

Validity of the long-wave approximation in periodically layered media

Borge Arntsen*

Statoil, Postuttak 7005, Trondheim, Norway

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ABSTRACT

In seismic modelling, a stack of thin layers is often replaced by an effective equivalent anisotropic homogeneous slab. For waves with finite wavelength, this is an approximation, and the error thus introduced can be quantified by considering the relative error in the phase velocity between the layer stack and the effective medium. For periodic layering, the relative phase-velocity error can be expressed in closed form as a function of wavelength, reflection coefficients and layer thicknesses. By comparing the relative phase-velocity error with laboratory measurements and numerical simulations, we find that the difference in seismic response between a periodic layer stack and an equivalent effective medium depends not only on wavelength, but it also depends significantly on reflection coefficients and the ratio between layer thicknesses. For a 1% relative error in the phase velocity, and if all layers have the same thickness measured in vertical traveltime, we find that the wavelength must be larger than approximately three times the layer period for a reflection coefficient of 0.1, but this increases to 13 times the layer period for a reflection coefficient of 0.9, which is highly unrealistic in a geological setting.

INTRODUCTION

A common procedure in synthetic seismogram generation is the replacement of a stack of thin layers with an anisotropic homogeneous medium (Bruggeman 1937; Riznichenko 1949; Postma 1955; Helbig 1958; Backus 1962). This approach is strictly valid only in the limit of infinite wavelength, but in practice it is used for seismic waves at finite wavelengths. We refer to this as the long-wave approximation. A problem that naturally arises is how to evaluate the errors in phase and amplitude thus introduced as a function of wavelength, layer thicknesses and layer parameters. We propose using the relative difference in phase velocity between the layer stack and the effective medium as an error function of the long-wave approximation. In order to gain new insight, we restrict ourselves to the case of a periodically layered medium. This is sufficiently simple to yield a closed-form error function, which can be used to determine when a layer stack can be replaced by an homogeneous effective medium.

Many of the sedimentary processes are, by their very nature, cyclic (Anstey and O'Doherty 2002a,b). Although perfect periodic layering is rare, layer sequences are sometimes nearly periodic and can be approximated by a periodic medium. Morlet *et al.* (1982) showed how a real well log can be approximated by a periodic medium, and how wave propagation through a periodic medium can be used to aid interpretation of seismic data in the vicinity of a well.

The theory of periodically layered elastic media was fully developed by Rytov (1956), who also suggested using the second-order terms in the low-frequency expansion of the dispersion relationship to establish the conditions of validity for the long-wave approximation. However, Rytov (1956) did not extend this discussion. Christensen (1979) derived an expansion for the dispersion relationship of a periodically layered medium up to the second order in wavenumber, but he did not consider numerical examples and he did not use it to establish the limits of the validity of the long-wave approximation.

Several authors (Helbig 1984; Melia and Carlson 1984; Carcione *et al.* 1991; Marion *et al.* 1994; Hovem 1995) have tried to quantify the error in the long-wave approximation by

*E-mail: barn@statiol.com

attempting to establish an approximate minimum wavelength λ_0 at which the amplitude and phase of waves propagating in a layered medium can be approximated by the amplitude and phase of waves propagating in an anisotropic homogeneous medium. Instead of using the wavelength λ_0 directly, it is more convenient to use the ratio $R_0 = \lambda_0/d$, where d is a typical layer thickness.

Helbig (1984) found a theoretical value of $R_0 > 3$, while Melia and Carlson (1984) concluded from laboratory experiments that R_0 lies between 10 and 100. Marion *et al.* (1994) also performed laboratory experiments and found that R_0 lies in the range between 8 and 15, while Carcione *et al.* (1991) found from numerical experiments that R_0 depends on the reflection coefficients of the medium and that the approximate value of R_0 lies in the range of 5 to 8. Folstad and Schoenberg (1992) found from numerical experiments that R_0 had an approximate value of 10, while Hovem (1995) suggested, from theoretical considerations, a value of $R_0 = 4$. Frazer (1994) considered a random binary medium and used three different approximate methods to derive the effective long-wave wave-velocity, but made no attempt to estimate a value for R_0 .

From the works cited above, it is clear that different approaches and different models seem to give different answers as to what conditions have to be satisfied before a finely layered medium can be approximately replaced by a homogeneous medium. In the following sections, we give a simple closed-form expression for the difference between a periodic finely layered medium and an equivalent homogeneous medium in terms of an error function which depends on wavelength (or frequency), layer thickness and the reflection coefficients. We do this by obtaining an approximate fourth-order dispersion relationship for a periodically layered medium and use this relationship to compute the phase-velocity error of the effective medium relative to the periodically layered original medium. We also briefly discuss how these results apply to cyclic stratification which is not exactly periodic. Numerical examples give new insight into the error made in replacing a finely layered medium with a homogeneous medium, and how the error depends both on the magnitude of the reflection coefficients and on the ratio R of the wavelength to layer thickness. In the last section we discuss the limitations of the proposed error function and the geological significance of the numerical examples. For simplicity, this work is limited to wave propagation normal, or near normal, to the layering of the medium. The medium can then be treated as purely acoustic and the anisotropy of the effective medium does not enter the discussion.

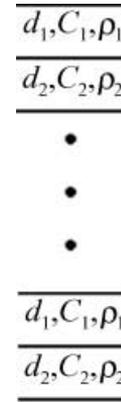


Figure 1 Periodically layered medium.

WAVE PROPAGATION IN A PERIODIC STACK OF PLANE LAYERS

Consider a horizontally plane-layered medium composed of layers periodically alternating N times; one layer is characterized by wave velocity c_1 , density ρ_1 and thickness d_1 , the other layer is characterized by wave velocity c_2 , density ρ_2 and thickness d_2 . The resulting medium is shown in Fig. 1. Since the main objective of this paper is the study of the properties of waves propagating in the direction normal to the layering, elastic effects are neglected and all layers are assumed to be purely acoustic.

Waves at the top of one period can be related to the waves at the top of the next period through a change of the phase Θ , which must be of the general form $\Theta = \omega\tau$, where ω is the circular frequency and τ is given by

$$\tau = d/C(\omega), \quad (1)$$

where $d = (d_1 + d_2)$ is the thickness of one period and $C(\omega)$ is the frequency-dependent velocity of the wave.

The value of $\Theta = \omega\tau$ is given implicitly by the dispersion relationship (Rytov 1956),

$$\cos(\omega\tau) = \cos(\omega\tau_1 + \omega\tau_2) - \frac{2r^2}{1-r^2} \sin(\omega\tau_1) \sin(\omega\tau_2), \quad (2)$$

where $\tau_1 = d_1/c_1$ and $\tau_2 = d_2/c_2$.

The reflection coefficient r is given by

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad (3)$$

where the impedance Z_i is defined by $Z_i = \rho_i c_i$, $i = 1, 2$. Equation (2) was originally derived by Rytov (1956), but expressed in terms of densities, moduli and layer thicknesses instead of reflection coefficients and traveltimes.

In the limit of very low frequency or very long wavelength, the dispersion equation (2) gives the wave velocity C_0 of an ‘effective’ homogeneous medium equivalent to the periodic layered medium, i.e.

$$\frac{1}{C_0^2} = \frac{1}{d^2} \left[(\tau_1 + \tau_2)^2 + \left(\frac{4r^2}{1-r^2} \right) \tau_1 \tau_2 \right]. \quad (4)$$

Expressing equation (4) in terms of layer parameters gives

$$\frac{1}{C_0^2} = \langle \rho c_{33}^{-1} \rangle, \quad (5)$$

where $\langle \rangle$ denotes thickness-weighted averages, and the stiffness $c_{33} = \rho c^2$.

It is also useful to define a wavelength $\lambda = 2\pi C_0/\omega$ and the ratio of wavelength to layer spacing by the dimensionless number R given by $R = \lambda/d$.

Sedimentary rocks sometimes exhibit layers showing cyclic successions on many different scales. These successions are usually associated with cyclic changes of sea-level, which might in turn be accounted for by climatic change and plate tectonics. These patterns appear in well logs and on seismic data (Anstey and O’Doherty 1971, 2002a,b). The well logs show a typical cyclic pattern of reflection coefficients with alternating sign, which is also the most important feature of a periodic medium. However, the cycles in real logs are rarely exactly periodic, but can rather be characterized as quasi-periodic or nearly periodic. The dispersion relationship given above is based on an exactly periodic medium and cannot easily be generalized. However, a suggestion is to use equation (2) as a rough approximation to the dispersion relationship of a quasi-periodic medium if the reflection coefficients of the quasi-periodic medium are constant and the layer thicknesses d_l , $l = 1, \dots, N$, of the quasi-periodic medium can be expressed as $d_l = d_k + \delta d_l$, $k = 1, 2$, where the deviations δd_l can be considered to be random and small with zero mean value. The resulting fluctuations in the amplitude and phase of a wave travelling through the quasi-periodic medium would then be small and tend to cancel each other out. As a crude approximation, equation (7) would then still be approximately valid with the layer thicknesses estimated by averaging.

THE PHASE-VELOCITY ERROR

The objective of this section is to study the behaviour of waves in a plane-layered periodic medium when the wavelength is long compared to the period length; specifically, it is to express $R = \lambda/d$, where λ is the wavelength, as a function of the difference in phase velocity between a periodically layered

medium and the effective homogeneous medium with velocity given by equation (4).

Consider a periodically layered medium with period d and wave velocity C_0 in the zero-frequency limit given by equation (4). For a finite frequency, the layered medium is assumed to have wave velocity $C(\omega)$ given by equation (2). The relative phase-velocity error ϵ is defined as

$$\epsilon = \frac{C_0 - C(\omega)}{C_0}. \quad (6)$$

The dispersion equation (2) can be solved numerically to obtain $R = \lambda/d$ as a function of ϵ . An expression in closed form is also desirable for ease of interpretation. In the Appendix it is shown how the dispersion equation (2) can be solved to give a closed-form expression in terms of R . The solution is valid only for large values of R and is given by equation (A8), i.e.

$$R = \lambda/d = \frac{\pi}{\sqrt{3}} \sqrt{\frac{(1-\epsilon)^{-4} - \beta}{(1-\epsilon)^{-2} - 1}}, \quad (7)$$

where the quantity β is given by equation (A5) as

$$\beta = \frac{\left[(1 + \tau_2/\tau_1)^4 + \left(\frac{8r^2}{1-r^2} \right) (\tau_2/\tau_1 + (\tau_2/\tau_1)^3) \right]}{\left[(1 + \tau_2/\tau_1)^2 + \left(\frac{4r^2}{1-r^2} \right) (\tau_2/\tau_1) \right]^2}. \quad (8)$$

Christensen (1979) also derived a long-wavelength expression for the dispersion relationship of a periodically layered medium, but this expression was based on an approximate solution of the wavefield. The expression given in equation (7) is the main result of this section since it expresses the relationship between R and the relative phase-velocity difference ϵ between the zero-frequency limit C_0 of the phase velocity C and the phase velocity itself. By specifying a largest acceptable relative phase-velocity error between a periodically layered medium and the effective medium, equation (7) can be used to compute a corresponding minimum value for R .

NUMERICAL RESULTS

Model 1 given in Table 1 is similar to that used in the laboratory experiment performed by Marion *et al.* (1994); only the thicknesses of the layers are slightly changed. Fig. 2 shows R (circles) for this model, computed as a function of $\epsilon = \frac{C_0 - C}{C_0}$ using equation (7); the line shows the corresponding numerical solution of equation (2).

Fig. 2 shows that the value of R is relatively insensitive to the value of ϵ when ϵ is larger than 0.01–0.02. However, when ϵ is smaller than approximately 0.01, R increases rapidly as ϵ decreases. In the limit when ϵ approaches zero, R approaches

Table 1 Material parameters for models 1,2 and 3

Model	Material	Thickness (mm)	Velocity (m/s)	Density (kg/m ³)
1	Plastic	0.5	2487	1210
	Steel	1.0	5535	7900
2	Epoxy	0.50	2530	1120
	Glass	0.50	5560	2510
3	Epoxy	0.50	2530	1815
	Glass	0.50	5560	1815

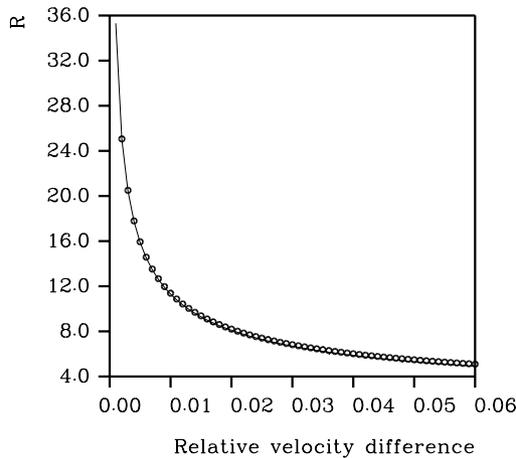


Figure 2 R as a function of relative phase-velocity difference, $\epsilon = (C_0 - C)/C_0$, for model 1. The circles show the approximate relationship given by equation (7), while the line is the numerical solution of the dispersion equation (2).

infinity. Fig. 2 indicates that, at a value of R equal to approximately 11.0 corresponding to a relative difference $\epsilon = 0.01$, the periodically layered medium is close to a homogeneous medium with wave velocity given by the long-wave approximation.

The laboratory measurements of Marion *et al.* (1994) were made in the time domain using source pulses containing a narrow range of frequencies. Strictly, equation (7) (and equation (2)) is only valid for monochromatic plane waves. As an additional check of equation (7), time-domain transmission responses were computed with numerical modelling using model 1 defined in Table 1. The modelling was implemented in the frequency–wavenumber domain, followed by a Fourier transform to the time domain, and it provides an exact solution, within numerical errors (Ursin 1983). A total of 248 layers was used. This choice makes the resulting layer stack large enough to avoid interference caused by internal reflec-

tions between boundaries at the top and bottom. Such internal reflections are unwanted and would distort the numerical results. A larger number of layers could have been used, but this would not change the numerical results. In addition, half-spaces were added below and above the periodically layered medium, with wave velocity defined by the average given in equation (4). This ensures acoustic coupling in the infinite wavelength limit.

Figs 3 and 4 show numerical simulations, with the dominant frequencies of the source corresponding to $R = 4.0$ and $R = 11.0$, respectively. The source pulse was the derivative of a Gaussian time function where the spectrum is proportional to $\omega \exp(-\omega^2/4\omega_0^2)$ and the peak of the spectrum is located at $\omega = \sqrt{2}\omega_0$. The solid line shows the corresponding numerical simulation when the periodically layered medium was replaced with a completely homogeneous medium with wave velocity given by the average defined in equation (4). Thus Fig. 4 shows that a 1% error in the phase velocity is small enough for the time response of a wave propagating through the periodic medium given by model 1 to be very close to the time response of a wave propagating through a homogeneous medium.

In Fig. 5, a semblance measure has been used to compare additional simulations for model 1 with simulations for the corresponding homogeneous model, using a range of source peak frequencies corresponding to values of R from 4.0 to

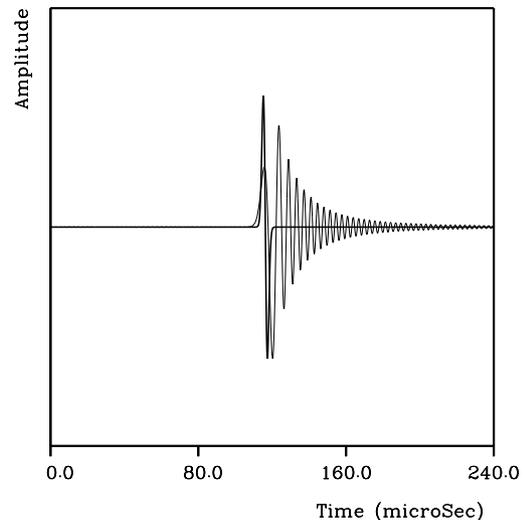


Figure 3 Simulation of a periodically layered medium defined by model 1 (thin line) and of a corresponding homogeneous medium (solid line). The dominant wavelength of the source pulse corresponds to $R = \lambda/d = 4.0$.

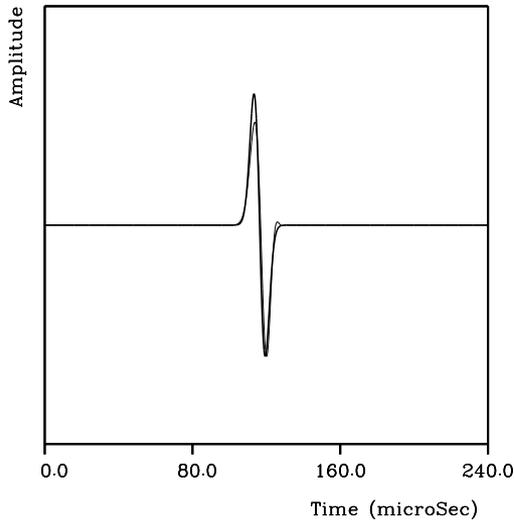


Figure 4 Simulation of a periodic layered medium defined by model 1 (thin line) and of a corresponding homogeneous medium (solid line). The dominant wavelength of the source pulse corresponds to $R = \lambda/d = 11.0$.

20.0. The semblance is computed using

$$S = \frac{\sum (a_i + b_i)^2}{2 \sum (a_i^2 + b_i^2)}, \quad (9)$$

where a_i and b_i are the samples of the simulated seismograms and the summation is over the number of samples (Carcione *et al.* 1991).

Fig. 5 shows that the semblance value at $R = 11.0$ is slightly less than 1.0, approaching 1.0 at $R \approx 15$. The laboratory measurements made by Marion *et al.* (1994) showed that the

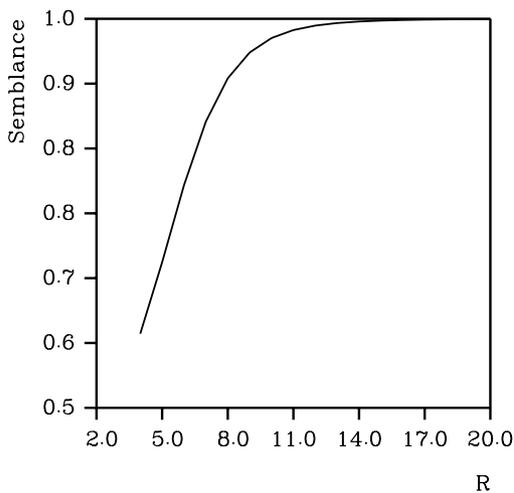


Figure 5 Semblance as a function of R for model 1 (see Table 1).

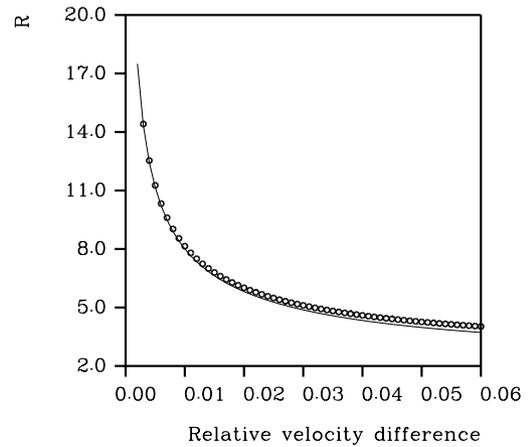


Figure 6 R as a function of relative phase-velocity difference, $\epsilon = (C_0 - C)/C_0$, for model 2. The circles show the approximate relationship given by equation (7), while the line is the numerical solution of the dispersion relationship given by equation (2).

transition from short- to long-wavelength behaviour occurs between $R = 8$ and $R = 15$. Thus, laboratory measurements and numerical simulation are in agreement with the predictions obtained from equation (7).

Although Fig. 5 contains the same information as Fig. 2, the time-domain simulations used to compute Fig. 5 more closely resemble the experimental situation described by Marion *et al.* (1994), and serve as a check that the monochromatic plane-wave expressions used to obtain Fig. 2 are relevant.

Models 2 and 3 defined in Table 1 coincide with two of the models used by Carcione *et al.* (1991); these were based on several physical models used by Melia and Carlson (1984).

Figs 6 and 7 show R as a function of ϵ (circles) computed using equation (7) for models 2 and 3. The line shows the corresponding numerical solution obtained from the dispersion equation (2). At the point $\epsilon = 0.01$, Figs 6 and 7 give values of R of approximately 8 and 5, respectively. This indicates that the periodic medium can be approximated with an effective homogeneous medium for values of R larger than 5 and 8 for models 2 and 3, respectively. Carcione *et al.* (1991) concluded from numerical experiments that the long-wavelength approximation is valid for values of R larger than 5–6 and 8, respectively. The predictions using equation (7) are therefore in agreement with the numerical experiments of Carcione *et al.* (1991).

Note that the expression given by equation (7) is an approximation valid for $R > 2\pi$, i.e long wavelengths, and will deviate from the results obtained from the exact dispersion relationship given by equation (2) when R becomes less than

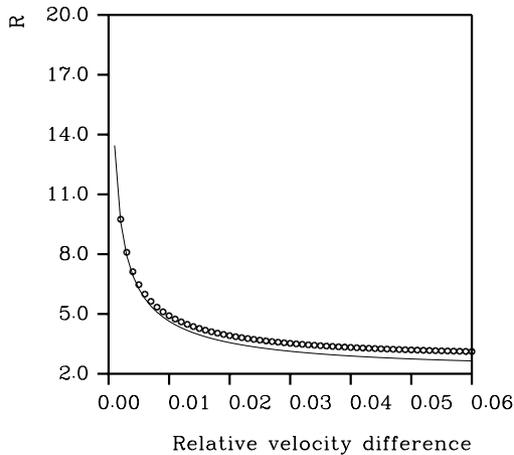


Figure 7 R as a function of relative phase-velocity difference, $\epsilon = (C_0 - C)/C_0$, for model 3. The circles show the approximate relationship given by equation (7), while the line is the numerical solution of the dispersion relationship given by equation (2).

2π . This is clearly shown in Figs 6 and 7, where the circles (approximation using equation (7)) increasingly deviate from the line (exact result using equation (2)) for values of R less than approximately 6. This is also, of course, the case for the results shown in Fig. 2, but is much less visible because the dispersion relationship in this case is only shown for relatively large values of R .

The main difference between models 1, 2 and 3, used to compute the results shown in Figs 2, 6 and 7, respectively, is the density contrast between the layers. Thus the reflection coefficient for model 1 is larger in magnitude than the reflection coefficient for model 2, which again is larger than the reflection coefficient for model 3. The difference in reflection coefficient between the models leads to different relationships between the relative velocity difference ϵ and R , and hence different transition points to an effective homogeneous medium. This can be further investigated by keeping ϵ and the ratio of layer thicknesses d_2/d_1 fixed, and using equation (7) (or the numerical solution of equation (2)) to predict how the minimum value of R changes with changing reflection coefficient. Fig. 8 shows R (circles) as a function of the reflection coefficient r at a constant value of $\epsilon = 0.01$, computed using model 1 and changing only the densities. The line shows the corresponding numerical solution obtained from the dispersion equation (2).

Taking the transition point to an effective homogeneous medium at a relative velocity difference of $\epsilon = 0.01$, Fig. 8 shows the transition point, from dispersive wave propagation in a periodically layered medium to non-dispersive wave prop-

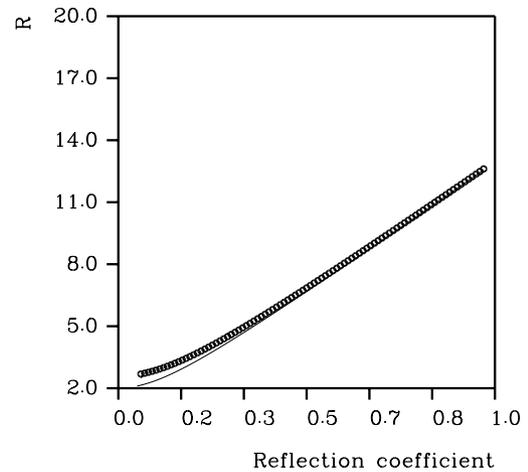


Figure 8 R as a function of reflection coefficient at a relative phase-velocity difference of $\epsilon = (C_0 - C)/C_0 = 0.01$ for model 1. The circles show the approximate relationship given by equation (7), while the line is the numerical solution of the dispersion relationship given by equation (2).

agation in a homogeneous effective medium, as a function of reflection coefficient. The interesting observation here is that the long-wave approximation is satisfied for smaller values of R if the reflection coefficient is small than is the case if the reflection coefficient is large. In cyclic sediments the reflection coefficients are rarely larger than 0.1, so from Fig. 8 it can be seen that this corresponds to a value of R of less than approximately 3. For many real sediments the long-wave approximation would then be satisfied for wavelengths longer than or equal to three times the layer period.

Fig. 9 also illustrates the effect of different ratios between layer thicknesses for fixed reflection coefficients. The parameters are those given for model 2 in Table 1 and R is plotted here as a function of the material fraction of epoxy, given by $d_1/(d_1 + d_2)$, and ranging from 0.02 to 0.9.

DISCUSSION AND CONCLUSION

The results in the preceding sections show that, for a seismic wave propagating through a periodically layered medium, the ratio R of the wavelength λ to the period thickness d can be expressed as a function of reflection coefficient r , the ratio $\tau_2/\tau_1 = (d_2/d_1)(c_1/c_2)$ (see equation (7)) and the relative difference in the phase velocities of the periodically layered medium and the effective medium. For example, Fig. 8 shows that, for a 1% relative error in the phase velocity, the wavelength must be larger than approximately 3 for a reflection

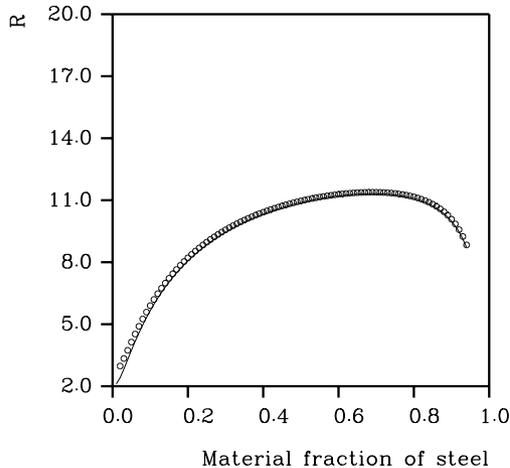


Figure 9 R as a function of volume fraction of the second layer $d_2/(d_1 + d_2)$ (steel fraction) at a relative phase-velocity difference of $\epsilon = (C_0 - C)/C_0 = 0.01$ for model 1. The circles show the approximate relationship given by equation (7), while the line is the numerical solution of the dispersion relationship given by equation (2).

coefficient of 0.1 but it increases to 13 for a reflection coefficient of 0.9, corresponding to an increase in velocity by a factor of 10, and an increase in stiffness by a factor of 100. Also, Fig. 9 shows that, for a 1% relative error in the phase velocity and for a fixed reflection coefficient, the wavelength must be larger than approximately 5 for a layer thickness ratio, $d_2/(d_1 + d_2)$, of epoxy equal to 0.05, but it increases to approximately 9 for a layer thickness ratio of 0.9.

Using equation (7), or the corresponding numerical solution obtained from the dispersion equation (2), it is now possible to relate the seemingly different estimates obtained for the minimum wavelength λ_0 at which an effective homogeneous medium can be regarded as a good replacement of a periodic stack of layers. For example, Helbig (1984) obtained $\lambda_0/d = 3$, where Carcione *et al.* (1991) obtained $\lambda_0/d = 5-8$ and Marion *et al.* (1994) obtained $\lambda_0/d = 8-15$. These results are different because the reflection coefficients and the traveltimes ratios are different and equation (7) provides the link between these previous studies. Moreover, Marion *et al.* (1994) obtained larger values for λ/d mainly because they considered models with larger density contrast and hence larger reflection coefficients than Helbig (1984) and Carcione *et al.* (1991). In fact, the transition from dispersive wave propagation to non-dispersive wave propagation in a periodically layered medium explicitly depends on the magnitude of the reflection coefficient. Equation (7) incorporates this essential feature of a periodically layered medium

and provides a simple way to understand the laboratory measurements.

The results derived in the previous sections are specific to a periodic medium composed of two types of material, and the numerical examples chosen to make comparisons with laboratory experiments possible. Sedimentary sequences with exact periodicity are not common, but sedimentary sequences are, in many cases, cyclic (see Anstey and O'Doherty 2002a,b; Wilson 1997).

As discussed previously, equation (7) is proposed as a crude approximation for a near-periodic cyclic medium, with the layer thicknesses estimated by averaging. Fig. 8 then shows that, for real near-periodic cyclic sedimentary sequences with small reflection coefficients, the long-wave approximation would be satisfied for wavelengths equal to or longer than three times the layer period.

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APPENDIX

The low-frequency dispersion relationship

When the frequency ω approaches zero, the right- and left-hand sides of equation (2) may be expanded in a Taylor series. Equation (2) gives, to second order,

$$1 - \frac{1}{2}\omega^2\tau^2 = 1 - \frac{1}{2}(\omega\tau_1 + \omega\tau_2)^2 - \frac{2r^2}{1-r^2}\omega\tau_1\omega\tau_2. \quad (\text{A1})$$

This is equal to

$$\tau_0^2 = (\tau_1 + \tau_2)^2 + \left(\frac{4r^2}{1-r^2}\right)\tau_1\tau_2, \quad (\text{A2})$$

where $\tau_0 = d/C_0$, and the subscript 0 indicates that this is the zero-frequency limit. Equation (A2) can also be written as

$$\frac{1}{C_0^2} = \frac{1}{d^2} \left[(\tau_1 + \tau_2)^2 + \left(\frac{4r^2}{1-r^2}\right)\tau_1\tau_2 \right]. \quad (\text{A3})$$

Expanding equation (2) up to fourth order yields

$$\frac{1}{24}(\omega\tau)^4 - \frac{1}{2}(\omega\tau)^2 = \frac{1}{24}(\omega\tau_0)^4\beta - \frac{1}{2}(\omega\tau_0)^2, \quad (\text{A4})$$

where $\tau = d/C$ and

$$\beta = \frac{\left[(1 + \tau_2/\tau_1)^4 + \left(\frac{8r^2}{1-r^2}\right)(\tau_2/\tau_1 + (\tau_2/\tau_1)^3) \right]}{\left[(1 + \tau_2/\tau_1)^2 + \left(\frac{4r^2}{1-r^2}\right)(\tau_2/\tau_1) \right]^2}. \quad (\text{A5})$$

Solving equation (A4) for $\omega\tau_0$ leads to

$$\omega\tau_0 = 2\sqrt{3}\sqrt{\frac{(\tau/\tau_0)^2 - 1}{(\tau/\tau_0)^4 - \beta}} \quad (\text{A6})$$

or

$$\lambda/d = \frac{\pi}{\sqrt{3}}\sqrt{\frac{(C_0/C)^4 - \beta}{(C_0/C)^2 - 1}}, \quad (\text{A7})$$

where $\lambda = 2\pi C_0/\omega$. Equation (A7) can be rewritten as

$$\lambda/d = \frac{\pi}{\sqrt{3}}\sqrt{\frac{(1-\epsilon)^4 - \beta}{(1-\epsilon)^2 - 1}}, \quad (\text{A8})$$

where $(C_0 - C)/C_0 = \epsilon$.