for applications of this thick sand model to aquifers considered homogeneous.

MATHEMATICAL CONSIDERATIONS

The partial differential equation governing unsteady-state flow of a slightly compressible fluid in the geometry shown in Fig. 2b appears as[2]

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + k_R \frac{\partial^2 p}{\partial z^2} = \frac{\mu \phi c}{k}\frac{\partial p}{\partial t} \quad . \; . \quad (1)$$

where k_R is the ratio of vertical effective permeability k_V to horizontal permeability k. Definition of the new variables,

$$r_D = r/r_b, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

$$y = z/r_b \sqrt{k_R} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

$$t_D = kt/\mu\phi c r_b^2{}^*, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

*When the units given in the Nomenclature are used, $t_D = 6.33\,kt/\mu\phi c r_b^2$.

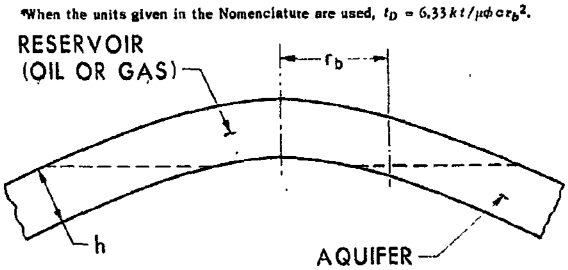

FIG. 1a — EDGE-WATER-DRIVE FLOW SYSTEM.

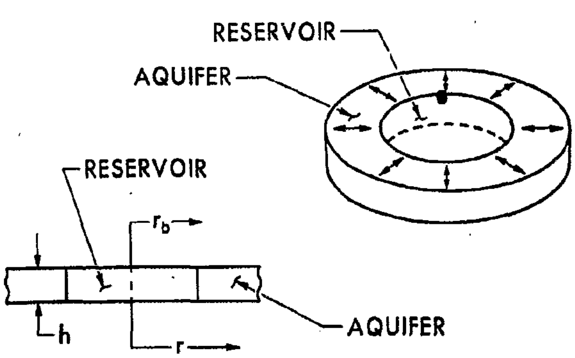

FIG. 1b — IDEALIZED FLOW MODEL FOR EDGE-WATER-DRIVE SYSTEM.

and

$$P(r_D, y, t_D) = p_i(y) - p(r_D, y, t_D), \quad . \; . \; . \quad (5)$$

allows Eq. 1 to be written

$$\frac{\partial^2 P}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P}{\partial r_D} + \frac{\partial^2 P}{\partial y^2} = \frac{\partial P}{\partial t_D} \quad . \; . \; . \quad (6)$$

The pressure $p_i(y)$ is the initial aquifer pressure which is assumed to be constant except for the vertical variation due to gravity. That is,

$$p_i(y) = p_o + \rho \frac{g}{g_c} z \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

where p_o is a constant equal to the initial aquifer pressure at the horizontal plane of the reservoir, $z = 0$ (see Fig. 2b).

Eq. 6 is solved here for the case of an infinite aquifer (i.e., $r_e = \infty$) and for the "constant rate case"[1] wherein the rate of water flow across the reservoir-aquifer interface ($z = 0, r < r_b$) is specified. The basic solution is obtained for a constant rate of water influx, while the general solution for an arbitrary time-dependent rate is obtained by application of Duhamel's superposition principle[3] to the basic solution. The velocity of water flow vertically into the reservoir is given by Darcy's law as

$$u = \frac{k_v}{\mu}\left(\frac{\partial p}{\partial z} - \rho \frac{g}{g_c}\right)_{z=0} \quad . \; . \; . \; . \; . \quad (8)$$

If this velocity is considered constant over the area of the reservoir ($0 < r < r_b$), then the volumetric rate of water influx, e_w is given by

$$e_w = \pi r_b^2 \frac{k_v}{\mu}\left(\frac{\partial p}{\partial z} - \rho \frac{g}{g_c}\right)_{z=0}, \quad . \; . \; . \quad (9)$$

which is equivalent to

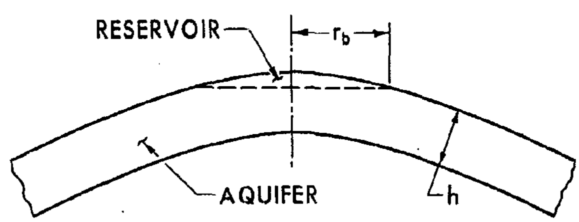

FIG. 2a — BOTTOM-WATER-DRIVE FLOW SYSTEM.

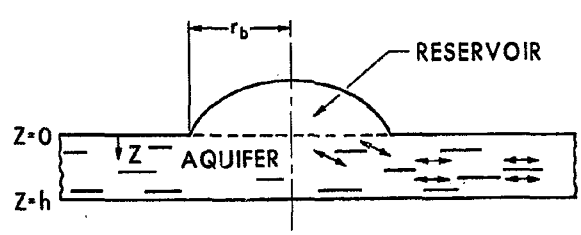

FIG. 2b — IDEALIZED FLOW MODEL FOR BOTTOM-WATER-DRIVE SYSTEM.

$$e_w = \frac{\pi r_b k \sqrt{k_R}}{\mu} \left(\frac{\partial P}{\partial y}\right)_{y=0} \quad \ldots \ldots (10)$$

Thus, the boundary condition at $z = 0$ for the basic (*constant* rate) solution is

$$\left(\frac{\partial P}{\partial y}\right)_{y=0} = f(r_D) \begin{cases} \dfrac{-\rho\, e_w\, \mu}{\pi r_b k \sqrt{k_R}}, & 0 \le r_D < 1 \\ 0, & 1 \le r_D \end{cases} \quad \ldots \ldots (11)$$

where e_w is considered constant. The initial condition and other boundary conditions corresponding to impermeable lower boundaries, finiteness of P at $r = 0$ and equilibrium at $r = \infty$ are

$$P(r_D, y, 0) = 0, \quad \ldots \ldots (12)$$

$$\left(\frac{\partial P}{\partial y}\right)_{y = b/r_b \sqrt{k_R}} = 0 \quad \ldots \ldots (13)$$

$$\lim_{r_D \to \infty} P(r_D, y, t_D) = 0, \quad \ldots \ldots (14)$$

and

$$\lim_{r_D \to 0} P(r_D, y, t_D) = \text{finite} \quad \ldots \ldots (15)$$

The assumption of a constant influx velocity u (Eq. 8) over the reservoir area $r < r_b$ is actually open to little objection since, if u were considered a function of radius r_D, the solution obtained would show the same time dependence but would differ by a multiplicative constant. Since the solution (Eq. 16) contains the multiplicative constant $\mu/\pi r_b k \sqrt{k_R}$, the appearance of another constant is immaterial for two reasons: (1) the factors k/μ, r_b and $\sqrt{k_R}$ are not generally known exactly; and (2) the constant $\mu/\pi r_b k \sqrt{k_R}$ will be chosen in a practical case by matching predicted pressures with available field data.

The solution to Eq. 6 for the conditions of Eqs. 11 through 15 is derived in the Appendix and appears as

$$P(r_D, o, t_D) = E \int_0^\infty \frac{J_1(x)}{x} \left[\coth Mx - \frac{e^{-x^2 t_D}}{Mx} - \frac{2x}{M} \sum_{m=1}^{m=\infty} \frac{e^{-(x^2 + a_m^2) t_D}}{x^2 + a_m^2}\right] J_0(r_D x)\, dx \quad \ldots \ldots (16)$$

where y has been set equal to zero,

$$E = \frac{e_w\, \mu^*}{\pi r_b k \sqrt{k_R}}, \text{ and}$$

$$M = b/r_b \sqrt{k_R}.$$

Since this solution varies with radius r_D, the question arises as to what value of r_D between 0 and 1 should be chosen for numerical evaluation of P. Rather than choosing a single value of r_D, the solution Eq. 16 was integrated over the radius to obtain the "areal mean" dimensionless reservoir-pressure drop.

$$\overline{P}(t_D) = \frac{1}{E} \frac{\int_0^1 2\pi r_D\, P(r_D, o, t_D)\, dr_D}{\pi (1)^2}, \quad \ldots (17)$$

or

$$\overline{P}(t_D) = 2 \int_0^\infty \frac{J_1^2(x)}{x^2} \left[\coth Mx - \frac{e^{-x^2 t_D}}{Mx} - \frac{2x}{M} \sum_{m=1}^{m=\infty} \frac{e^{-(x^2 + a_m^2) t_D}}{x^2 + a_m^2}\right] dx \quad \ldots (18)$$

$\overline{P}$ was numerically integrated on an IBM 704 digital computer for several values of the parameter M, and the results are listed in Table 1. Although the integrations were carried out to a dimensionless time of 1,600, the results can be approximated quite closely* for $t_D > 10$ by

$$P = A + \frac{1}{4M} \ln t_D \quad \ldots \ldots (19)$$

where A is a constant dependent upon M as shown in Table 2 and Fig. 3. Values of $\overline{P}$, for $t_D < 10$ and for values of M not listed in Table 1, can be found by interpolating between curves of $\overline{P}$ vs t_D as shown in Fig. 4. Cross plots of $\overline{P}$ vs M with t_D as parameter would also serve this purpose.

WORKING EQUATIONS

Combination of Eqs. 5, 7, 16 and 17 yields

$$p = p_o - \frac{.0502\, e_w\, \mu}{r_b k \sqrt{k_R}} \overline{P}(t_D), \quad \ldots \ldots (20)$$

which gives reservoir pressure p as a function of time for a constant rate of water influx e_w. The constant π has been absorbed into other unit conversion factors to give the constant .0502; units of the variables present are given in the Nomenclature. Application of Duhamel's superposition principle[3] to the constant-rate-case solution (Eq. 20) gives the "variable-rate-case" solution

$$p_j = p_o - \frac{.0502\, \mu}{k\, r_b \sqrt{k_R}} \sum_{i=0}^{i=j-1} \Delta \overline{e}_i\, \overline{P}_{j-i} \quad \ldots (21)$$

Since $\Delta \overline{e}_i$ is defined as $\overline{e}_{i+1} - \overline{e}_i$ and $\overline{e}_i$ as the

*E is $.0502\, e_w \mu / r_b k \sqrt{k_R}$ when units given in the Nomenclature are used.

*The error between values of $\overline{P}$ tabulated in Table 1 and calculated from Eq. 19 is less than 1 per cent at t_D = 10 and decreases rapidly with increasing time.

TABLE 1 — DIMENSIONLESS PRESSURE DROP VS DIMENSIONLESS TIME FOR THICK SAND MODEL

$\overline{P}$ Values for Different Values of M

t_D	M=.05	M=.1	M=.3	M=.5	M=.7	M=.9	M=1.0
.1	1.433	.741	.329	.287	.285	.285	.285
.2	2.675	1.362	.536	.412	.383	.377	.377
.3	3.649	1.849	.699	.509	.454	.436	.436
.4	4.464	2.257	.835	.591	.512	.483	.483
.5	5.166	2.608	.952	.661	.562	.522	.521
1	7.693	3.871	1.373	.914	.743	.663	.648
2	10.590	5.319	1.856	1.204	.950	.824	.792
3	12.442	6.245	2.164	1.389	1.082	.927	.885
4	13.789	6.919	2.339	1.523	1.178	1.001	.952
5	14.848	7.448	2.565	1.629	1.254	1.060	1.005
6	15.722	7.885	2.711	1.717	1.316	1.109	1.049
7	16.465	8.257	2.835	1.791	1.369	1.150	1.086
8	17.111	8.580	2.942	1.856	1.416	1.186	1.118
9	17.684	8.866	3.038	1.913	1.456	1.218	1.147
10	18.198	9.123	3.123	1.964	1.493	1.246	1.173
12	19.089	9.569	3.272	2.053	1.557	1.296	1.217
14	19.846	9.947	3.398	2.129	1.611	1.338	1.255
16	20.503	10.276	3.508	2.195	1.658	1.374	1.288
18	21.083	10.566	3.604	2.253	1.699	1.407	1.317
20	21.603	10.826	3.691	2.305	1.736	1.435	1.343
24	22.505	11.277	3.841	2.395	1.801	1.486	1.388
28	23.269	11.659	3.969	2.471	1.855	1.528	1.426
32	23.931	11.990	4.079	2.538	1.903	1.565	1.459
36	24.516	12.282	4.177	2.596	1.944	1.597	1.489
40	25.039	12.544	4.264	2.648	1.982	1.626	1.515
45	25.625	12.837	4.361	2.707	2.024	1.659	1.544
50	26.149	13.099	4.449	2.759	2.061	1.688	1.570
55	26.623	13.336	4.528	2.807	2.095	1.714	1.594
60	27.057	13.553	4.600	2.850	2.126	1.738	1.616
70	27.824	13.937	4.728	2.927	2.181	1.781	1.654
80	28.490	14.269	4.839	2.994	2.228	1.818	1.687
90	29.077	14.563	4.937	3.052	2.270	1.851	1.717
100	29.603	14.826	5.024	3.105	2.308	1.880	1.743
120	30.512	15.281	5.176	3.196	2.373	1.930	1.788
140	31.282	15.665	5.304	3.272	2.428	1.973	1.827
160	31.948	15.998	5.414	3.339	2.475	2.010	1.860
180	32.536	16.292	5.513	3.398	2.517	2.043	1.890
200	33.062	15.556	5.601	3.451	2.555	2.072	1.916
220	33.538	16.794	5.680	3.498	2.589	2.099	1.940
240	33.973	17.011	5.753	3.542	2.620	2.123	1.961
260	34.373	17.211	5.819	3.582	2.649	2.145	1.981
280	34.743	17.396	5.881	3.619	2.675	2.165	2.000
300	35.088	17.568	5.938	3.653	2.700	2.185	2.017
330	35.564	17.806	6.018	3.701	2.734	2.211	2.041
360	35.999	18.024	6.090	3.744	2.765	2.235	2.063
390	36.399	18.224	6.157	3.784	2.793	2.257	2.083
420	36.769	18.409	6.219	3.821	2.820	2.278	2.101
450	37.114	18.581	6.276	3.856	2.844	2.297	2.119
480	37.436	18.742	6.330	3.888	2.867	2.315	2.135
510	37.739	18.894	6.380	3.918	2.889	2.332	2.150
540	38.025	19.037	6.428	3.947	2.909	2.348	2.164
570	38.295	19.172	6.473	3.974	2.929	2.363	2.178
600	38.551	19.300	6.515	4.000	2.947	2.377	2.190
650	38.951	19.500	6.582	4.040	2.976	2.399	2.210
700	39.322	19.685	6.644	4.077	3.002	2.420	2.229
750	39.667	19.858	6.702	4.111	3.027	2.439	2.246
800	39.989	20.019	6.755	4.143	3.050	2.457	2.262
850	40.292	20.171	6.806	4.174	3.071	2.474	2.277
900	40.578	20.313	6.854	4.202	3.092	2.490	2.292
950	40.840	20.449	6.899	4.229	3.111	2.505	2.305
1000	41.105	20.577	6.941	4.255	3.129	2.519	2.318
1050	41.349	20.699	6.982	4.279	3.147	2.532	2.330
1100	41.581	20.815	7.021	4.303	3.163	2.545	2.342
1150	41.803	20.926	7.058	4.325	3.179	2.558	2.353
1200	42.016	21.032	7.093	4.346	3.194	2.570	2.364
1250	42.220	21.134	7.127	4.367	3.209	2.581	2.374
1300	42.416	21.233	7.160	4.386	3.223	2.592	2.384
1350	42.605	21.327	7.191	4.405	3.237	2.602	2.393
1400	42.787	21.418	7.222	4.423	3.250	2.612	2.402
1450	42.962	21.505	7.251	4.441	3.262	2.622	2.411
1500	43.132	21.590	7.279	4.458	3.274	2.631	2.419
1550	43.296	21.672	7.306	4.474	3.286	2.641	2.428
1600	43.454	21.752	7.33291	4.490	3.300	2.649	2.436

TABLE 2 — DEPENDENCE OF A UPON M

M	A
.05	6.5644
.10	3.3065
.30	1.1846
.50	0.8010
.70	0.6622
.90	0.6000
1.00	0.5911

average rate of water influx during the time increment from $(i-1)\Delta t$ to $i\Delta t$, $\overline{e}_i$ is simply

$$\overline{e}_i = (V_{i-1} - V_i)/\Delta t \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (22)$$

and

$$\Delta\overline{e}_i = \overline{e}_{i+1} - \overline{e}_i = (2V_i - V_{i-1} - V_{i+1})/\Delta t \equiv \Delta V_i/\Delta t.$$

Therefore, Eq. 21 can be written as

$$p_j = p_o - \frac{.0502\,\mu}{r_b k\sqrt{k_R}\,\Delta t}\sum_{i=0}^{i=j-1}\Delta V_i \overline{P}_{j-i} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (23)$$

and gives reservoir pressure p as a function of time for an arbitrary, time-variant water-influx rate or, equivalently, reservoir pore volume variation.

The solution for the case of constant reservoir pressure, while not obtained directly here, can be approximated by re-arranging Eq. 23 as

$$p_o - p_j \equiv \Delta p = \frac{.0502\,\mu}{r_b k\sqrt{k_R}\Delta t}\left[\sum_{i=0}^{i=j-2}\Delta V_i \overline{P}_{j-i} + (2V_{j-1} - V_{j-2})\overline{P}_1 - V_j\,\overline{P}_1\right],$$

and

$$V_j = \frac{1}{\overline{P}_1}\left[\sum_{i=0}^{i=j-2}\Delta V_i \overline{P}_{j-1} + (2V_{j-1} - V_{j-2})\right]$$

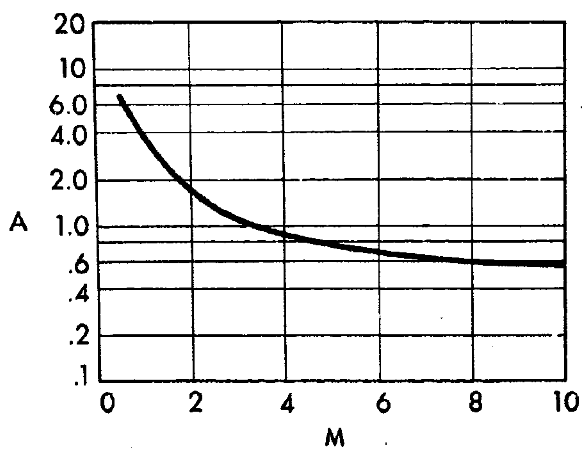

FIG. 3 — PLOT OF A VS M FOR THICK SAND MODEL.

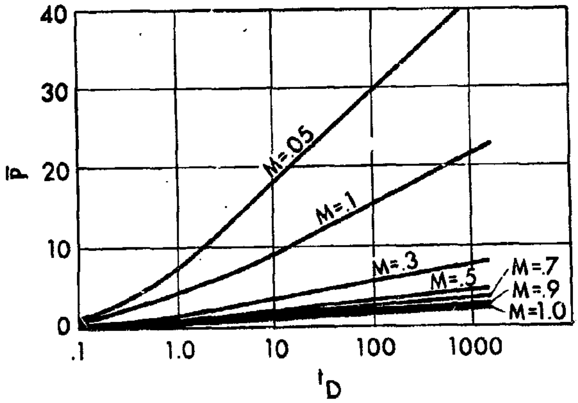

FIG. 4 — $\overline{P}$ AS A FUNCTION OF t_D AND M.

$$\overline{P}_1 - \frac{r_b k \sqrt{k_R}\,\Delta t\ \Delta p}{.0502\ \mu}\Bigg] \quad . \ . \ . \ . \ . \ . \ . \ (24)$$

Eq. 24 gives the reservoir pore volume V as a function of time when the reservoir pressure p is held at a constant value less than p_o by Δ_p. The cumulative volume of water influx into the reservoir is, of course,

$$W_e = V_o - V_j \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (25)$$

EXAMPLE PROBLEM 1

A gas storage reservoir has been created by injection of gas into an aquifer formation. The field has been grown to a radius r_b of 3,000 ft and has been shut in for a period of time sufficient to allow the reservoir and aquifer pressure to reach an approximately uniform value of 1,080 psia. Estimate the reservoir pressure as a function of time which must be maintained to grow the gas bubble at a constant rate of 80 Mcf of pore volume/D. The aquifer formation is 550-ft thick, core data indicate a permeability ratio k_R of .37, and water-pumping tests indicate an effective aquifer horizontal permeability of .310 darcies. Other available data are $\phi = .17$, $c = 7 \times 10^{-6}$ 1/psi, $\mu = 1$ cp.

SOLUTION

Since the rate of water movement is specified to be constant and bottom-water drive exists, the constant-rate Eq. 20 will be employed. The value M is

$$M = h/r_b\sqrt{k_R} = 550/3000\ \sqrt{.37} = 0.3,$$

so that $\overline{P}$ values corresponding to $M = .3$ will be read from Table 1. If pressures are calculated at 30-day intervals, then at the end of i 30-day periods,

$$t_D = i\ \frac{6.33k\ \Delta t}{\mu\ \phi\ c r_b^2}$$

$$= i\ \frac{6.33\ (.310)\ (30)}{1(.17)(7\times10^{-6})(3000)^2}$$

$$=\ 5.5\,i.$$

Interpolation in Table 1 at $M = .3$ then yields the first three columns of Table 3. From the data given,

$$\frac{.0502\ e_w\mu}{r_b k\ \ \sqrt{k_R}} = \frac{.0502(-80,000)(1)}{3000(.310)\ \sqrt{.37}} = -7.1$$

where e_w is negative because the bubble is being grown and water is moving away from the reservoir. Eq. 20 now becomes

$$p = p_o + 7.1\ \overline{P} = 1080 + 7.1\ \overline{P},$$

which allows calculation of the last column in Table 3. The required gas-injection schedule could be calculated from the gas equation of state $n = pV/zRT$ where p is reservoir pressure, V is reservoir pore volume, n is gas in place and z is the compressibility factor (the reservoir pore volume V_o at "zero time", when uniform pressure of 1,080 exists, would have to be known).

Eqs. 20, 23 and 24 are the basic "working equations" allowing calculation of reservoir pressure or volume from knowledge of the water-influx rate or reservoir pressure. In general, however, neither the influx rate nor reservoir pressure is known in advance; rather, a fluid-in-place (oil or gas) schedule is known or specified and an estimate of the reservoir performance (pressure and/or pore volume vs time) is desired. In this case, a relationship is needed between fluid in place, pore volume V and reservoir pressure p, i.e., a material-balance equation. In the case of gas reservoirs, this material balance is exceptionally simple and will be used in conjunction with Eq. 23 to yield a pressure-explicit equation for use in solution of the problem just stated. For a gas reservoir,

$$V_j = n_j RT\left(\frac{z}{p}\right)_j^{*}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (26)$$

and since z can be represented as a linear function of pressure over normal operating pressure ranges,

$$z_j = a + bp_j$$

and

$$V_j = n_j RT(b + a/p_j) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (27)$$

*The assumption is implied here that the reservoir pressure p is essentially uniform so that p_j in Eqs. 26 and 23 are in the same pressure.

TABLE 3

Time (months)	t_D	$\overline{P}$	Reservoir Pressure p (psi)
0	0	0	1080
1	5.5	2.638	1098.7
2	11	3.197	1102.7
3	16.5	3.532	1105.1
4	22	3.766	1106.8
5	27.5	3.953	1108
6	33.0	4.103	1109.1

Eq. 23 can be written

$$p_j = p_o - \frac{.0502\ \mu}{r_b k\ \sqrt{k_R}\Delta t}\left[\sum_{i=0}^{i=j-2} \Delta V_i\ \overline{P}_{j-i} + (2V_{j-1} - V_{j-2})\,\overline{P}_1 - V_j\ \overline{P}_1\right], \quad . . (28)$$

and elimination of V_j between Eq. 27 and Eq. 28 yields a quadratic in p_j which gives

$$p_j = C_j + D_j \quad . . . . (29)$$

where

$$C_j^* = \frac{1}{2}\left[p_o + F\overline{P}_1 RTbn_j - F\left(\sum_{i=0}^{i=j-2} \Delta V_i \overline{P}_{j-i} + (2V_{j-1} - V_{j-2})\overline{P}_1\right)\right],$$

$$D_j = C_j^2 + F\overline{P}_1 RTn_j a,$$

and

$$F = .0502\ \mu/r_b k\ \sqrt{k_R}\ \Delta t.$$

Eq. 29 allows direct calculation of gas reservoir pressure at successive times of Δt, $2\Delta t$, $3\Delta t$, . . . , for any given gas-in-place schedule n_j. Coats, Tek and Katz[4] derived an equation similar to Eq. 29 from the constant-rate-case solution given by van Everdingen and Hurst[1] for horizontal, radial aquifer water flow.

EXAMPLE PROBLEM 2

A newly discovered gas reservoir is to be converted to storage use after a period of production. The projected schedule of cumulative gas production is plotted in Fig. 5. The following data are available from various tests and sources; $V_o = 4.4 \times 10^8$ cu ft, $p_o = 466$ psia, $r_b = 1{,}880$ ft, $k = .0296$ darcies, $\mu = 1.2$ cp, $RT = 5{,}570$ psia cu ft/lb mole, $\phi = 0.2$, $c = 7 \times 10^{-6}$ 1/psi, $h = 565$ ft, $k_R = 1.0$, $a = .998$, and $b = -.00016$. Estimate the pressure and pore-volume behavior of the reservoir using the thick sand model, and compare this behavior to that predicted by the horizontal, radial-flow model.

*For $j = 1$, $\sum_{i=0}^{i=j-1} \Delta V_i \overline{P}_{j-i} = 0$, $V_{j-2} = V_{-1} = V_o$

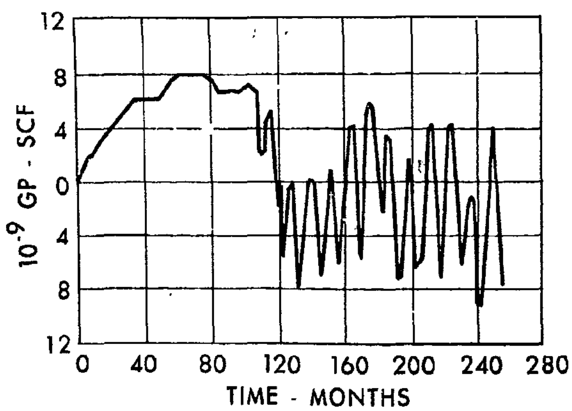

FIG. 5 — CUMULATIVE GAS PRODUCTION AS A FUNCTION OF TIME.

SOLUTION

The production schedule plotted in Fig. 5 consists of monthly values of cumulative gas produced over a period of 254 months. A time increment Δt of one month, or 30.4 days, will be chosen therefore. From the data and Eq. 27,

$$\begin{aligned} n_o &= V_o/RT\ (b + a/p_o) \\ &= 4.4 \times 10^8/5570(-.00016 + .998/466) \\ &= 39.9 \times 10^6 \text{ lb mole.} \end{aligned}$$

The gas in place at time $j\Delta t$ is then

$$\begin{aligned} n_j &= n_o - G_{pj}\, p_{\text{base}}/RT_{\text{base}} \\ &= n_o - G_{pj}\,[15.025/10.73\ (520)] \end{aligned}$$

where G_{pj} is the cumulative standard cubic feet of gas produced at the end of j months. An IBM 704 computer program was written to accept the gas-production schedule, the afore-mentioned data, and a table of $\overline{P}$ values vs dimensionless time t_D for $M = h/r_b\ \sqrt{k_R} = 565/1880\ (1) = .3$ to calculate the gas-in-place n_j; and solve Eq. 29 for the predicted pressure p_j. These pressure and the corresponding pore volumes, calculated from Eq. 27, are plotted as the solid curves in Figs. 6 and 7.

The variable-pressure-case solution for the horizontal, radial-flow model[1] is

$$W_e = \pi h \phi c r_b^2 \sum_{i=0}^{i=j-1} \Delta p_i Q_{j-i}$$

where $\Delta p_i = p_{i-1} - p_{i+1}$, $\Delta p_o = p_o - p_1$ and Q_{j-i} is the dimensionless influx quantity Q at $t_D = (j-i)\ \Delta t_D$ tabulated by van Everdingen and Hurst. Combination of this equation with Figs. 25 and 27 gives a pressure-explicit equation[4] similar to Eq. 29. Solution of this equation for the same data previously given, the same gas-in-place schedule and for an infinite (in radial extent) aquifer gave the dashed curves shown in Figs. 6 and 7.

DISCUSSION AND CONCLUSIONS

Eq. 16 or Eq. 20 is, to the author's knowledge, the only solution available to the diffusivity equation governing aquifer water movement about a bottom-water-drive reservoir. While this solution is valid for radially infinite aquifers, other solutions can be obtained by use of finite Hankel transforms (see Appendix) for aquifers of various degrees of finiteness. Example Problem 2 shows that significant differences may arise between field performances predicted by the thick sand and horizontal radial-flow models. Figs. 6 and 7 show quantitatively the differences between reservoir pressures and pore volumes calculated from the two models.

In recent years increasing emphasis has been placed on the "resistance-curve" technique as opposed to the flow-model approach. The latter approach by necessity involves various idealizations pertinent to reservoir and aquifer geometry and aquifer homogeneity. The objection thus arises that most practical cases violate significantly one or more of these idealizations. The resistance-curve method meets this objection by requiring no assumptions concerning aquifer geometry and homogeneity but, rather, involves the determination of a resistance-curve analogous to the $\overline{P}$ vs t_D curve of this paper or the Q_t vs t_D function of van Everdingen and Hurst[1] directly from field data. The paper by Hutchinson and Sikora[5] is an example of this method.

It is the author's opinion that the resistance-curve method will, in the long run, replace the model approach with resultant increased ease and accuracy in calculation of water movement. However, this method has been of little value to date in studies of edge-water-drive gas fields carried out at the U. of Michigan, and requires development far beyond its present state. Various degrees of success in matching field behavior have been achieved by use of the horizontal radial-flow model, reflecting the various degrees to which the edge-water-drive fields studied satisfy the idealizations involved in that model. At the present time, flow models provide a useful tool in prediction of reservoir performance and, in the future, should remain useful as standards for comparison to resistance curves obtained from field data by a reliable method. For example, conclusions pertinent to aquifer geometry and/or extrapolation (in time) of resistance curves might be accomplished by comparing these curves to those corresponding to various models.

ACKNOWLEDGMENT

Suggestions and incentive from D. L. Katz, chairman of the Dept. of Chemical and Metallurgical Engineering, greatly aided this study. The author gratefully acknowledges the permission of R. C. F. Bartels, director of the U. of Michigan Computing Center, to carry out calculations on the University's IBM 704 computer, and the American Gas Assn. Project No. 31 for its financial assistance.

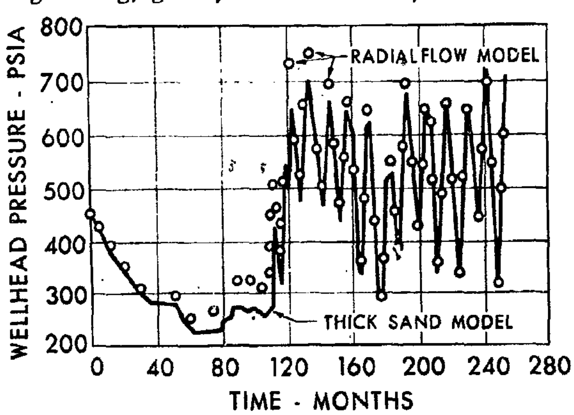

FIG. 6 — WELLHEAD PRESSURE AS A FUNCTION OF TIME.

NOMENCLATURE

a = constant in equation $z = a + bp$, dimensionless

b = constant in equation $z = a + bp$, 1/psi

c = compressibility of aquifer water and formation, 1/psi

E = $e_w \mu / \pi r_b k \sqrt{k_R}$

e_w = rate of water influx, cu ft/day

$\overline{e}_i$ = average rate of water influx from time $(i-1)\Delta t$ to $i\Delta t$, cu ft/day

$\Delta\overline{e}_i$ = $\overline{e}_{i+1} - \overline{e}_i$

g_c = conversion constant, 32.17 ft-lb mass/lb force-second^2

h = aquifer thickness, ft

J_0, J_1 = Bessel functions of first kind, of order 0 1, respectively

k = aquifer formation permeability in horizontal direction, darcies

k_R = ratio of vertical-to-horizontal aquifer permeability

k_v = aquifer formation permeability in vertical direction, darcies

M = parameter, $h/r_b\sqrt{k_R}$

n_j = gas in place in reservoir at time $j\Delta t$, lb mole

p = pressure, psia

$p_i(y)$ = initial aquifer pressure, psia

p_i = reservoir pressure at time $i\Delta t$, psia

p_o = initial aquifer (and reservoir) pressure at reservoir depth, psia

$\overline{P}$ = dimensionless pressure-drop function

$\overline{P}_i$ = value of $\overline{P}$ at $t_D = i\Delta t_D$

Δp_i = $p_{i-1} - p_{i+1}$

r = radius, ft

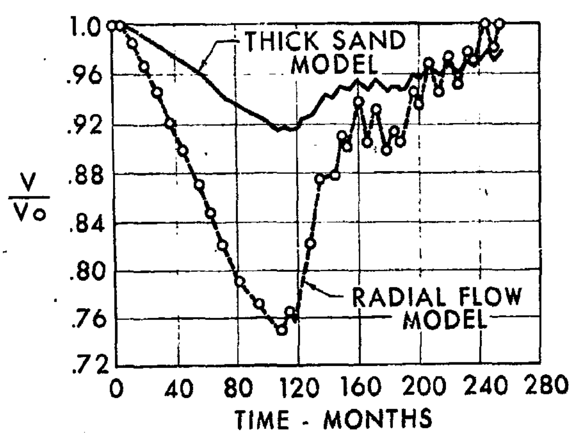

FIG. 7 — RESERVOIR-PORE-VOLUME RATIO AS A FUNCTION OF TIME.

r_D = dimensionless radius, r/r_b
r_b = radius of reservoir, ft
r_e = exterior radius of aquifer, ft
R = gas constant, 10.73 psia-cu ft/lb mole-°R
t = time, days
Δt = time increment, days
t_D = dimensionless time, $6.33\ kt/\mu\phi c r_b^2$
T = reservoir temperature, °R
u = velocity of aquifer water flow, ft/day
V = reservoir pore volume, cu ft
V_i = reservoir pore volume at time $i\Delta t$; cu ft, $V_{-1} = V_o$
$\Delta V_i = 2V_i - V_{i-1} - V_{i+1}$; $\Delta V_o = V_o - V_1$
V_o = initial reservoir pore volume, cu ft
W_e = cumulative water influx, cu ft
y = dimensionless vertical distance, $z/r_b\sqrt{k_R}$
z = vertical distance co-ordinate, ft, or gas compressibility factor, dimensionless
ρ = density, lb mass/cu ft
ϕ = aquifer formation porosity
$\alpha_m = m\pi/M$
μ = aquifer water viscosity, cp

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APPENDIX

The solution to Eq. 6 for conditions of Eqs. 11 through 15 is obtained by use of the infinite Hankel transform[6] defined by

$$U(x, y, t_D) = \int_0^\infty r_D P(r_D, y, t_D) J_0(xr_D)\, dr_D \quad . \ (30)$$

Following Sneddon,[6] the Hankel transform of $\frac{\partial^2 P}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P}{\partial r_D}$ is

$$\int_0^\infty r_D \left(\frac{\partial^2 P}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P}{\partial r_D}\right) J_0(xr_D)\, dr_D = -x^2 U(x, y, t_D), \quad \ldots \ (31)$$

so that multiplication of both sides of Eq. 6 by $r_D J_0(xr_D)\,dr_D$ and integration from zero to ∞ yields

$$\frac{\partial^2 U}{\partial y^2} - x^2 U = \frac{\partial U}{\partial t_D} \quad \ldots \ (32)$$

Solution of Eq. 32 by separation of variables, $U(x, y, t_D) = Y(y)\,\theta(t_D)$, yields

$$U = A \cosh xy + B \sinh xy + e^{-\lambda^2 t_D}\left(C \cos\sqrt{\lambda^2 - x^2}\, y + D \sin\sqrt{\lambda^2 - x^2}\, y\right) \quad \ldots \ (33)$$

In order that Eq. 11 be satisfied,

$$\left.\frac{\partial U}{\partial y}\right|_{y=0} = \bar{f}(x)$$

where $\bar{f}(x)$ is the Hankel transform of $f(r_D)$, so that

$$Bx + D\sqrt{\lambda^2 - x^2}\, e^{-\lambda^2 t_D} = \bar{f}(x)$$

$$= \int_0^\infty r_D f(r_D) J_0(xr_D)\, dr_D$$

$$= -\int_0^1 E J_0(xr_D) r_D\, dr_D = -E\frac{J_1(x)}{x}, \quad \ldots \ (34)$$

and D must equal zero and B must be $-EJ_1(x)/x^2$ if Eq. 34 is to hold for all time. Thus, U is now given by

$$U = A \cosh xy - \frac{EJ_1(x)}{x^2}\sinh xy + C\cos\sqrt{\lambda^2 - x^2}\, y\ e^{-\lambda^2 t_D}.$$

In order that Eq. 13 be satisfied,

$$\left.\frac{\partial U}{\partial y}\right|_{y=M} = 0$$

$$= A \sinh Mx - \frac{EJ_1(x)}{x}\cosh Mx - C\sqrt{\lambda^2 - x^2} \cdot \sin(\sqrt{\lambda^2 - x^2}\, M)\, e^{-\lambda^2 t_D},$$

which requires

$$A = \frac{EJ_1(x)}{x^2}\coth Mx \quad \ldots \ (35)$$

$$\sqrt{\lambda^2 - x^2} = m\pi/M \equiv \alpha_m,\ m = 0, 1, 2, \ldots \quad (36)$$

Thus, U is now given by

$$U = \frac{EJ_1(x)}{x^2}\left\{\frac{\cosh[x(M-y)]}{\sinh Mx}\right\}$$

$$+\sum_{m=0}^{m=\infty} C_m \cos a_m y \, e^{-\lambda^2{}_m t_D} \quad . \quad . \quad . \quad . \quad (37)$$

where $\lambda_m^2 = x^2 + a_m^2$ and the summation is imposed in order to satisfy the initial condition of Eq. 12. This initial condition is $U(x, y, 0) \cong 0$ or

$$\sum_{m=0}^{m=\infty} C_m \cos a_m y = -\frac{E J_1(x)}{x^2} \frac{\cosh [x(M=y)]}{\sinh Mx} \quad . \quad . \quad . \quad (38)$$

The terms $\cos a_m y$ form an orthogonal set over the interval $(0, M)$, so that multiplication of both sides of Eq. 38 by $\cos (a_m y) dy$ and integration from 0 to M yields

$$C_m = -\frac{2EJ_1(x)}{(a_m{}^2 + x^2)Mx} \; ,$$

and

$$C_o = -\frac{EJ_1(x)}{Mx^3} \; .$$

The final solution for U now appears.

$$U(x,y,t_D) = \frac{EJ_1(x)}{x^2} \left\{ \frac{\cosh [x(M-y)]}{\sinh Mx} - \frac{e^{-x^2 t_D}}{M\cdot} - \frac{2x}{M} \sum_{m=0}^{m-\infty} \frac{e^{-\lambda^2{}_m t_D}}{a_m{}^2 + x^2} \cos a_m y \right\} . \; (39)$$

Since the Hankel inversion integral is[6]

$$P(r_D, y, t_D) = \int_0^\infty x \, U(x, y, t_D) J_0(r_D x) dx, \quad . \quad . \; (40)$$

the final solution for P is

$$P(r_D, y, t_D) = E \int_0^\infty \frac{J_1(x)}{x} \left\{ \frac{\cosh[x(M-y)]}{\sinh Mx} \frac{e^{-x^2 t_D}}{Mx} - \frac{2x}{M} \sum_{m=1}^{m=\infty} \frac{e^{-(x^2 + a_m^2) t_D}}{x^2 + a_m^2} \cos a_m y \right\} J_0(r_D x) dx,$$

which becomes identical to Eq. 16 when y is set equal to zero. ***

Further Discussion of Paper Published in
***Society of Petroleum Engineers Journal*, March, 1962**

A Mathematical Model for Water Movement about Bottom-Water-Drive Reservoirs

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(Published on Page 44)

DISCUSSION

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The mathematical problem considered by the author[1] can be given another physical interpretation which is of some practical significance. The alternative physical problem involves the approximate behavior of a single well which is producing at a constant rate through an axially symmetric, horizontal fracture of infinite flow capacity which is located at the center, top or bottom of a uniformly thick, horizontal, homogeneous, anisotropic reservoir of infinite lateral extent which contains a single, slightly compressible fluid. Using Coats' asymptotic results, the wellbore pressure in the fractured system is given by the following equation.

$$p_w = p_1 - \frac{162.5\, q\, \mu\, B}{k\, h}\left\{\log\left(\frac{.00633\, k\, t}{\phi\, \mu\, c\, r_w^2}\right) + .351 + .87\, S^*\right\};\quad t \geq 1580\, \phi\, \mu\, c\, r_f^2/k \qquad (1)^{**}$$

where

$$S^* = \begin{cases} MA(M/2) - \ln(r_f/r_w) - .4045, & \text{center,} \\ 2MA(M) - \ln(r_f/r_w) - .4045, & \text{top or bottom,} \end{cases}$$

$A(M)$ = geometric constant (numerical values are given in Table 2 of the subject paper),

$M = h\sqrt{k/k_v}\,/r_f$,

k = permeability in the horizontal direction, and

k_v = permeability in the vertical direction.

It is obvious that Eq. 1 differs from the usual equation for radial flow into a wellbore by the geometric parameter S^*; this result is in accord with the basic assumption made by Hartsock and Warren[2] in a prior steady-state study of fractured systems. Transient S^*-values based on Coats' results and steady-state values computed by the method of Hartsock and Warren are compared in Fig. D-1; the small, systematic deviations from the ideal correlation are the result of the nonuniform distribution of the flux over the fracture surface.

Because of the mathematical equivalence of the two physical problems, the following supplementary conclusions can be drawn.

1. By including the pseudo-skin resistance S^*, the performance of a reservoir with a bottom-water

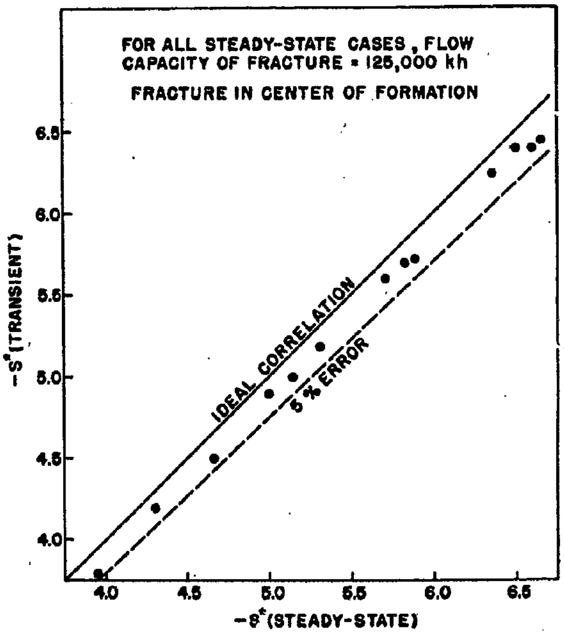

FIG. D-1—COMPARISON OF TRANSIENT AND STEADY-STATE RESULTS.

[1]References given at end of paper.

**Standard AIME nomenclature is used unless otherwise indicated; units are psi, STB/D, cp, md, ft, reservoir bbl/STB, $(psi)^{-1}$ and days.

drive can be asymptotically approximated by utilizing conventional methods which assume radial influx; the values of $A(M)$ given by Coats can be used for limited aquifers if the external radius of the aquifer is at least four times as large as the radius of the region initially occupied by oil, r_o, and $t \geq 1580\ \phi\ \mu\ c\ r_o^2/k$.

2. If the radius of a high-capacity fracture is less than one-fourth of the drainage radius, the steady-state values of S^* controls the performance of the well for $t \geq 1580\ \phi\ \mu\ c\ r_f^2/k$; post-fracturing production tests will indicate erroneously high PI's if the duration of the production period is less than $1580\ \phi\ \mu\ c\ r_f^2/k$.

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AUTHOR'S REPLY TO J. E. WARREN

The equivalence pointed out between the bottom-drive reservoir and fractured well problems is of significant interest and is an outstanding example of consolidation of distinct research results into a more meaningful whole. Only one observation is made here. Warren's Conclusion 1 states that bottom-drive *reservoir* performance can be asymptotically approximated by conventional methods involving radial influx if the S^* factor is included. This conclusion applies with more meaning to the fractured *well* problem where production rate is held constant. In the case of a reservoir (especially gas-storage reservoirs where cycling occurs), the production (influx) rate may vary considerably with time which requires that superposition be applied to Eq. 1. The result of this superposition is that dimensionless pressure-drop values at small times are required and these values cannot be approximated by the asymptotic expression employed in Eq. 1. Recourse must then be made to an equation such as Eq. 21 of the subject paper, and $\bar{P}$ values at dimensionless time less than 10 must be obtained from the tables.

***