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Prediction of Waterflood Behavior in a Stratified System

J. E. WARREN
J. J. COSGROVE
MEMBERS AIME

KUWAIT OIL CO. LTD.
AHMADI, KUWAIT
GULF RESEARCH & DEVELOPMENT CO.
PITTSBURGH, PA.

ABSTRACT

A general model which approximates the effect of cross-flow has been developed to give a practical method for predicting the waterflood behavior of a stratified reservoir. The model is based on a modification of Dietz's theory and allows for variations in both the permeability and hydrocarbon pore volume.

In the particular cases considered, it is assumed that the permeability can be characterized by a log-normal distribution and the hydrocarbon pore volume by a normal distribution. A simple graphical method which enables the practicing engineer to predict the behavior of a stratified system is presented. The results obtained by the proposed method are compared with those obtained by the Dykstra-Parsons method.

As a result of this study the following conclusions have been drawn: (1) The effect of cross-flow in a stratified system can be appreciable, particularly at very favorable or very unfavorable mobility ratios; (2) Under normal conditions, the effect of variations in the hydrocarbon pore volume can be neglected; and (3) The failure to use all of the available permeability data can lead to large errors in the prediction of the behavior of a stratified reservoir.

INTRODUCTION

Various methods have been proposed to characterize and to predict the waterflood behavior of a stratified reservoir. Most of the methods assume that the reservoir is composed of discrete homogeneous continuous layers. With this model, the degree of stratification can be measured by several parameters based on core-analysis data. Among these are the Lorenz coefficient[1] and the variation of a log-normal permeability distribution.[2]

The behavior of stratified systems is usually predicted by the Stiles,[3] Dykstra-Parsons[4] method or some modification of these. In both of these methods the reservoir is divided into discrete homogeneous layers with no cross-flow between layers. In the Stiles method the mobility ratio is assumed to be equal to unity while in the Dykstra-Parsons method it is allowed to take on any value.

Other predictive methods have been proposed; Hiatt's[5] method allows for cross-flow between the beds and a method by Schmalz and Rahme[6] correlates the recovery directly with the Lorenz coefficient.

In this new approach, which is essentially a continuous analog of Hiatt's method, the effects of both mobility ratio and cross-flow between the beds have been included. It is assumed that the permeability can be represented by a log-normal distribution and the hydrocarbon pore volume or porosity by a normal distribution. These types of distributions have been observed by several authors,[1,2,4] and most field data seem to confirm their observations; e.g., if these distributions are assumed and there is a 1:1 correspondence between porosity and permeability samples, the commonly encountered linear relationship between porosity and the log of permeability is obtained.

$$\phi = \phi_A + \frac{\sigma_\phi}{\sigma_K} \ln (K/K_M) \quad . \; . \; . \; . \; . \; . \; (1)$$

Since, in many field cases the permeability and porosity data are truncated or cut-off at predetermined upper and/or lower values, the effect of discarding part of the data was investigated with the proposed method. A better technique for truncating the data is suggested.

THEORY

In the derivation of the method to be described in this paper the following assumptions are made:

1. Capillary forces are negligible per se; their effects are only manifest in the relative permeability curves.
2. The fluids are immiscible, incompressible and homogeneous.
3. The reservoir is horizontal and uniformly thick; it is initially liquid saturated i.e., connate water and oil.
4. The reservoir is composed of discrete layers, each having its own permeability, porosity and connate water saturation; each layer is homogeneous,

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[1]References given at end of paper.

isotropic and uniform in thickness. The permeability can be represented by a log-normal distribution and the hydrocarbon pore volume by a normal distribution.

5. Despite differences in absolute permeability, the same relative permeability curves, based on the hydrocarbon pore volume, apply to all layers.

6. The displacement process can be represented by the movement of a sharp pseudo-interface; only oil flows on one side of the interface and only water flows on the other.

7. The porous medium is quasi-linear, i.e., the cross-sectional area normal to the flow can be represented as a function of the distance travelled by the front.

8. The gravitational effects are negligible relative to the viscous effects.

9. The displacement efficiency and mobility ratio remain constant throughout the life of the project.

With these assumptions,* it is shown in the Appendix that the fractional flow of displacing phase f_D is given by

$$f_D = \frac{1}{1+\frac{1}{M}\left[\frac{F(K)}{1-F(K)}\right]} \quad \ldots\ldots (2)$$

and the average reduced saturation (vertical sweep) by

$$\bar{S}_D = S_D + (1-f_D)/(df_D/dS_D) \quad \ldots (3)$$

where

$$S_D = 1 - P(K) + \frac{\sigma_\phi \exp(-\ln(K/K_M)^2/(2\sigma_K^2))}{\sqrt{2\pi}\ \phi_A} \quad \ldots\ldots (4)$$

and

$$\frac{df_D}{dS_D} = \frac{MK\exp(-.5\sigma_K^2)}{\left[M-(M-1)F(K)\right]^2 K_M}\left[\frac{1}{1+(\sigma_\phi/(\sigma_K\phi_A))\ln(K/K_M)}\right] \quad \ldots\ldots (5)$$

$$\text{Recovery}\,(R) = \bar{S}_D E_D \quad \ldots\ldots (6)$$

where

E_D = displacement efficiency

The displacement efficiency E_D is the fraction of the total volume of oil which is displaced from the volume of oil which is affected by the injected fluid. In this application of the modified Dietz method, f_D, the fractional flow of the displacing phase, represents the ratio of the displacing flow to the total flow across any vertical plane perpendicular to the direction of flow. The reduced displacing phase saturation S_D is the integrated reduced saturation in any vertical plane; and, the average reduced displacing phase saturation $\bar{S}_D$ is the integrated value of S_D behind the front — the vertical sweep. If the hydrocarbon pore volume, ϕ or $\phi'(1 - S_{wc})$, is assumed to be constant, then

$$f_D = \frac{1}{1+\frac{1}{M}\left[\frac{F(K)}{1-F(K)}\right]} \quad \ldots (7)$$

$$\bar{S}_D = S_D + (1-f_D)/(df_D/dS_D) \quad \ldots (8)$$

where

$$S_D = 1 - P(K) \quad \ldots\ldots (9)$$

and

$$\frac{df_D}{dS_D} = \frac{MK\exp(-.5\sigma_K^2)}{\left[M-(M-1)F(K)\right]^2 K_M} \quad \ldots (10)$$

Furthermore, if the hydrocarbon pore volume is assumed constant and the permeability data are truncated at a lower value K_1, and an upper value K_2,

$$f_D = \frac{1}{1+\frac{1}{M}\left[\frac{F(K)-F(K_1)}{F(K_2)-F(K)}\right]} \quad \ldots (11)$$

$$\bar{S}_D = S_D + (1-f_D)/(df_D/dS_D) \quad . \ (12)$$

where

$$S_D = 1 - \left[\frac{P(K)-P(K_1)}{P(K_2)-P(K_1)}\right] \quad \ldots (13)$$

*It is apparent that any displacement process which satisfies these assumptions can be described by the derived equations; e.g., cycling performance.

and

$$\frac{df_D}{dS_D} = \frac{M\left[F(K_2) - F(K_1)\right] K \exp(-.5\sigma_K^2)(P(K_2) - P(K_1))}{\left[MF(K_2) + F(K)(1-M) - F(K_1)\right]^2 K_M}$$

It is now possible, by using the above equations, to predict the behavior of a stratified system. However, before the calculations can be performed, values of $P(K)$, $F(K)$ and M have to be determined.

The distribution function $P(K)$ can be characterized by its variation V. The variation can either be determined graphically from the permeability data or from its Lorenz coefficient L. The Lorenz coefficient is related to the variation V by the following expression:

$$L = \text{erf}\ (1/(1-V)). \quad \dots\dots (14)$$

A graph of this function is given on Fig. 1. If the harmonic average and the arithmetic average of the permeabilities are known, the variation V can be approximated by

$$V \cong 1 - \exp\left(-\sqrt{\ln (K_A/K_H)}\right) \quad \dots\dots (15)$$

and the mean permeability by

$$K_M \cong \sqrt{K_H K_A}$$

The first moment $F(K)$ of the permeability data can be obtained from $P(K)$ by a simple graphical technique which is demonstrated in the section entitled "Example Problem".

The mobility ratio M which is assumed to remain constant throughout the life of the project is given by

$$M = \frac{\mu_o \bar{k}_{rD}}{\mu_D k_{ro}} \quad \dots\dots (16)$$

where $\bar{k}_{rD}$ is the average relative permeability to the displacing phase behind the front. It can be approximated by the value of k_{rd} at the average reduced saturation determined from a Buckley-Leverett calculation. If $\mu_o/\mu_D < 20$ and the Naar-Henderson[7] approximations (imbibition) are used for relative permeability then

$$\frac{\bar{k}_{rD}}{k_{ro}} = .16$$

and the mobility ratio can be approximated by

$$M = \frac{1}{16} \frac{\mu_o}{\mu_D} \quad \dots\dots (17)$$

The displacement efficiency E_D also has to be determined. Stahl[9] suggests that E_D be based on the difference between the initial and residual saturations; however, this gives an upper limit for the recovery. A lower limit on E_D can be obtained by using the average reduced saturation at breakthrough from a Buckley-Leverett calculation. If $\mu_o/\mu_D < 20$ and the Naar-Henderson approximations for relative permeability are used, then the displacement efficiency E_D is equal to 0.5.

DISCUSSION OF RESULTS

The new method of predicting the behavior of a stratified system was compared with the Dykstra-Parsons (DP) method. The basis of comparison was that used by Stahl.[9] In this comparison the coverage C from DP method was compared with $\bar{S}_D$ since both quantities indicate the vertical sweep efficiency. The results are shown in Fig. 2 for a permeability variation of 0.8. For unfavor-

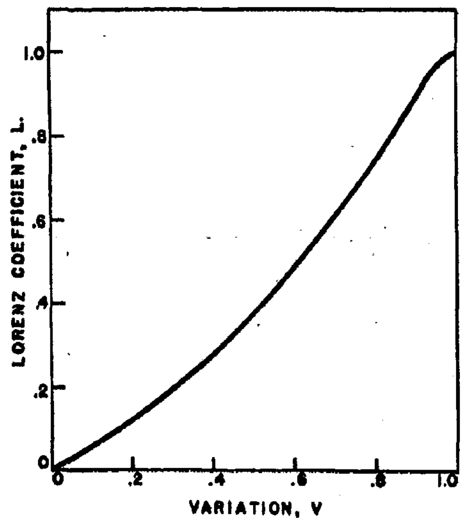

FIG. 1 — CORRELATION OF LORENZ COEFFICIENT AND VARIATION.

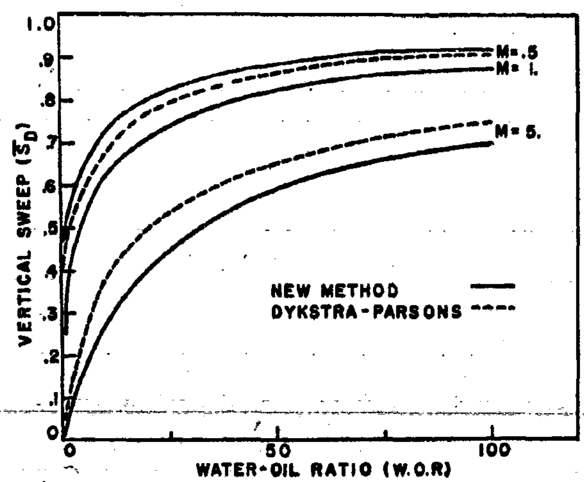

FIG. 2 — COMPARISON OF PREDICTIVE METHODS (VARIATION $V = 0.8$).

able mobility ratios, the new method gives a lower $\bar{S}_D$ at a given water-oil ratio than the DP method. For a mobility ratio of unity, the two methods coincide, but for favorable ratios, the new method gives a higher $\bar{S}_D$ at a given water-oil ratio.

The apparent anomaly is caused by cross-flow. At unfavorable mobility ratios, the pressure distribution is such that cross-flow occurs in the direction which tends to cause the flood front to become unstable; hence, the recovery is lower. At favorable mobility ratios, the reverse is true; cross-flow tends to stabilize the front and improves the recovery. Fig. 3 shows the results for a variation of 0.5. The trend of the results is the same as

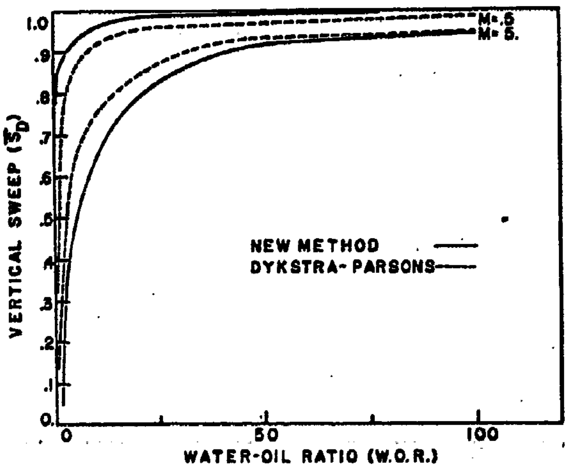

FIG. 3 — COMPARISON OF PREDICTIVE METHODS (V = 0.5).

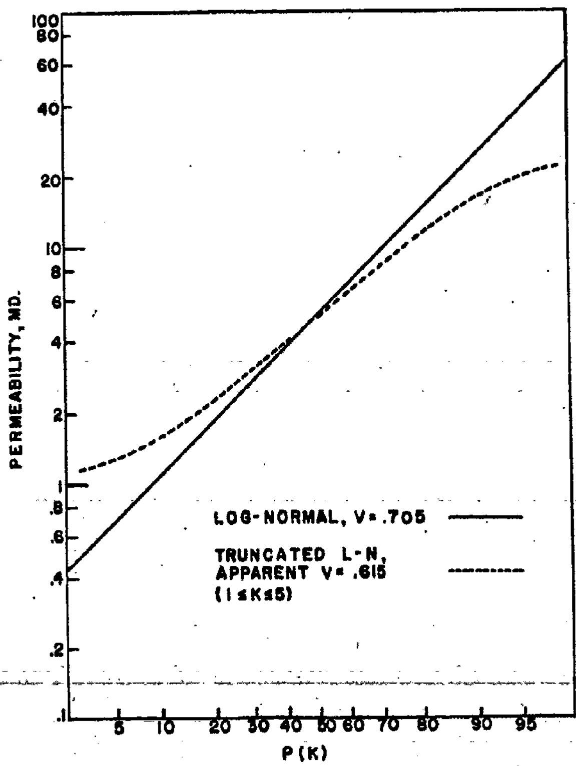

FIG. 4 — EFFECT OF CUT-OFFS.

those at higher variation, but the differences between the methods are reduced.

Figs. 4 through 6 show the effect of truncating the permeability data. The solid curve on Fig. 4 shows a log-normal permeability distribution with a median value of 5 md and a variation of 0.705. The usual method of truncating the data is first to select the cut-off values, and then determine the distribution.

For example, in the above case, if the data are truncated at K = 1 and K = 25 the dashed curve is obtained. Now, if the best straight line is drawn through these points, a variation of 0.615 is obtained. Thus, if the data are truncated in this manner, errors will be introduced. Actually, exponential and linear distributions can be represented as degenerate forms of log-normal distributions which result from truncation.

The correct method of truncating data is to use all the available data to construct the permeability

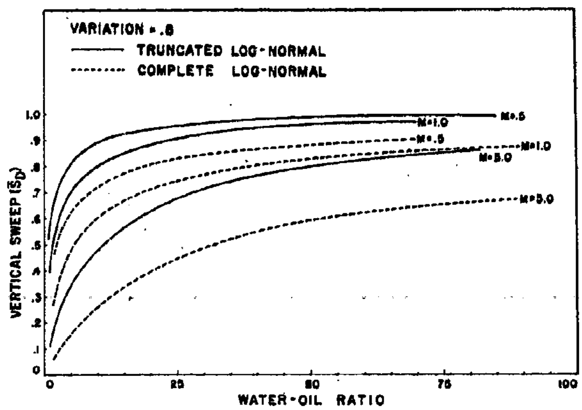

FIG. 5 — EFFECT OF CUT-OFF (V = 0.8).

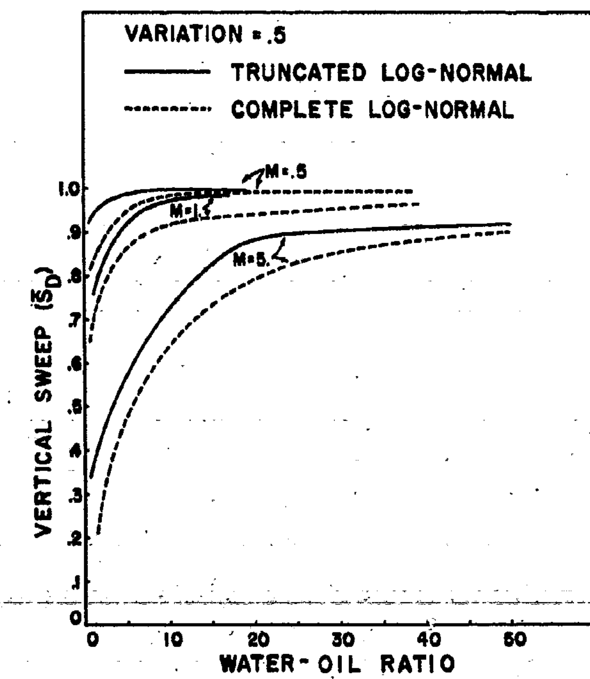

FIG. 6 — EFFECT OF CUT-OFF (V = 0.5).

distribution curve, to select the points of truncation, and to use Eqs. 11, 12 and 13 to predict the behavior.

The results obtained using proper truncation are shown in Figs. 5 and 6; in both cases, the points of truncation were chosen at $P(K) = 0.1$ for the lower value and $P(K) = 0.9$ for the upper value. Fig. 4 indicates results for a variation of 0.8 and various values of mobility ratio M. In all cases the truncated data indicate a higher recovery. Truncating the data increases the calculated recovery; this is particularly true for high values of the variation because in these cases the high permeability values are eliminated, thus retarding the advance of the flood front. This effect is greatly reduced as the variation decreases; which is demonstrated in Fig. 6 for a variation of 0.5.

The effects of a variable hydrocarbon pore volume are shown in Figs. 7 through 9. These figures show the results for various combinations of hydrocarbon pore volume, variation of the permeability data and mobility ratios. For all the cases considered, the effect of changes in the hydrocarbon pore volume could be neglected.[11] If the porosity and connate water saturation can be expressed as functions of permeability and if constant values of porosity and connate water are to be used, the appropriate values are those which occur at the median or the geometric mean of the permeability data.

Finally, it should be pointed out that if the system is assumed to have a constant permeability (this value of permeability should be the median or geometric mean value of the core samples) and gravity effects are included, behavior similar to that of a stratified system can be obtained. Thus, field behavior resembling that described in this paper does not necessarily indicate that the reservoir is stratified.

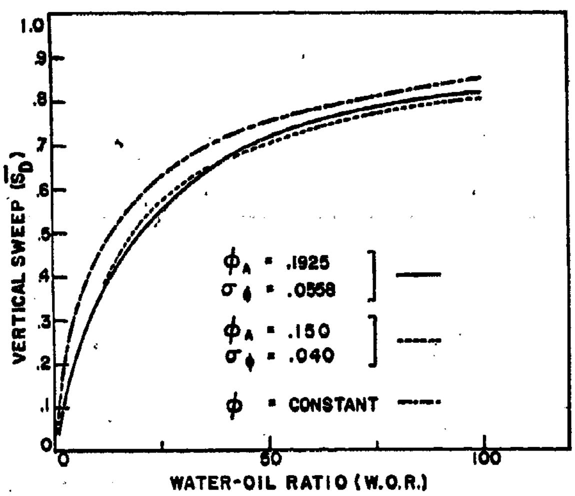

FIG. 7 — EFFECT OF VARIABLE HYDROCARBON PORE VOLUME (V = 0.8, MOBILITY RATIO = 5).

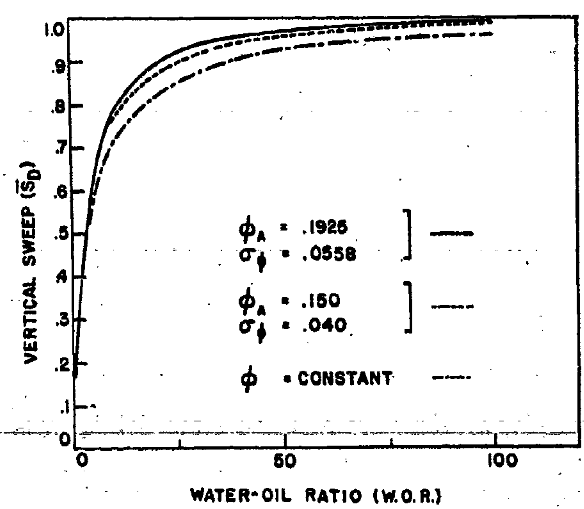

FIG. 8 — EFFECT OF VARIABLE HYDROCARBON PORE VOLUME (V = 0.5, MOBILITY RATIO = 5).

EXAMPLE PROBLEM

In the example problem it will be assumed that the porosity is constant, that the permeability data which are tabulated in increasing order in Table 1 and a mobility ratio of 2.04 are given. The procedure to be employed is the following:

CHARACTERIZE PERMEABILITY DISTRIBUTION

1. Plot permeability data on log-probability paper (Fig. 10); this is the $P(K)$ curve.
2. Calculate the variation V which is given by

$$V = (K_{84.1\%} - K_{50\%}) / K_{84.1\%}$$

$$= \frac{550 - 252}{550}$$

$$= .542$$

3. Calculate the standard deviation σ_K.

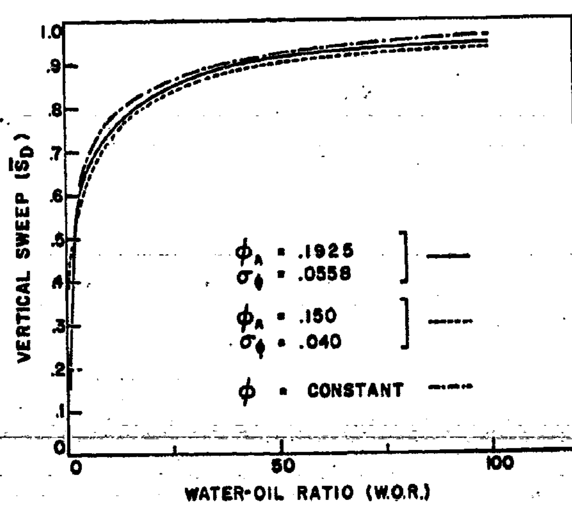

FIG. 9 — EFFECT OF VARIABLE HYDROCARBON PORE VOLUME (V = 0.8, MOBILITY RATIO = 1).

TABLE 1 — PERMEABILITY DISTRIBUTION

h ft	Σh	K_{MD}	$P(K)$ (% less than)
1	1	35	0
1	2	47	2
1	3	58	4
1	4	77	6
1	5	83	8
1	6	100	10
1	7	107	12
1	8	110	14
2	10	119	16
3	13	140	20
3	16	160	26
2	18	170	32
3	21	200	36
3	24	210	42
1	25	250	48
1	26	260	50
3	29	270	52
2	31	310	58
5	36	330	62
1	37	400	72
3	40	410	74
1	41	500	80
2	43	520	82
1	44	600	86
2	46	640	88
4	50	730	92

Note: K values are not given at equal h increments. Therefore, when computing per cent less than, the data must be weighted accordingly; i.e., for K = 170, per cent less than $= \frac{16}{50} \times 100 = 32$ per cent.

$$\sigma_K = \ln \frac{1}{1-V} = .784$$

CONSTRUCT THE FIRST MOMENT CURVE, $F(K)$

1. Read the permeability value K at $P(K)$ = 50 per cent from Fig. 10.
2. Calculate $K_{50\%} \exp(\sigma^2_K)$
3. Plot $K_{50\%} \exp(\sigma^2_K)$ at $P(K)$ = 50 per cent.
4. Draw line parallel to $P(K)$ curve through this point. This is the $F(K)$ curve. Now, with the $P(K)$ and $F(K)$ curves, it is possible to predict the behavior of the system using Eqs. 7, 8 and 9.

The calculated results are listed in Table 2 and Fig. 11.

CONCLUSIONS

As a result of this study, the following conclusions can be drawn.

1. The effect of cross-flow in a stratified system can be appreciable particularly at very favorable or very unfavorable mobility ratios.
2. Under normal conditions, the effect of variations in the hydrocarbon pore volume can be neglected.
3. The failure to use all available permeability data can lead to large errors in the prediction of the behavior of a stratified reservoir.

NOMENCLATURE

ϕ = hydrocarbon pore volume = $\phi'(1 - S_{wc})$,

TABLE 2 — CALCULATED VALUES FOR FIRST MOMENT CURVE, $F(K)$

S_D	$P(K)$	K_{MD}	$F(K)$	f_D	$\frac{df_D}{ds_D}$	$\bar{S}_D$	WOR
.95	.05	70	.008	.996	.048	—	249
.9	.1	92	.02	.990	.135	.974	99
.8	.2	130	.05	.975	.197	.927	39
.7	.3	168	.09	.954	.266	.873	21
.6	.4	208	.15	.921	.351	.825	11.6
.5	.5	252	.22	.879	.458	.764	7.26
.4	.6	309	.30	.826	.616	.682	4.75
.3	.7	379	.40	.754	.855	.588	3.1
.2	.8	485	.52	.653	1.277	.472	1.88
.1	.9	690	.69	.478	2.707	.293	.92
.05	.95	900	.80	.338	3.602	.234	.51

ϕ' = porosity,

$P(\phi)$ = distribution function for hydrocarbon pore volume data, dimensionless,

ϕ_A = arithmetic mean hydrocarbon pore volume, dimensionless,

σ_ϕ = standard deviation of hydrocarbon pore volume data, dimensionless,

K = permeability, L^2,

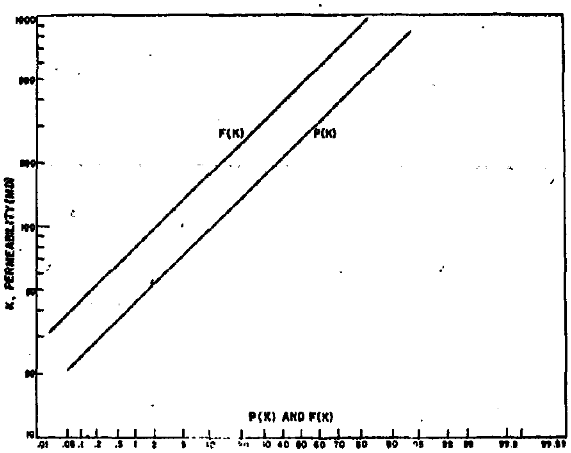

FIG. 10 — PERMEABILITY DISTRIBUTION FOR EXAMPLE PROBLEM.

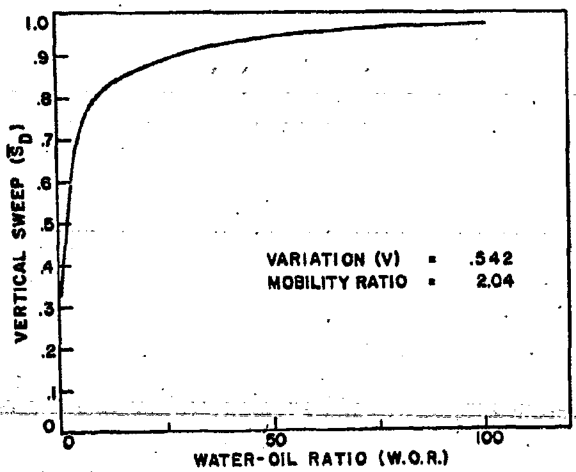

FIG. 11 — PREDICTION FOR EXAMPLE PROBLEM [(V) = 0.542, MOBILITY RATIO = 2.04].

$P(K)$ = distribution function of permeability data, dimensionless,

$F(K)$ = first moment of $P(K)$, dimensionless,

K_M = geometric mean of permeability data, L^2,

K_H = harmonic mean of the permeability data, L^2,

σ_K = standard deviation of permeability data, dimensionless,

f_D = fractional flow of displacing phase, dimensionless,

μ_o = viscosity of oil, M/LT,

μ_D = viscosity of displacing phase, M/LT,

k_{ro} = relative permeability to oil, dimensionless,

k_{rD} = relative permeability to displacing phase, dimensionless,

$\bar{k}_{rD}$ = average relative permeability to displacing phase, dimensionless,

$M = \dfrac{\bar{k}_{rD}\,\mu_o}{k_{ro}\,\mu_D}$ mobility ratio, dimensionless,

S_D = integrated reduced saturation in any plane perpendicular to direction of flow, dimensionless,

$\bar{S}_D$ = average reduced saturation behind the displacing front (vertical sweep), dimensionless,

E_D = displacement efficiency for hydrocarbon pore volume, dimensionless,

R = recovery, dimensionless,

S_{wc} = connate water saturation, dimensionless,

L = Lorenz coefficient, dimensionless,

V = variation, dimensionless,

U_o = oil rate, L^3/T,

U_D = displacing phase rate, L^3/T,

U = total rate, L^3/T,

P_D = pressure in displacing phase, M/LT^2,

P_o = pressure in oil phase, M/LT^2,

h = reservoir thickness, L,

X = coordinate parallel to flow, L,

y = coordinate perpendicular to flow, L,

Y = fraction of reservoir thickness invaded by displacing phase in a plane perpendicular to flow, dimensionless.

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APPENDIX

The model proposed in this paper is shown schematically in Fig. 12. The actual displacement front may be irregular (fingering) but when it is averaged over the cross-section a smooth front is obtained. The distribution function[10] for the permeability $P(K)$ is given by

$$P(K) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\ln(K/K_M)/\sigma_K\sqrt{2}\right)\right] \qquad (1a)$$

and that for the porosity or hydrocarbon pore volume by

$$P(\phi) = \frac{1}{2}\left[1 + \operatorname{erf}\left((\phi - \phi_A)/\sigma_\phi\sqrt{2}\right)\right]^* \qquad (2a)$$

Now, since there is a 1:1 correspondence between the porosity and permeability samples

$$P(K) = P(\phi) \,.\,.\,.\,.\,.\,.\,.\,.\,.\,. \qquad (3a)$$

and

$$\frac{\ln(K/K_M)}{\sigma_K} = \frac{(\phi - \phi_A)}{\sigma_\phi}$$

or

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*It is obvious that this relationship is only an approximation since the range of ϕ is $0 \leq \phi \leq 1$ rather than $-\infty < \phi < \infty$ (the condition required for normality); but if $\sigma_\phi < 0.316\ \phi_A$ we have the following inquality:

$$0.999 < \left[P(1) - P(0)\right] < 1.000$$

Thus, the approximate distribution function is acceptable if $\sigma_\phi < 0.316\ \phi_A$.

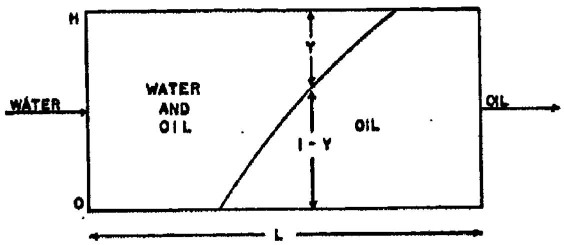

FIG. 12 — PROPOSED MODEL.

$$\phi = \phi_A + (\sigma_\phi / \sigma_K) \ln (K / K_M) . \quad \text{(4a)}$$

Let us consider the flow across any plane through the interface. Owing to the assumptions concerning the dominance of viscous forces, the displacing fluid will enter the most permeable layers first; and, the reduced displacing phase saturation can be obtained by integration; e.g.,

$$S_D = \frac{1}{\phi_A} \int_Y^1 \phi \, d(y/h) . \quad \text{(5a)}$$

But,

$$Y = P(\phi) = P(K)$$

Therefore, the following expression obtains:

$$S_D = \frac{1}{\phi_A} \int_\phi^\infty \phi \frac{dP(\phi)}{d\phi} d\phi$$

Or, alternatively,

$$S_D = 1 - \frac{1}{\phi_A} \int_{-\infty}^\phi \phi \frac{dP(\phi)}{d\phi} d\phi . \quad \text{(6a)}$$

Using the moment generating function,

$$\int_{-\infty}^\phi \phi \frac{dP(\phi)}{d\phi} d\phi = \phi_A P(\phi) - \sigma_\phi^2 \frac{dP(\phi)}{d\phi}$$

Consequently,

$$S_D = 1 - \frac{1}{\phi_A} \left[\phi_A P(\phi) - \sigma_\phi^2 \frac{dP(\phi)}{d\phi} \right] . \quad \text{(7a)}$$

From 3a and 4a, S_D can be defined as follows:

$$S_D = 1 - P(K) + \frac{\sigma_\phi}{\sqrt{2\pi}\, \phi_A} \exp \left(- \frac{\ln (K/K_M)^2}{2\sigma_K^2} \right) \quad \text{(8a)}$$

The flow rates can now be obtained in a similar manner by integrating over the cross-section; i.e.,

$$U_D = - \left[\frac{k_{rD}}{\mu_D} \int_Y^1 K d(y/h) \right] \frac{\partial P_D}{\partial X}$$

An equivalent form is that which follows:

$$U_D = - \left[\frac{k_{rD}}{\mu_D} \int_K^\infty K \frac{dP(K)}{dK} dK \right] \frac{\partial P_D}{\partial X} . \quad \text{(9a)}$$

Also,

$$U_o = - \left[\frac{k_{ro}}{\mu_o} \int_o^K K \frac{dP(K)}{dK} dK \right] \frac{\partial P_o}{\partial X} . \quad \text{(10a)}$$

From continuity of volume, we have

$$\frac{\partial U_D}{\partial X} = - \frac{\partial}{\partial t} (S_D \phi_A) \quad \text{(11a)}$$

and,

$$\frac{\partial U_o}{\partial X} = - \frac{\partial}{\partial t} ((1 - S_D) \phi_A) \quad \text{(12a)}$$

Since the fluids are incompressible and capillary effects have been neglected

$$U = U_o + U_D$$

and

$$\frac{\partial P_D}{\partial X} = \frac{\partial P_o}{\partial X}$$

Therefore,

$$f_D = \frac{U_D}{U} = \frac{1}{1 + \frac{k_{ro} \mu_D}{k_{rD} \mu_o} \left[Z \right]} \quad \text{(13a)}$$

where

$$Z = \frac{\int_o^K K \frac{dP(K)}{dK} dK}{\int_K^\infty K \frac{dP(K)}{dK} dK}$$

Let the first moment of a log-normal distribution $F(K)$ be defined as follows:

$$F(K) = \frac{\int_o^K K \frac{dP(K)}{dK} dK}{\int_o^\infty K \frac{dP(K)}{dK} dK}$$

Therefore,

$$Z = \frac{F(K)}{1 - F(K)}$$

And, the fractional flow of the displacing phase is given by

$$f_D = \frac{1}{1 + \frac{1}{M} \left[\frac{F(K)}{1 - F(K)} \right]} \quad \text{(14a)}$$

By analogy with the Buckley-Leverett theory, the average reduced displacing phase saturation or vertical sweep can be obtained; i.e.,

$$\bar{S}_D = S_D + (1-f_D)/(df_D/dS_D) \quad \text{. (15a)}$$

where

$$\frac{df_D}{dS_D} = \frac{df_D}{dF(K)}\,\frac{dF(K)}{dK}\,\frac{dK}{dP(K)}\,\frac{dP(K)}{dK}\,\frac{dK}{dS_D}$$

The derivatives can be evaluated to give the following:

$$\frac{df_D}{dS_D} = \frac{MK\exp(-.5\sigma^2_K)}{\left[M-(M-1)F(K)\right]^2 K_M}\left[\frac{1}{1+(\sigma_\phi/(\sigma_K\phi_A))\ln(K/K_M)}\right] \quad \text{. . . . . (16a)}$$

From the definition of f_D,

$$W.O.R. = f_D/(1-f_D)$$

For those cases in which hydrocarbon pore volume has a constant value,

$$\phi = \phi_o$$

Then

$$f_D = \frac{1}{1+\frac{1}{M}\left[\frac{F(K)}{1-F(K)}\right]} \quad \text{. . . . . . . (17a)}$$

$$S_D = 1 - P(K). \quad \text{. . . . . (18a)}$$

$$\frac{df_D}{dS_D} = \frac{MK\exp(-.5\sigma^2_K)}{\left[M-(M-1)F(K)\right]^2 K_M} \quad \text{. . . . (19a)}$$

$$\bar{S}_D = S_D + (1-f_D)/(df_D/dS_D) \quad \text{. . . (20a)}$$

For those cases in which the permeability data are truncated at a lower value K_1 and an upper value K_2, let us define a new permeability distribution function.

$$P'(K) = \frac{P(K)-P(K_1)}{P(K_2)-P(K_1)} \quad \text{. . . . . (21a)}$$

Then

$$f_D = \frac{1}{1+\frac{1}{M}\left[\frac{F(K)-F(K_1)}{F(K_2)-F(K)}\right]} \quad \text{. . . . . (22a)}$$

$$S_D = 1 - \left[\frac{P(K)-P(K_1)}{P(K_2)-P(K_1)}\right] \quad \text{. . . . . (23a)}$$

$$\frac{df_D}{dS_D} = \frac{M(F(K_2)-F(K_1))K\exp(-.5\sigma^2_K)(P(K_2)-P(K_1))}{\left[MF(K_2)+(1-M)F(K)-F(K_1)\right]^2 K_M} \quad \text{. . . . . (24a)}$$

and

$$\bar{S}_D = S_D + (1-f_D)/(df_D/dS_D) \quad \text{. . . (25a)}$$

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