The reflection and transmission responses of a periodic layered medium

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Introduction

Marion and Coudin (1992) presented an interesting experiment on acoustic wave propagation in a periodic layered medium with two materials. They observed that when the ratio of wavelength to layer period is large, the resulting velocity is that of an effective medium. And when the ratio of wavelength to layer period is small, the velocity of the medium is close to the time-average velocity. They found that the transition occurs at a critical value of wavelength to layer period equal to 10. Carcione, Kosloff and Behle (1991) found the critical value to be about 8 for epoxy and glass and to be 6 to 7 for sandstone and limestone. Helbig (1984) found a critical value equal to 3. Hovem (1995) used an eigenvalue analysis of the propagator matrix to show that the critical value depends on the contrast in acoustic impedance between the two media. Stovas and Arntsen (2003) showed that there is a transition zone from effective medium to time-average medium which depends on the strength of the reflection coefficient in a finely layered medium.

Here we consider a periodic layered medium with two materials. We use the eigenvalue decomposition of Hovem (1995) to obtain theoretical values for the wavelength to layer thickness values for effective medium, time-average medium and the transition band in between. These values depend strongly on the reflection coefficient between the two media.

Multilayer reflection and transmission responses

We consider one cycle of a binary medium with velocities v_1 and v_2, densities ρ_1 and ρ_2 and the thicknesses d_1 and d_2 as shown in Figure 1. For given a frequency f the phase factors are: $\theta_k = 2\pi f d_k / v_k = 2\pi f \Delta t_k$, where Δt_k is the traveltime in medium k for one cycle. The normal incident reflection coefficient at the interface between the layers is given by

$$r = \frac{\rho_2 v_2 - \rho_1 v_1}{\rho_2 v_2 + \rho_1 v_1} \qquad (1)$$

The propagator matrix for one cycle is computed for an input at the bottom of the layers (Hovem, 1995)

$$\mathbf{S} = \frac{1}{1-r^2}\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1}\end{pmatrix}\begin{pmatrix} 1 & r \\ r & 1\end{pmatrix}\begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{-i\theta_2}\end{pmatrix}\begin{pmatrix} 1 & -r \\ -r & 1\end{pmatrix} = \begin{pmatrix} a & b \\ b^* & a^*\end{pmatrix} \qquad (2)$$

where * denotes complex conjugate, and

$$a = \frac{e^{i(\theta_1+\theta_2)}\left(1-r^2 e^{-2i\theta_2}\right)}{1-r^2}$$
$$b = -\frac{re^{i(\theta_1+\theta_2)}\left(1-e^{-2i\theta_2}\right)}{1-r^2} = \frac{2ir\sin\theta_2}{1-r^2}e^{i\theta_1} \qquad (3)$$

We note, that $\det\mathbf{S} = |a|^2 - |b|^2 = 1$, but that the propagator matrix is not unitary. The propagator matrix can be represented by the eigenvalue decomposition (Hovem, 1995)

$$\mathbf{S} = \mathbf{U\Lambda U}^{-1}, \qquad (4)$$

where $\mathbf{\Lambda} = diag\left[\lambda_1, \lambda_2\right]$ with

$$\lambda_{1,2} = \begin{cases} \mathrm{Re}\, a \pm i\sqrt{1-(\mathrm{Re}\, a)^2} & for\ |\mathrm{Re}\, a| < 1 \\ \mathrm{Re}\, a \mp \sqrt{(\mathrm{Re}\, a)^2 - 1} & for\ |\mathrm{Re}\, a| \ge 1 \end{cases} \qquad (5)$$

and the matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 1 \\ (\lambda_1 - a)/b & (\lambda_2 - a)/b \end{pmatrix} \qquad (6)$$

A stack of M cycles of total thickness $D = Md = M(d_1 + d_2)$ has the propagator matrix

$$\mathbf{Q} = \mathbf{S}^M = \mathbf{U\Lambda}^M\mathbf{U}^{-1} = \frac{1}{u_{22}-u_{21}}\begin{pmatrix} \lambda_1^M u_{22} - \lambda_2^M u_{21} & \lambda_2^M - \lambda_1^M \\ -u_{21}u_{22}\left(\lambda_2^M - \lambda_1^M\right) & \lambda_2^M u_{22} - \lambda_1^M u_{21}\end{pmatrix} \qquad (7)$$

with $u_{21} = (\lambda_1 - a)/b$ and $u_{22} = (\lambda_2 - a)/b$.

The transmission and reflection responses for a down-going wave at the top of the layers are (Ursin, 1983)

$$t_D = q_{22}^{-1} = \frac{\lambda_2 - \lambda_1}{(\lambda_2 - a)\lambda_2^M - (\lambda_1 - a)\lambda_1^M}$$
$$r_D = q_{12}q_{22}^{-1} = \frac{\left(\lambda_2^M - \lambda_1^M\right)b}{(\lambda_2 - a)\lambda_2^M - (\lambda_1 - a)\lambda_1^M} \qquad (8)$$

which can be written as

$$t_D = \frac{e^{i\alpha}}{\sqrt{1+C^2}}, \quad r_D = \frac{Ce^{i\left(\alpha+\theta_1+\frac{\pi}{2}\right)}}{\sqrt{1+C^2}} \qquad (9)$$

When $|\mathrm{Re}\, a| < 1$, the eigenvalues give a complex phase-shift, representing a propagating regime. Then equation (5) gives

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$$\lambda_{1,2} = e^{\pm i\varphi} \qquad (10)$$

with $\cos\varphi = \mathrm{Re}\, a$. Then we use

$$\cos\alpha = \frac{\cos M\varphi}{\sqrt{1+C^2}}, \qquad C = |b| \frac{\sin M\varphi}{\sin\varphi}, \qquad (11)$$

in equation (9).

When $|\mathrm{Re}\, a| \geq 1$, the eigenvalues are a damped or increasing exponential function, representing an attenuating regime. Then equation (5) gives

$$\lambda_{1,2} = e^{\mp\varphi} \qquad (12)$$

with $\cosh\varphi = \mathrm{Re}\, a$. Then the reflection and transmission responses are still given by equation (9) but with phase and amplitudes factors now given by

$$\cos\alpha = \frac{\cosh M\varphi}{\sqrt{1+C^2}}, \qquad C = |b| \frac{\sinh M\varphi}{\sinh\varphi}. \qquad (13)$$

For the limiting cases with $|\mathrm{Re}\, a| = 1$, there is a double root

$$\lambda_{1,2} = \mathrm{Re}\, a \qquad (14)$$

and then we must use

$$\cos\alpha = \frac{1}{\sqrt{1+|b|^2 M^2}}, \qquad C = |b| M \qquad (15)$$

In equation (9), with $|b|^2 = (\mathrm{Im}\, a)^2$.

Equivalent time-average and effective medium

Now we investigate the effect of increasing the number of layers, but keeping the total thickness $D = Md$ fixed. The total behavior of the reflection and transmission responses is determined by $\mathrm{Re}\, a$ which is plotted in Figure 2 for different values of *M*. The peaks are at $\mathrm{Re}\, a = 1$ given by the equation

$$\tan\frac{\theta_1}{2} + \frac{1 \mp |r|}{1 \pm |r|} \tan\frac{\theta_2}{2} = 0 \qquad (16)$$

The boundaries between a propagating and attenuating regime are at $\mathrm{Re}\, a = -1$ given by the equation

$$\tan\frac{\theta_1}{2} \tan\frac{\theta_2}{2} = \frac{1 \mp |r|}{1 \pm |r|} \qquad (17)$$

For low frequencies the stack of layers behaves as an effective medium with a velocity defined by (Backus, 1962; Hovem, 1995)

$$\frac{1}{v_{EM}^2} = \frac{1}{v_{TA}^2} + \frac{4 d_1 d_2}{(d_1 + d_2)^2} \frac{1}{v_1 v_2} \frac{r^2}{1 - r^2} \qquad (18)$$

For high frequencies the stack of layers is characterized by the time-average velocity defined by

$$\frac{d_1 + d_2}{v_{TA}} = \frac{d_1}{v_1} + \frac{d_2}{v_2} \qquad (19)$$

This occurs for frequencies above the second root of the equation $\mathrm{Re}\, a = -1$. There is a transition zone between these two roots in which the stack of layers partly blocks the transmitted wave.

The behavior of the medium is characterized by the ratio between wavelength and layer thickness. This is given by

$$\gamma = \frac{v_{TA}}{f(d_1 + d_2)} = \frac{1}{f \Delta t}, \qquad (20)$$

where Δt is the traveltime through the two single layers. To estimate the critical ratio of wavelength to layer thickness we assume $\Delta t_1 = \Delta t_2 = \Delta t / 2$. The effective medium limit then occurs at

$$\gamma_1 = \pi \left(a\tan\sqrt{\frac{1-|r|}{1+|r|}} \right)^{-1} \qquad (21)$$

and the time-average limit occurs at

$$\gamma_2 = \pi \left(\frac{\pi}{2} - a\tan\sqrt{\frac{1-|r|}{1+|r|}} \right)^{-1} \qquad (22)$$

For small values of r we obtain

$$\gamma_{1,2} = \pi \left(\frac{\pi}{4} \mp a\tan\frac{|r|}{2} \right)^{-1} \approx 4 \left(1 \pm \frac{2|r|}{\pi} \right). \qquad (23)$$

Numerical examples

To test our equations above we use a similar model as in Marion and Coudin (1992) with three different reflection coefficients: the original $r = 0.87$ and $r = 0.48$ and $r = 0.16$. We use m and Hz instead of mm and kHz.

The total thickness of the layered medium is $D = M_k d_k = 51m$ is constant. M_k, $k = 1, 2, 4, \ldots, 64$ is the number of cycles in the layered medium, so that the individual layer thickness is decreasing as *k* is increasing. The ratio $\theta_1 / \theta_2 = \Delta t_1 / \Delta t_2 = (d_1 v_2)/(d_2 v_1) = 0.91$. The other model parameters are given in Table 1.

The very important parameter that controls the regime is $\mathrm{Re}\, a$. The plots of $\mathrm{Re}\, a$ versus frequency are given in Figure 2 for different models. One can see that the propagating and attenuating regimes are periodically repeated in frequency. The higher reflectivity the more narrow frequency bands are related to propagating regime ($|\mathrm{Re}\, a| < 1$). One can also follow that the first effective medium zone is widening as the index of model increases, and that the ratio γ is the parameter which controls the regime. The exact transmission response is computed using

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a layer recursive algorithm (Ursin and Stovas, 2002). We use a Ricker wavelet with a central frequency of 500 Hz. The transmission responses are shown in Figure 3. No amplitude scaling was used. The upper seismogram in Figure 3 is similar to the Marion and Coudin (1992) experiment and the Hovem (1995) simulations. The effects related to the effective medium (difference between the first arrival traveltime for model M_1 and M_{64} and the transition between effective and time average medium, models $M_4 - M_{16}$) are the more pronounced for the high reflectivity model. For this model the first two traces (models M_1 and M_2) are composed of separate events, and then the events become more and more interferential as the thickness of the layers decrease. Model M_8 gives trains of nearly sinusoidal waves (tuning effect). The transmission response for model M_{16} is strongly attenuated, and models M_{32} and M_{64} behave as the effective medium. From Figure 2 and the transmission responses (Figure 3) one can distinguish between time average, effective medium and transition behavior. This behavior can be seen for any reflectivity, but a decrease in the reflection coefficient results in the convergence of the traveltimes for time-average and effective medium. This makes the effective medium arrival very close to the time average one. Note also that for the very much-pronounced effective medium (model M_{64}, $r = 0.87$) one can see the effective medium multiple on the time about 0.085s. The effective medium is related to the propagating regime. The effective medium velocity depends on the reflection coefficient. The lesser contrast the higher effective medium velocity (the more close to the time average velocity). One can also distinguish between effective medium, transition and time average frequency bands (Figure 2). These bands are separated by the frequencies given by conditions from the first two roots of equation $\operatorname{Re} a = -1$. The first root gives the limiting frequency for effective medium. The second one gives the limiting frequency for time average medium. From Figure 4 one can see that the transition zone converges to the limit $\gamma = 4$ with decreasing reflection coefficient.

Conclusions

We have derived explicit expressions for the reflected and transmitted wave field in a periodic medium. We found that there are two important parameters which control the wave propagation in the periodic medium: the reflection coefficient r and $\Delta t_1 / \Delta t_2$, the ratio between one-way traveltimes of the two parts of the cyclic layered medium. For low frequencies (large values of wavelength to layer thickness), the layered structure behaves as an effective medium, then there is a transition zone, and for high frequencies (small values of wavelength to layer thickness) the medium is described by the time-average velocity. The width of the transition zone increases with larger values of the reflection coefficient.

Acknowledgements

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	v_1 $\frac{km}{s}$	v_2 $\frac{km}{s}$	ρ_1 $\frac{g}{cm^3}$	ρ_2 $\frac{g}{cm^3}$	r	v_{TA} $\frac{km}{s}$	v_{EM} $\frac{km}{s}$
I	5.5	2.55	7.9	1.2	0.87	3.97	1.97
II	5.5	2.55	4.7	3.6	0.48	3.97	3.5
III	5.5	2.55	3.1	4.8	0.16	3.97	3.9

Table 1. Model parameters (I – large contrast model, II – medium contrast model, III – weak contrast model).

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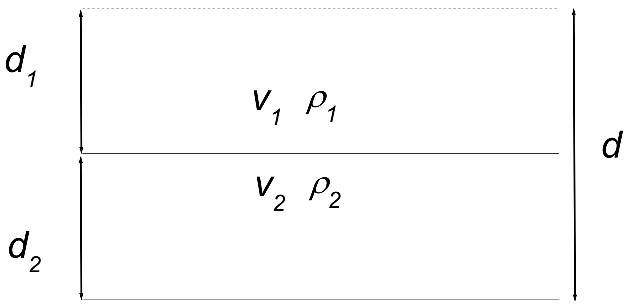

Figure 1. Single cycle of the periodic medium.

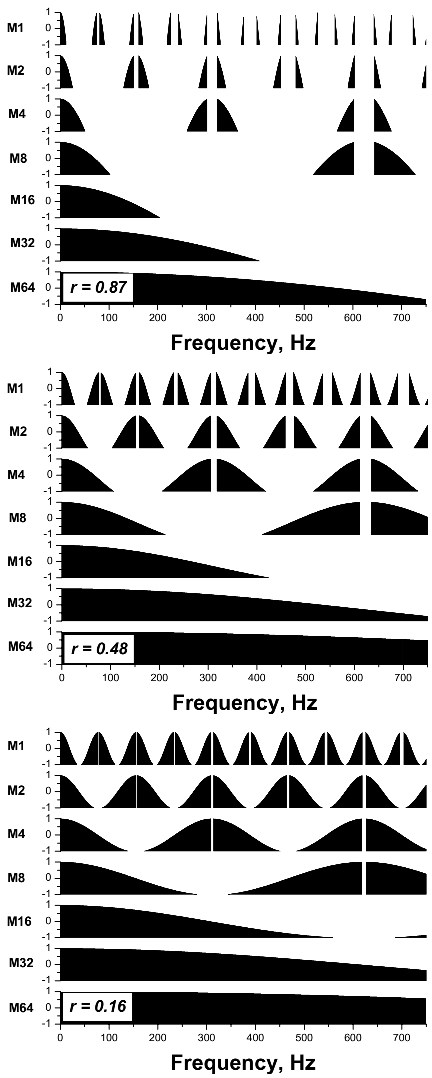

Figure 2. The real part of *a* versus frequency.

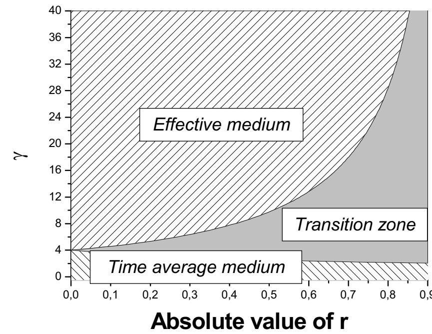

Figure 4. The critical γ ratio vs. reflection coefficient ($\theta_1/\theta_2 = \Delta t_1/\Delta t_2 = 1$).

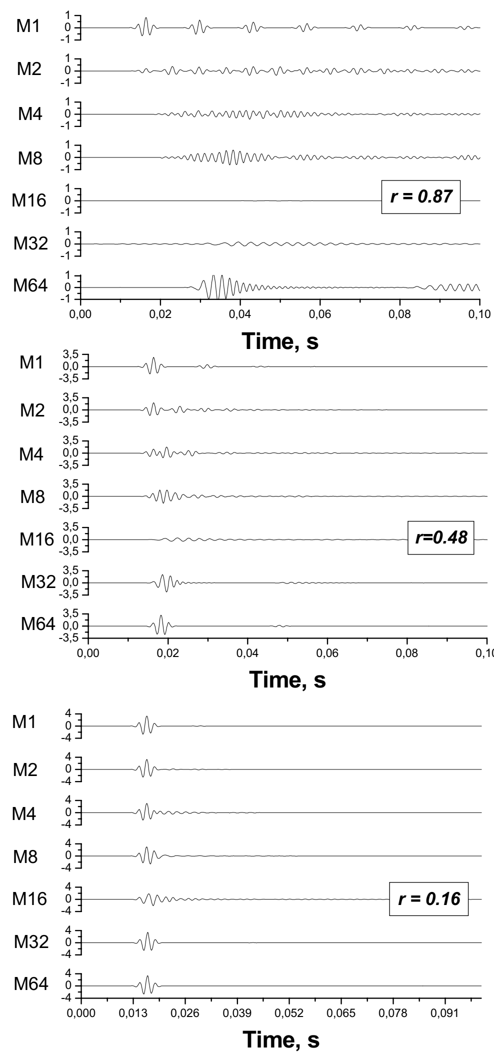

Figure 3. Numerical simulations of the transmission response.

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