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Application of Buckley-Leverett Displacement Theory to Noncommunicating Layered Systems

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ABSTRACT

This paper presents the results of applying the Buckley-Leverett[1] displacement theory to petroleum reservoirs consisting of a finite number of layers. The layers are assumed to communicate only in the wellbores, and the reservoir may be represented as a linear system. Most previous investigations of this nature were limited by assumptions and by inconsistent calculation techniques. This study improves on previous work by applying the Buckley-Leverett displacement theory to a noncommunicating layered system where permeability, porosity, initial saturation, residual saturation and relative permeability vary from layer to layer in a logical and consistent manner. Gravity and capillary-pressure effects are neglected. A modification of the Higgins-Leighton calculation method was used in this study. Waterflood predictions were made with all properties varying, and then with only permeability varying using several mobility ratios. These results were compared with the Stiles and Dykstra-Parsons predictions. It is shown that the latter methods generally give poor values for the breakthrough recovery and pessimistic predictions for the performance after breakthrough. Similar results were obtained for a gas-displacement case.

INTRODUCTION

Field experience with immiscible displacement usually shows constant producing conditions until breakthrough of the displacing fluid. Then oil production continues at increasing displacing-to-displaced fluid ratios until the economic limit is reached. Three different ideal mechanisms are known that will produce this behavior: (1) relative permeability effects as described by Buckley-Leverett frontal advance theory,[1] (2) vertical stratification as considered by Stiles,[2] Dykstra and Parsons[3] and others and (3) different path lengths involved in areal (two-dimensional) flow between wells as described by Dyes *et al.*[4] Without question, a combination of these factors modified by formation heterogeneity and other known and unknown factors actually does control the behavior of real systems. This paper presents results of an investigation of certain factors that should affect performance but which have received little attention to date.

In 1944, Law[5] demonstrated that porosity and permeability are often found to have normal and logarithmic-normal distributions, respectively, throughout cored intervals in natural formations. This led to the concept of the noncommunicating, multilayered reservoir model for immiscible displacement. This model assumes that the reservoir is composed of a number of layers that communicate only at the wellbores. Each layer is individually homogeneous, but may be different from every other layer. Stiles[2] presented one of the earliest applications of this model to waterflood performance. In addition, Stiles assumed that the initial saturations and relative permeabilities were the same for each layer, porosity was the same, displacement was piston-like, fluids were incompressible and injection into each layer was proportional to that layer's permeability capacity (permeability-thickness product). The last assumption would be true if the mobility ratio for the displacement were unity.[21] Dykstra and Parsons[3] used the same model as Stiles, but rigorously included mobility ratios other than unity for piston-like displacement. Dykstra and Parsons used their general result to produce charts for log-normal permeability distributions between layers. Similarly, Muskat[6] published analytical solutions for linear and exponential permeability distributions.

In 1959, Roberts[7] described a scheme for calculating water-drive performance for the noncommunicating, layered reservoir model which considered two-phase flow in the displaced region. Roberts used the same model and assumed that the injection rate into a layer was proportional to that layer's permeability capacity, but that flood front locations could be evaluated from the Dykstra-Parsons results. These assumptions are inconsistent, and a material balance cannot be maintained except for a mobility ratio of unity. At the same time, Kufus and Lynch[8] coupled Buckley-Leverett displacement theory with the layered model to provide an improvement of the Dykstra-Parsons method that was consistent.

In 1960, Higgins and Leighton[9] presented a numerical method for calculating waterflood performance also considering two-phase flow in the displaced region. The result was used to investigate variation in absolute permeability and oil viscosity. An excellent, detailed history of using the noncommunicating, layered reservoir model was presented by Nielsen.[10]

The preceding techniques (and many related ones) were similar in that differences in initial saturations, residual saturations and relative permeabilities from layer to layer were neglected. It is well known that the irreducible water saturation is an important function of absolute permeability. Calhoun[11] showed that the irreducible water saturation

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[1]References given at end of paper.

increases by about 5 percent of pore volume each time the permeability is cut in half for a variety of rocks. Similar results are given by Amyx, Bass and Whiting.[12]

Naar and Henderson[14] showed that the residual nonwetting phase saturation under imbibition is approximately one-half the initial saturation of that phase. Thus, the residual nonwetting-phase saturation should be affected by the residual water saturation. Finally, the work of Corey[15] and Naar and Henderson indicates an attendant change in relative permeabilities for both imbibition and drainage processes as a consequence of irreducible water and residual nonwetting-phase changes. In addition to changes in phase saturations just described, it would also be reasonable to expect variations in initial saturations if the layers had been subject to depletion. Sheldon[16] found a wide range in initial gas saturations for depletion of the noncommunicating layered system.

DESCRIPTION OF METHOD

A model similar to that of Higgins and Leighton[9] was used in this study. The reservoir was considered to be composed primarily of a finite number of homogeneous layers. The following properties were allowed to vary between layers: absolute permeability, porosity, thickness, initial saturations, residual saturations and relative permeability-saturation relations. The following assumptions were made: (1) constant width and length for all layers, (2) negligible capillary and gravity forces, (3) constant outlet pressure, (4) constant pressure drop for all layers at a given time, (5) constant injection rate for the reservoir (for ease of comparison with other models), (6) incompressible and immiscible fluid flow and (7) no crossflow between layers.

Each layer was divided into a number of flow cells (Fig. 1). Injecting an incremental volume of water — used here for the injection fluid, although both water injection and gas injection were studied — into the first cell causes water and oil to move from cell to cell resulting in production from the last cell. This results in a change of saturation. The fractional flow of oil and water from cell to cell may be determined from relative permeability and viscosity ratios. The specific resistivity for flow in cell i is

$$\text{res}_i = \left[\frac{\Delta x}{\frac{k_{ro}}{\mu_o} + \frac{k_{rw}}{\mu_w}}\right]_i \qquad (1)$$

and the total resistance to flow in layer j is

$$R_j = \frac{\sum_{i=1}^{NC} \text{res}_i}{k_j A_j} \qquad (2)$$

where NC = total number of cells in the jth layer. The total resistance to flow for the entire reservoir is

$$R_t = \frac{1}{\sum_{j=1}^{NL} \frac{1}{R_j}} \qquad (3)$$

where NL = total number of layers in the reservoir. Since the total pressure drop is assumed to be the same across each layer, it is possible to calculate the injectivity of the jth layer at a given time by the following proportion:

$$\frac{i_{wj}}{i_{tw}} = \frac{i_{wj}}{\sum_{j=1}^{NL} i_{wj}} = \frac{R_t}{R_j} \qquad (4)$$

Clearly, this ratio will be a function of time for each layer when the mobility ratio is other than unity. By fixing the outlet pressure and the total injection rate it is possible to calculate the reservoir injection pressure and the injection rate for each layer as a function of time.

The incremental volume of water injected into the first cell of each layer is the product of that layer's injection rate and a small increment of time. Time increments on the order of 0.1 to 0.5 days were used in this study, although values outside this range were examined for accuracy of the solution. The fractional flow of water from cell to cell during the time period was calculated from saturation conditions existing at the start of the period using the fractional flow equation

$$f_w = \frac{1}{1 + \frac{k_o \mu_w}{k_w \mu_o}} \qquad (5)$$

The calculations require use of a digital computer; however, the programming is not particularly difficult.

The method used, a modification of the Higgins-Leighton method, is actually a numerical solution to the Buckley-Leverett frontal advance equation using an explicit approach and neglecting gravity and capillarity. Perhaps a more sophisticated numerical calculation could have been used, but the one chosen lends itself to easy understanding and seems to be adequate for purposes of this study. There was some difficulty in choosing a sufficiently fine cell length (mesh spacing) to track the front accurately. Mesh sizes of the order used by Higgins and Leighton[9] were used on a problem almost identical with theirs and gave fractional injectivities that oscillated with time. To reach a balance between reasonable computation time and a sufficiently small mesh to track the front accurately, it was necessary to use a finer spacing in the vicinity of the front and a wider spacing elsewhere. Fractional injectivities accurate to four figures were then obtained with no oscillations. Further discussion of the numerical technique is included in the Appendix.

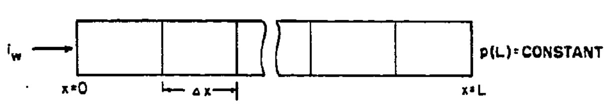

Fig. 1—Calculation Model.

WATERFLOOD

The waterflood performance of a reservoir consisting of 10 layers was studied using the method previously outlined. The results of this method (hereafter called Buckley-Leverett solution) are compared with results obtained using both the Stiles and Dykstra-Parsons techniques.

The permeability in the model reservoir was considered to be log-normally distributed with a variation of 0.5 as defined by Standing, Lindblad and Parsons.[17] Porosity was assumed to be normally distributed and related to the absolute permeability by the following equation taken from Warren and Skiba.[18]

$$\phi = \overline{\phi_A} + \frac{\sigma_\phi}{\sigma_k} \ln (k/k_M) \qquad (6)$$

The specific relation used in this study is shown in Fig. 2 and Table 1. Connate water, considered synonymous with residual water, is a function of absolute permeability (Fig. 2, Table 1). This relation was adapted from Fig.

TABLE 1 — VARIATION OF SATURATIONS AND POROSITY WITH ABSOLUTE PERMEABILITY

Layer	Permeability (md)	Connate Water Saturation (percent)	Porosity (percent)	Initial Gas Saturation* (percent)
1	750	23.7	20.3	6.3
2	560	27.3	18.6	4.8
3	455	29.8	17.4	3.9
4	380	32.1	16.4	3.3
5	325	34.0	15.5	2.9
6	275	36.0	14.5	2.4
7	235	38.0	13.6	2.1
8	195	40.3	12.5	1.8
9	160	42.7	11.4	1.4
10	120	46.2	9.7	1.1

*Gas injection case only.

3-27 of Amyx, Bass and Whiting,[12] and may be stated analytically as

$$S_{wc} = \overline{S}_{wc} - \frac{\sigma_{S_{wc}}}{\sigma_k} \ln (k/k_M) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The thickness was not varied between layers in order to facilitate checking our results with those of other methods. The initial water saturations were set equal to the connate water saturations for the same reason.

The basic reservoir and fluid properties for the waterflood cases are given in Table 2. The mobility ratio M is defined as the ratio of the water mobility at residual oil saturation to the oil mobility at residual water saturation. This mobility ratio was used in the Stiles and Dykstra-Parsons calculations. Other methods[19-20] for determining water mobility for this ratio were not examined in detail since this was not the main purpose of the study. The residual oil saturation is a function of S_{wc} for the waterflood (imbibition) cases.

TABLE 2 — RESERVOIR AND FLUID PROPERTIES, 10-LAYER WATERFLOOD MODEL

	Case A	Case B	Case C
M	0.125	1.0	10.0
L, ft	600	600	600
W, ft	600	600	600
h, ft	15	15	15
$p(L)$, psia	2,000	2,000	2,000
μ_o, cp	2.0	4.0	40.0
μ_w, cp	1.0	1.0	1.0
B_o, res. bbl/STB	1.3	1.3	1.3
B_w, res. bbl/STB	1.0	1.0	1.0
i_w, STB/D	1,000	1,000	1,000
k_{ro}	. . . $(1-2S)^{3/2}\,[2-(1-2S)^{1/2}]$ . . .		
k_{rw}	S^4	S^2	S^2
V	0.5	0.5	0.5
k_M, md	300	300	300
$\overline{\phi}$	0.15	0.15	0.15
$\overline{S}_{wc}$	0.35	0.35	0.35
σ_ϕ	0.04	0.04	0.04
$\sigma_{S_{wc}}$	0.0853	0.0853	0.0853

Figs. 3 and 4 compare prediction techniques for Case A where a favorable mobility ratio of 0.125 was used. "Recoverable oil" is the oil recovered when only residual oil remains. The Buckley-Leverett solution is shown for two models: Model 1 where all properties vary between layers, and Model 2 where permeability varies but porosity,

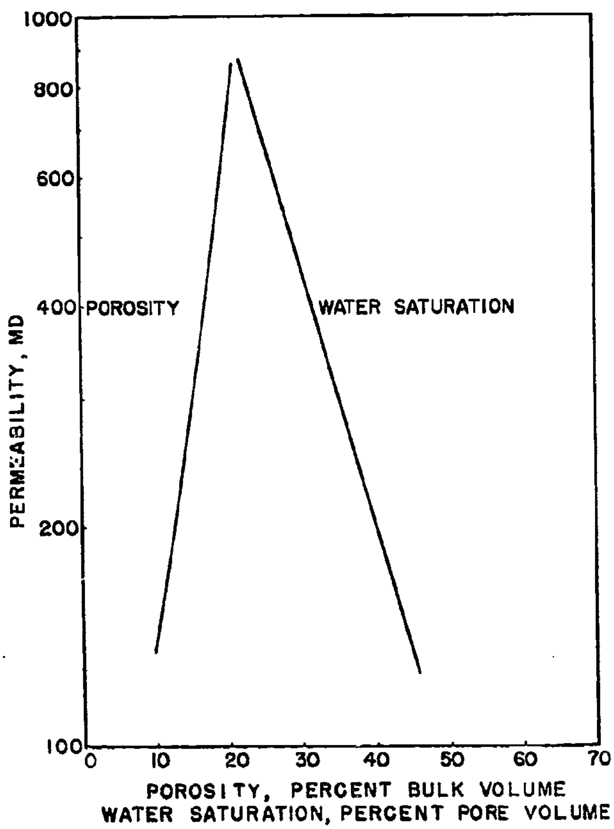

FIG. 2—WATER SATURATION AND POROSITY VS PERMEABILITY.

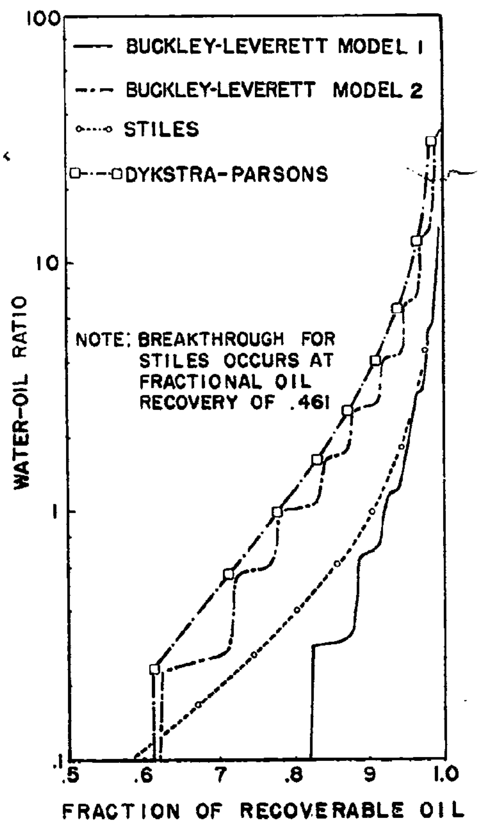

FIG. 3—WATERFLOOD CASE A (M=0.125), RECOVERY VS WOR.

connate water and residual oil are constant and equal to the average values used in the Stiles and Dykstra-Parsons calculations. Both the Stiles and Dykstra-Parsons methods give much earlier breakthrough recoveries compared with the Buckley-Leverett solution where all properties may vary (Model 1). The differences do not show up as clearly on the time plot (Fig. 4) because of the scale chosen. Further, there is a significant difference between the performance after breakthrough among the three methods although the Stiles and Buckley-Leverett predictions do approach one another during the late stages of the project. As might be expected, the Dykstra-Parsons prediction and the Buckley-Leverett Model 2 predictions (only permeability varies) are quite similar over the total life. In fact, the results show that the Dykstra-Parsons WOR curve should be drawn in a step-wise fashion rather than using a smooth curve through the points. Use of a step-wise WOR variation would make the time curves for the Dykstra-Parsons and Buckley-Leverett Model 2 agree more closely. This is because it is necessary in the Dykstra-Parsons technique to evaluate

$$\int_0^{N_p} (\text{WOR})\, dN_p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

to determine the time required to reach a given recovery.

Figs. 5 and 6 show the predictions for Case B using a mobility ratio of 1.0. The reservoir and fluid properties are the same as Case A with the following exceptions: $\mu_o =$ 4.0 cp and $k_{rw} = S^2$. The Stiles and Dykstra-Parsons methods are identical for a unity mobility ratio. Comments regarding the performance of reservoir Case A apply similarly to reservoir Case B. As expected, breakthrough occurs much earlier for Case B than for Case A due to increased mobility of the displacing phase.

The final waterflood prediction (Case C) for a mobility ratio of 10 is shown in Figs. 7 and 8. The reservoir and fluid properties are the same as Case B except that oil viscosity has been increased to 40 cp. The Buckley-Leverett Model 1, where all properties vary, gives a breakthrough recovery identical with that predicted by the Dykstra-Parsons method. This is just a coincidence, and calculations made with mobility ratios exceeding 10 show that the Dykstra-Parsons method will give breakthrough recoveries exceeding that of Buckley-Leverett Model 1.

As before, the performance after breakthrough as predicted by the Stiles and Dykstra-Parsons methods is conservative compared with the Buckley-Leverett solution where all properties vary and two-phase flow behind the front is considered. Hiatt[13] experienced this same effect in comparing predictions made by the former methods with actual performance of water-drive reservoirs. This difference is due not only to permeability variations and the effect of production behind the front, but by a combination of factors including the interrelation of absolute permeability, porosity, initial saturations, residual saturations and relative permeabilities.

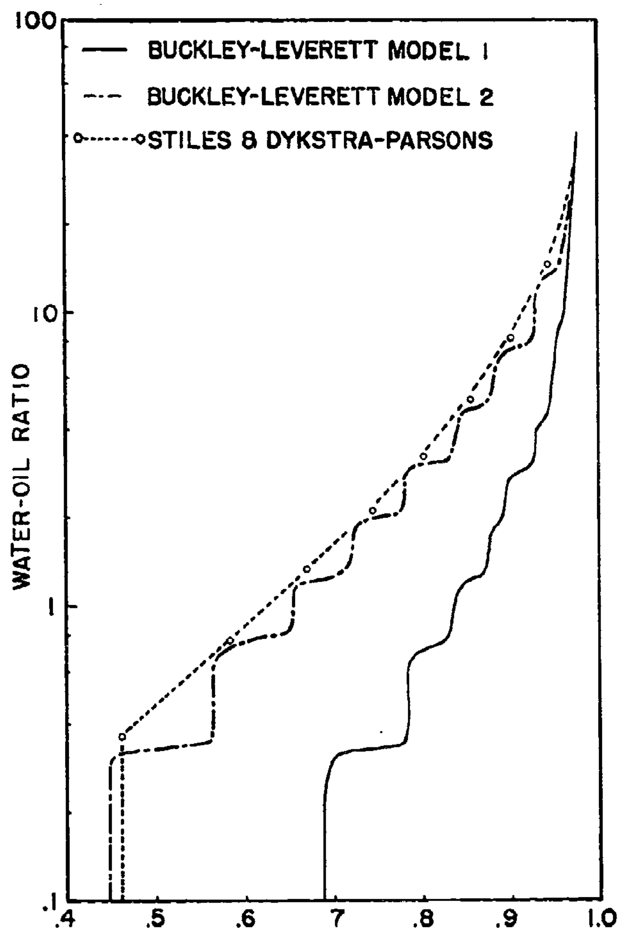

FIG. 5—WATERFLOOD CASE B ($M=1$), RECOVERY VS WOR.

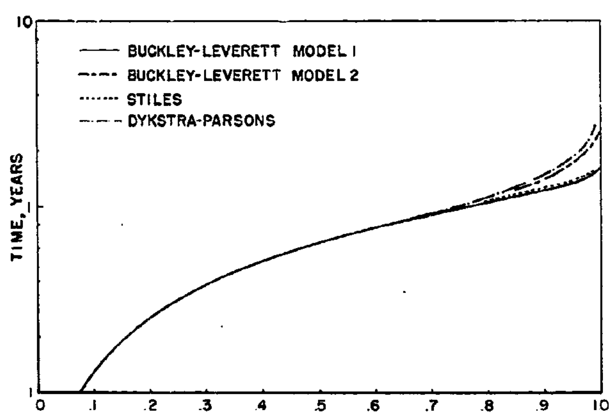

FIG. 4—WATERFLOOD CASE A ($M=0.125$), RECOVERY VS TIME.

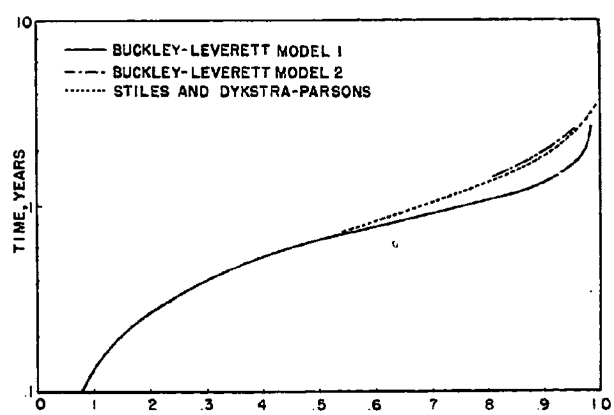

FIG. 6—WATERFLOOD CASE B ($M=1$), RECOVERY VS TIME.

GAS INJECTION

Performance of our model reservoir under gas injection was studied using the modified Higgins-Leighton method described previously. The same permeability-porosity-connate water relations were used as for the waterflood cases. One major difference was inclusion of an initial free gas saturation that varied between layers (Table 1). These saturations were determined by assuming an average free gas saturation of 3 percent in the reservoir and applying the method presented by Sheldon[10] to differentially deplete the layers until this average was reached. Reservoir and fluid properties for the gas injection case are given in Table 3.

Two models were examined: Model 1 where all properties and saturations were varied, and Model 2 where

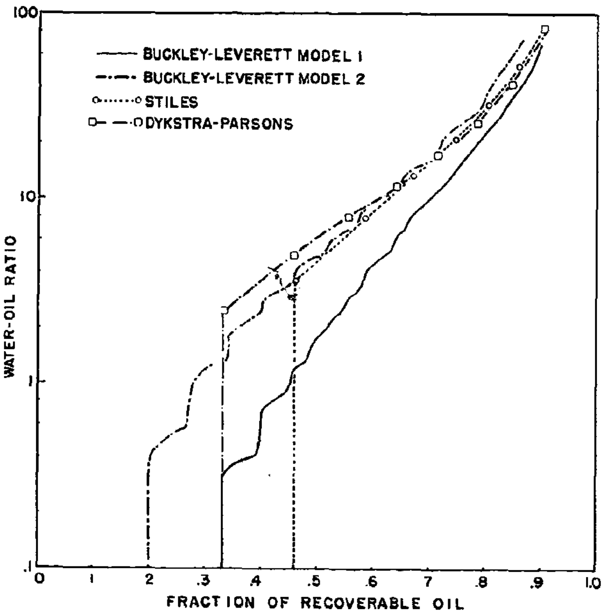

FIG. 7—WATERFLOOD CASE C ($M = 10$), RECOVERY VS WOR.

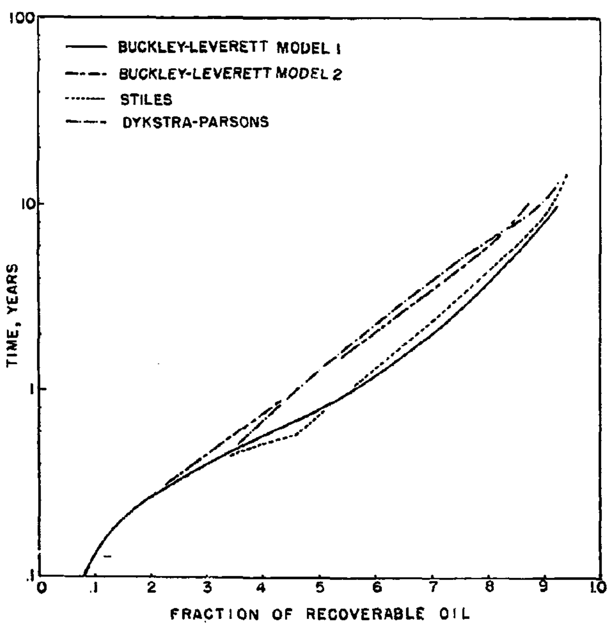

FIG. 8—WATERFLOOD CASE C ($M = 10$), RECOVERY VS TIME.

TABLE 3 — RESERVOIR AND FLUID PROPERTIES, 10-LAYER GAS INJECTION MODEL

	Case D
L, ft	600
W, ft	600
h, ft	15
$p(L)$, psia	2,000
μ_o, cp	1.0
μ_g, cp	0.02
B_o, res. bbl/STB	1.3
B_g, res. bbl/Mscf	1.0
i_g MMscf/D	1.0
k_{ro}	$(1-S)^4$
k_{rg}	$S^3(2-S)$
$\overline{S}_g$	0.03
V	0.5
$\overline{k}_H$, md	300
$\overline{\phi}$	0.15
$\overline{S}_{wi}$	0.35
σ_ϕ	0.04
$\sigma_{S_{wi}}$	0.0853

only permeability was varied and remaining properties were set at their average values. Figs. 9 and 10 present the results of studying these two cases with a limiting GOR of 50,000 scf/STB. The trends observed in the waterflood cases are also shown; i.e., a pessimistic prediction is obtained when only the variation of permeability is included in the calculations.

When this study began, it was considered that results obtained from Buckley-Leverett displacement in the improved layered model would give pessimistic results. In the case of gas injection it certainly would be expected

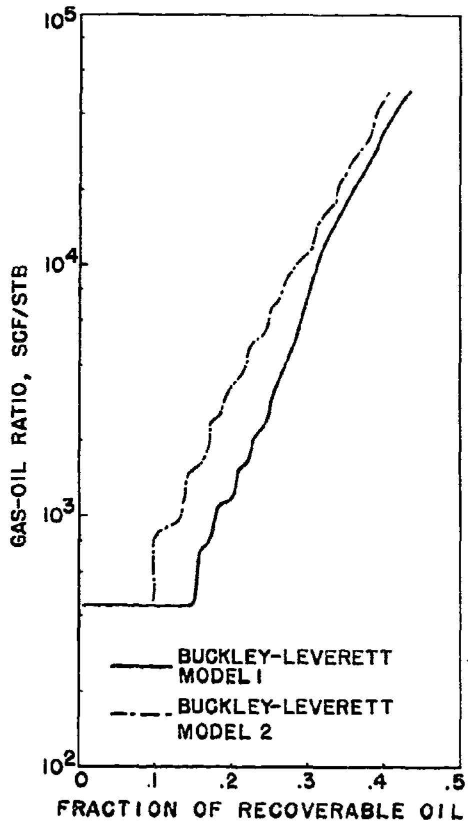

FIG. 9—GAS INJECTION CASE D, RECOVERY VS GOR.

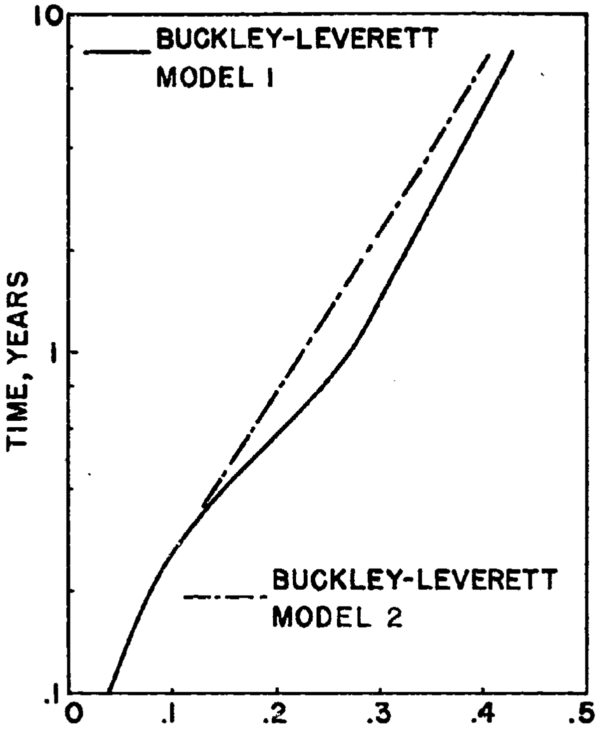

FIG. 10—GAS INJECTION CASE D. RECOVERY VS TIME.

that a high initial gas saturation in the high-permeability layer would lead to rapid breakthrough and a very rapid increase in GOR. Surprisingly, this did not happen. The low connate water saturation and higher porosity tended to slow the advance of the front and overrode the effect of initial gas saturation.

CONCLUSIONS

For both water and gas displacement calculations, the model for which all properties varied gave a more favorable performance prediction than the model for which only permeability varied from layer to layer. Insofar as the layered model (with all properties varying) used in this study matches any real reservoir, we have to conclude that previous immiscible designs using Stiles or Dykstra-Parsons predictions were probably pessimistic. Perhaps this is the most important conclusion from the study. In addition, the following conclusions appear warranted as a result of this work.

1. Using standard layered models (such as Stiles or Dykstra-Parsons) for displacement calculations may lead to erroneous breakthrough recoveries, depending on mobility ratio and rock properties. The predicted performance after breakthrough is generally quite conservative, if not overly pessimistic.

2. Using a model that accounts for changes in the many interrelated rock properties such as permeability, porosity, initial saturation, residual saturation and relative permeabilities should give a better estimate of the breakthrough recovery and the performance after breakthrough. This applies to gas injection as well as waterflooding, and assumes that the model adequately represents the reservoir.

3. The modified Higgins-Leighton method used here should apply equally well to predict the performance of, or allocate production from, several independent reservoir with a common aquifer and commingled production. Thi method should also guide engineers in allocating injecte volumes to noncommunicating reservoirs or strata serve by a common injection well(s).

4. In drawing a curve through the values of WOR v cumulative oil production from a Dykstra-Parsons cal culation, it appears better to draw a step function when th mobility ratio is less than one, and a smooth curve for mo bility ratios greater than one. This approach should giv a better prediction of recovery and WOR vs time.

NOMENCLATURE

L = reservoir length, ft
W = reservoir width, ft
h = layer thickness, ft
x = linear distance from injection face, ft
$p(L)$ = pressure at $x = L$, psia
V = permeability variation, dimensionless
k_M = median permeability, md
$\overline{\phi}$ = mean porosity, fraction
σ_k = standard deviation of permeability data,

$$\sigma_k = \ln \frac{1}{1-V}, \text{ dimensionless}$$

σ_ϕ = standard deviation of porosity data, dimensionless
$\sigma_{S_{wc}}$ = standard deviation of connate water saturatio data, dimensionless
$\overline{S}_g$ = mean gas saturation, fraction
S_D = saturation of displacing phase, fraction
$\overline{S}_{wc}$ = mean connate water saturation, fraction

$$S = \begin{cases} \dfrac{S_D - S_{wc}}{1 - S_{wc}} & \text{for imbibition} \\ \dfrac{S_D}{1 - S_{wc}} & \text{for drainage} \end{cases}$$

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APPENDIX

The modified Higgins-Leighton method used here is actually a numerical solution to the Buckley-Leverett displacement equation. Gravity and capillary effects were neglected, and end effects were considered negligible in this field-sized system. Solutions to the Buckley-Leverett equation (or a similar equation) have been presented by others, including the following: (1) Douglas *et al.*[22] used a finite difference solution which included gravity but not capillarity; (2) Sheldon *et al.*[23] used the method of characteristics and the concept of shocks while neglecting gravity and capillarity; (3) McEwen[24] presented a finite difference solution using a predictor-corrector method (He neglected gravity but included the capillary pressure term.); (4) Fayers and Sheldon[25] used finite difference approximations and included both gravity and capillarity; and (5) Hovanessian and Fayers[26] extended the work of Douglas *et al.*[22] to include gravity and pressure profiles.

The equations solved in this work are

$$\frac{\partial S_D}{\partial t} + f'(S_D)\,\frac{\partial S_D}{\partial x} = 0,\ 0 < x \leq L,$$

$$S_D(x,0) = g(x), \text{ and}$$

$$q_D(0,t) = h(t). \quad . \quad . \quad . \quad . \quad . \quad (\text{A-1})$$

The finite difference approximation used to solve Eq. A-1 is explicit in time and uses backward spatial differences. Therefore, there are restrictions to the time step size and mesh size to insure stability and accuracy of solution. A fine mesh was used in the vicinity of the front in each layer, while a coarse mesh was used in the rest of the region. The mesh size depends on the accuracy needed. Time step sizes on the order of 0.1 to 0.5 days were satisfactory for this study. It can be shown that the numerical method is stable if the ratio of time step to mesh size has the following upper bound.

$$\frac{\Delta t}{\Delta x_i} < \frac{2\phi A}{q_t\,[\max f'(S_D)]}\ ,\, 1 \leq i \leq NC \quad . \quad (\text{A-2})$$

This is not the maximum upper bound, but will ensure stability and serve as a starting point for numerical experimentation. From practical considerations, the upper bound to the time step-mesh size ratio would have to be less than that value that would allow one of the following to occur in a mesh cell: $S_w > 1 - S_{or}$ or $S_o < S_{or}$.

Eq. A-2 applied to waterflood Case A ($M = 0.125$) gives a ratio of 0.16 days/ft. The same equation, applied to a case where the mobility ratio was 20, gave a ratio of 0.14 days/ft. In the latter case, the more shallow slope of the fractional flow curve was offset by the increasing water injection rate in the high-permeability layer as the front advanced in that layer, changing the ratio only slightly.

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