

# Discussion

On “A new insight into the reciprocity principle” (Børge Arntsen and José M. Carcione, *Geophysics*, 65, 1604-1612).

## INTRODUCTION

Reciprocity is an important property of elastodynamic, electromagnetic, and acoustic wavefields. Combined with optimization techniques, reciprocity theorems can be regarded as providing the basic ingredients to imaging and inversion methods in geophysical exploration and remote sensing (de Hoop, M. V. and de Hoop, A. T., 2000). Furthermore, reciprocity serves as the basis for the elimination procedures of surface-related multiples in marine seismic data processing (Fokkema and van den Berg, 1993). In view of all this, a thorough and elucidating discussion on the configurations to and the conditions under which reciprocity applies, and what reciprocity leads to, like the recent paper by Arntsen and Carcione (2000), serves a useful purpose. In this paper, also a number of interesting applications are worked out in detail. The aim of this discussion is to indicate briefly how some of the results obtained in that paper are related to the ones that the present author has presented in de Hoop (1995), a reference that does not appear in Arntsen and Carcione (2000).

To comply with de Hoop (1995) [and largely with Arntsen and Carcione (2000) as well], the standard subscript notation for Cartesian vectors and tensors is used and the summation convention applies. Let  $\mathbf{x} = \{x_1, x_2, x_3\}$  specify position in space with respect to a fixed orthogonal Cartesian reference frame and let  $t$  denote the time coordinate. Then the dynamic stress  $\tau_{pq}$  and the particle velocity  $v_r$  of an elastodynamic disturbance satisfy the linear system of partial differential equations (de Hoop, 1995, 327)

$$-\Delta_{km pq}^+ \partial_m \tau_{pq} + \rho_{kr} \ast \partial_t v_r = f_k, \quad (1)$$

$$\Delta_{ij nr}^+ \partial_n v_r - S_{ij pq} \ast \partial_t \tau_{pq} = h_{ij}, \quad (2)$$

in which  $\ast$  denotes time convolution,  $\rho_{kr}$  is the tensorial inertia relaxation function,  $S_{ij pq}$  is the compliance relaxation function,  $f_k$  is the volume source density of body force,  $h_{ij}$  is the volume source density of deformation rate, and  $\Delta_{km pq}^+ = (\delta_{kp} \delta_{mq} + \delta_{kq} \delta_{mp})/2$ , with  $\delta_{km}$  the Kronecker tensor, is the symmetric unit tensor of rank four that is characteristic for elastodynamics (de Hoop, 1995, 311). Via the constitutive relaxation tensors  $\rho_{kr} = \rho_{kr}(\mathbf{x}, t)$  and  $S_{ij pq} = S_{ij pq}(\mathbf{x}, t)$  full inhomogeneity, anisotropy, time invariance, and linear relaxation behavior of the medium are taken into account. By restricting the temporal support of the relaxation functions to  $\{t \geq 0\}$ , also causality of the medium's response is guaranteed.

## GREEN'S TENSORS

By invoking the superposition principle and considering the volume source densities (supposed to have bounded spa-

tial supports) as a superposition of point sources, the introduction of the pertaining point-source solutions as the relevant Green's tensors leads to the following source-type integral representations for the wavefield quantities:

$$v_r(\mathbf{x}, t) = \int_{D^f} G_{rk}^{vf}(\mathbf{x}, \mathbf{x}', t) \ast f_k(\mathbf{x}', t) dV(\mathbf{x}') + \int_{D^h} G_{rij}^{vh}(\mathbf{x}, \mathbf{x}', t) \ast h_{ij}(\mathbf{x}', t) dV(\mathbf{x}'), \quad (3)$$

$$-\tau_{pq}(\mathbf{x}, t) = \int_{D^f} G_{pqk}^{\tau f}(\mathbf{x}, \mathbf{x}', t) \ast f_k(\mathbf{x}', t) dV(\mathbf{x}') + \int_{D^h} G_{pqij}^{\tau h}(\mathbf{x}, \mathbf{x}', t) \ast h_{ij}(\mathbf{x}', t) dV(\mathbf{x}'), \quad (4)$$

where  $D^f$  is the support of  $f_k = f_k(\mathbf{x}, t)$  and  $D^h$  is the support of  $h_{ij} = h_{ij}(\mathbf{x}, t)$ . These representations hold in the interior of any bounded domain if on one part of its boundary surface the normal component of the dynamic stress vanishes, while on the complementary part the particle velocity vanishes [which is the case considered by Arntsen and Carcione (2000)], whereas the representation holds anywhere in space for unbounded domains provided that the causality condition for outgoing waves (“radiation condition”) is invoked (de Hoop, 1995, 439-440). The minus sign on the left-hand side of equation (4) has been introduced because of the property that  $-\Delta_{mnpq}^+ \tau_{pq} v_r$  represents the area density of elastodynamic power flow (de Hoop, 1995, 331), which property makes the pair  $\{-\tau_{pq}, v_r\}$  the thermodynamically intensive wavefield quantities.

## TIME-DOMAIN RECIPROcity OF THE TIME-CONVOLUTION TYPE

The time-domain reciprocity theorem of the time-convolution type [which is the one considered by Arntsen and Carcione (2000)] that interrelates two states denoted by the superscripts  $A$  and  $B$  follows upon integrating the local interaction quantity

$$\Delta_{mnpq}^+ \partial_m (-\tau_{pq}^A \ast v_r^B + \tau_{pq}^B \ast v_r^A)$$

over the domain of application  $D$ , applying Gauss' integral theorem and observing that, under the stated conditions, the contribution from the surface integral over the boundary  $\partial D$  of  $D$  vanishes. On the condition that the media in the two states are each others adjoints, i.e., if  $\rho_{kr}^A = \rho_{rk}^B$  and  $S_{ij pq}^A = S_{pq ij}^B$ , the result is [for details, see de Hoop (1995, 437-438)]

$$\int_D (v_r^A \ast f_r^B + \tau_{pq}^A \ast h_{pq}^B) dV = \int_D (v_k^B \ast f_k^A + \tau_{ij}^B \ast h_{ij}^A) dV, \quad (5)$$

where the spatial integrations are to be extended of the supports of the relevant volume source densities only. Equation (4), but with the terms corresponding to the deformation rate sources absent, corresponds to equation (8) of Arntsen and Carcione (2000).

### RECIPROCITY OF THE GREEN'S TENSORS

The reciprocity properties of the Green's tensors follow from equation (5) upon taking the volume source densities to be the ones corresponding to point sources in space, with a delta-function behavior in time. We take

$$\{f_k^A, h_{ij}^A\} = \{a_k^A, b_{ij}^A\} \delta(\mathbf{x} - \mathbf{x}', t)$$

and

$$\{f_r^B, h_{pq}^B\} = \{a_r^B, b_{pq}^B\} \delta(\mathbf{x} - \mathbf{x}'', t),$$

where  $\delta(\mathbf{x} - \mathbf{x}', t)$  is the 4-D Dirac delta distribution operative at  $\mathbf{x} = \mathbf{x}'$  and  $t = 0$ . It is essential that  $\mathbf{x}' \neq \mathbf{x}''$ . Substituting these choices in equation (5), using the wavefield representations (3) and (4) for the two states and recalling that the resulting expression has to hold for arbitrary values of  $a_k^A, b_{ij}^A, a_r^B,$  and  $b_{pq}^B$ , it follows that (de Hoop, 1995, 472-473)

$$G_{rk}^{vf:A}(\mathbf{x}'', \mathbf{x}', t) = G_{kr}^{vf:B}(\mathbf{x}', \mathbf{x}'', t), \quad (6)$$

$$G_{rij}^{vh:A}(\mathbf{x}'', \mathbf{x}', t) = -G_{ijr}^{\tau f:B}(\mathbf{x}', \mathbf{x}'', t), \quad (7)$$

$$G_{pqk}^{\tau f:A}(\mathbf{x}'', \mathbf{x}', t) = -G_{kpq}^{vh:B}(\mathbf{x}', \mathbf{x}'', t), \quad (8)$$

$$G_{pqij}^{\tau h:A}(\mathbf{x}'', \mathbf{x}', t) = G_{ijpq}^{\tau h:B}(\mathbf{x}', \mathbf{x}'', t). \quad (9)$$

Equation (6) is equivalent to equation (16) of Arntsen and Carcione (2000). Equation (9) is equivalent to their equation (29), while equations (7) and (8) are only partly discussed by them. The special case

$$h_{ij}^A = q^A \delta_{ij} \delta(\mathbf{x} - \mathbf{x}', t), \quad h_{pq}^B = q^B \delta_{pq} \delta(\mathbf{x} - \mathbf{x}'', t)$$

covers the case of an omnidirectional dilatational source. This special case of equations (7) and (8) corresponds to equation (25) of Arntsen and Carcione (2000).

As to the results for the couples, single or double, with or without moments (Arntsen and Carcione, 2000, Figures 2-4), this author holds the opinion that they are superfluous. Their inclusion is derived from their supposed importance to the modeling of earthquake mechanisms. However, tectonic dynamic strike-slip faulting, which is commonly accepted to be the mechanism of the generation of earthquakes, causes a time-dependent strike-slip displacement along a surface, a

case that is covered by the Kirchhoff type representations for the action of surface sources, similar to equations (3) and (4) for volume sources. In these representations, only the jump in the normal component of the dynamic stress (i.e., a surface source density of force) and the jump in the deformation rate (i.e., a surface source density of deformation rate) across the surface occur. The resulting elastodynamic wavefield in space-time is again fully described through the propagation action of the Green's tensors introduced in my equations (3) and (4) (see de Hoop, 1995, 494-501). Note that the introduction of deformation rate sources as has been done in equation (2) is essential to this type of Kirchhoff representation and that in this respect it is of prime importance to characterize the (visco-)elastic properties of the medium through its tensorial compliance rather than through its tensorial stiffness (the inverse of the compliance). Observe that in this respect equation (2) of Arntsen and Carcione (2000) differs essentially from my equation (2). Another observation is that only as they are written, my equations (1) and (2) have the standard shape of simultaneous partial differential equations of the first order to which the mathematical theory of hyperbolic systems applies.

### FREQUENCY-DOMAIN RECIPROCITY OF THE TIME-CONVOLUTION TYPE

The frequency-domain reciprocity theorem of the time-convolution type follows from its time-domain counterpart by observing that the operation of time convolution corresponds to the operation of multiplication in the (complex) frequency-domain. The relevant results follow easily from what has been presented so far and will not be reproduced here. For details, the reader is referred to de Hoop (1995, 445-450 and 468-471).

### “VERIFICATION” OF THE RECIPROCITY PROPERTY

In their section on numerical experiments, Arntsen and Carcione (2000) state that “A set of numerical experiments verify the reciprocity relations, etc.” In my opinion, the situation in this respect is the other way around. The reciprocity theorem as expressed by equation (5) holds as long as the basic equations (1) and (2) for the elastodynamic wavefield hold and solutions to these equations exist. Any numerical experiment related to reciprocity therefore provides a necessary (but not sufficient) check on the correctness and the accuracy of the computer code involved, rather than that the computer code would verify reciprocity!

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## Reply by the authors to the discussion by A. T. de Hoop

The discussion by A. T. de Hoop is highly appreciated, and we regret to have omitted the reference to de Hoop (1995). Part of the developments in our paper are similar to those of de Hoop and Stam (1988). Mittet and Hokstad (1995) have also presented similar results for the purely elastic case. These are both properly referenced in the Introduction. The main remarks of de Hoop are:

- *Most of the equations are known.* Yes, but in its general form and mostly using a mathematical notation unknown to geophysicists. Our aim is to show particular situations, using a simple notation, which may be relevant for seismic applications.
- *The relations obtained for couples are superfluous. They are not relevant for modeling earthquake sources. Strike-slip faulting is commonly accepted to be the mechanism of the generation of earthquakes. The Kirchhoff type representation covers all the cases.* Strike-slip mechanisms are common in the United States (e.g., San Andreas Fault). In Italy, we have mainly dip-slip – normal and reverse – mechanisms, where the slip-strike is not dominant (Boschi et al., 1995). Kirchhoff type representation may be relevant for analytical or semi-analytical modeling methods (?). Most full-wave equation (direct methods) algorithms use double couples to initiate the earthquake source (e.g., Helmberger and Vidale, 1988). They constitute a good approximation of the far-field (Madariaga, 1983). The modeling group at OGS commonly uses double couples (e.g., Priolo, 1999). In seismic exploration shear-wave data is often acquired by sources using a single horizontal force. Two experiments with oppositely directed forces are then combined into a single measurement. This situation is naturally described by a single couple (Tatham and McCormack, 1991).
- *We are wrong when we state that we “verify the relations” by numerical modeling.* Yes, we could have phrased the statement in a different way. The algorithm has been tested with appropriate analytical solutions. We should change the statement “verify the relations” to “exemplify the relations.” It must also be understood that

our modeling algorithm is a direct discretization of the equations of motion and as such can be used to perform numerical tests on our theoretical expressions to screen out mistakes and errors. Although this by no means represents an exhaustive test, it is still useful as it increases the confidence in the results.

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