Aspect ratio histograms of 3D Ellipsoids and 2D Ellipses*

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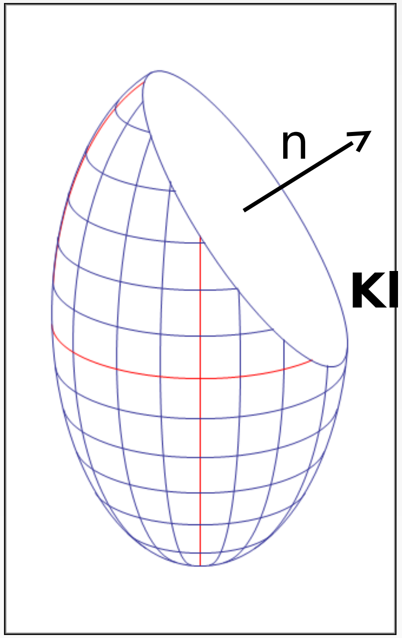
3D ellipsoids might represent…

- **Pore**
- **Crack**
- **Grain**

Cutting an ellipsoid with a plane



$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1 \qquad n=(n_1,n_2,n_3)^T$$

Klein (2012) shows that the cut is an ellipse with half axes

$$A=\sqrt{\frac{1-d}{\beta_1}} \qquad B=\sqrt{\frac{1-d}{\beta_2}} \qquad \alpha'=\sqrt{\frac{\beta_2}{\beta_1}}$$ 2D aspect ratio

$$\beta^2-\left[n_1^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+n_2^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+n_3^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right]\beta+\frac{n_1^2}{b^2c^2}+\frac{n_2^2}{a^2c^2}+\frac{n_3^2}{a^2b^2}=0$$

ecial case 1: two shortest semiaxes, a =b < c = $\alpha = c/a$ 3D aspect ratio

$$\alpha'=\frac{\alpha}{\sqrt{1+n_3^2(\alpha^2-1)}}$$

Simple relation between
2D and 3D aspect ratio

The histogram of the 2D aspect ratio

$$\alpha' = \frac{\alpha}{\sqrt{1 + n_3^2(\alpha^2 - 1)}}$$

Histogram given by ***the derivative*** of n_3 with respect to the 2D aspect ratio:

$$h(\alpha') \propto \frac{dn_3}{d\alpha'} = \frac{const \cdot \alpha^2}{\alpha'^2 \sqrt{(\alpha^2 - 1)(\alpha^2 - \alpha'^2)}}$$

$$h(\alpha') = \frac{N\alpha^2}{\alpha'^2 \sqrt{(\alpha^2 - 1)(\alpha^2 - \alpha'^2)}}$$

N = number of realization

Note: *h* is independent of n_3-distribution

The 3D to 2D transform:

$$h_{2D}(\alpha') = \int h_{3D}(\alpha) K(\alpha, \alpha') d\alpha$$

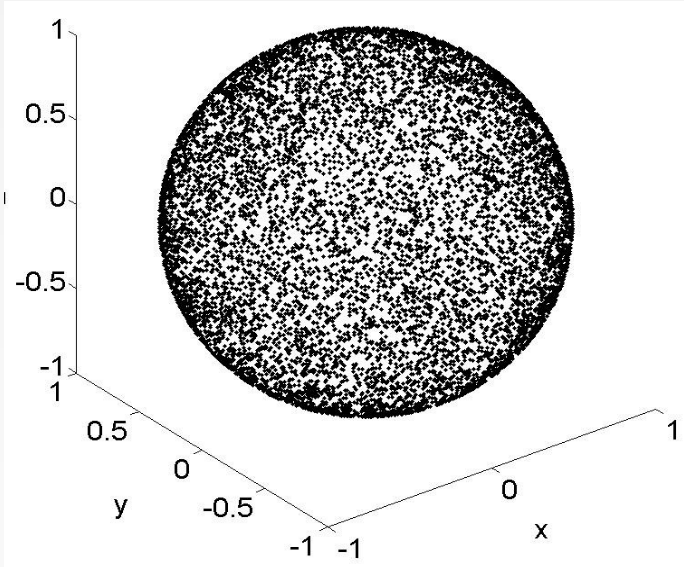
$$K(\alpha, \alpha') = \frac{\alpha^2}{\alpha'^2 \sqrt{(\alpha^2 - 1 + \varepsilon)(\alpha^2 - \alpha'^2 + \varepsilon)}}$$

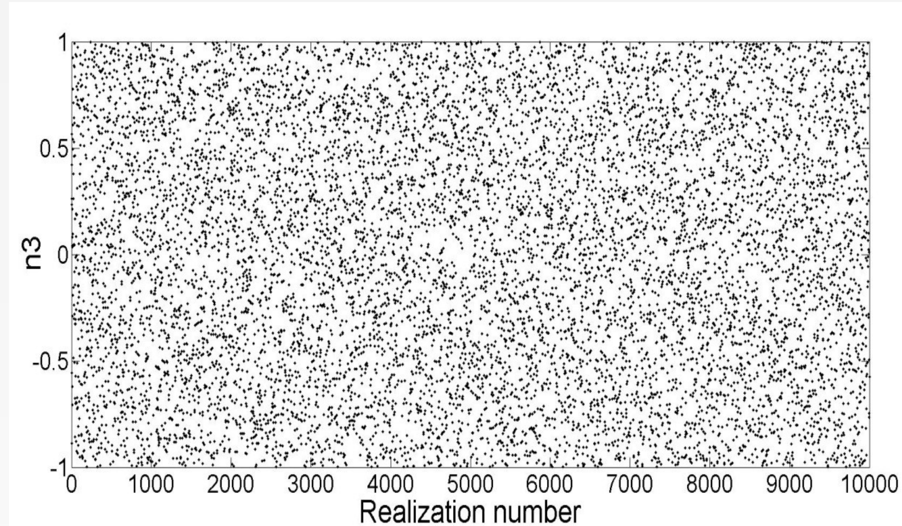
The 2D to 3D transform:

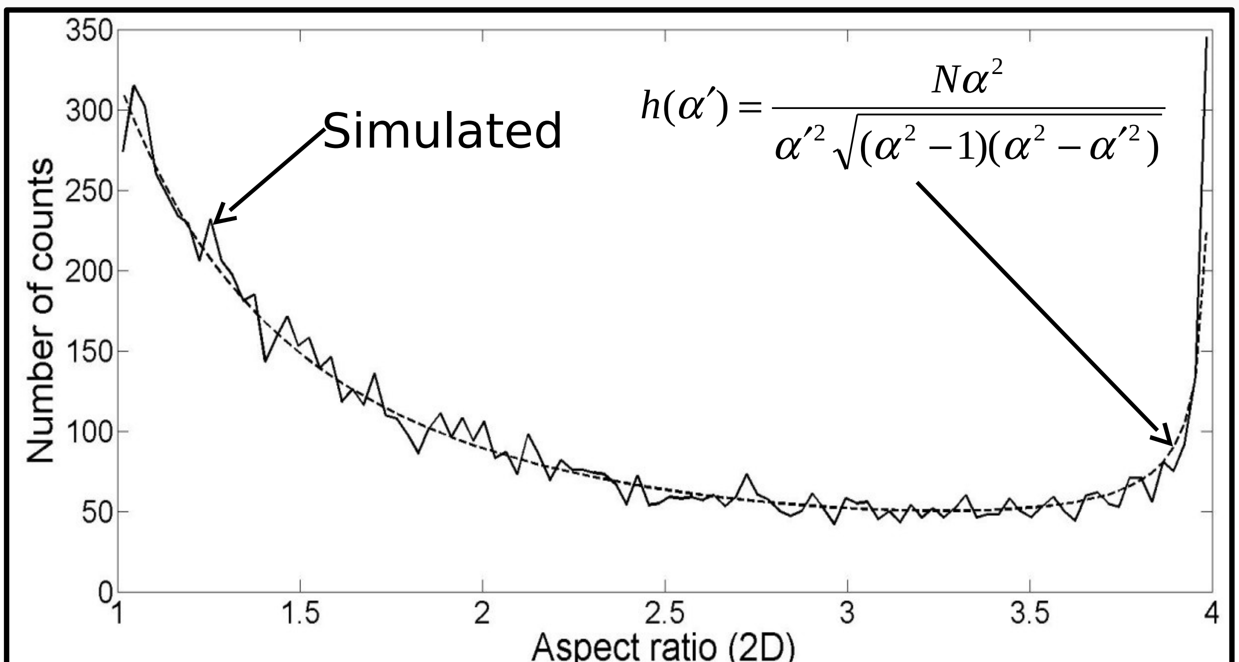
$$h_{3D}(\alpha) = \int h_{2D}(\alpha') K^{-1}(\alpha', \alpha) d\alpha'$$

$$K^{-1}(\alpha', \alpha) = \frac{\alpha\left(\alpha'^2 - 1\right)}{\alpha'\left(\alpha^2 - 1 + \varepsilon\right)^{\frac{3}{2}} \sqrt{(\alpha^2 - \alpha'^2 + \varepsilon)}}$$

Uniform distribution - 10 000 realization



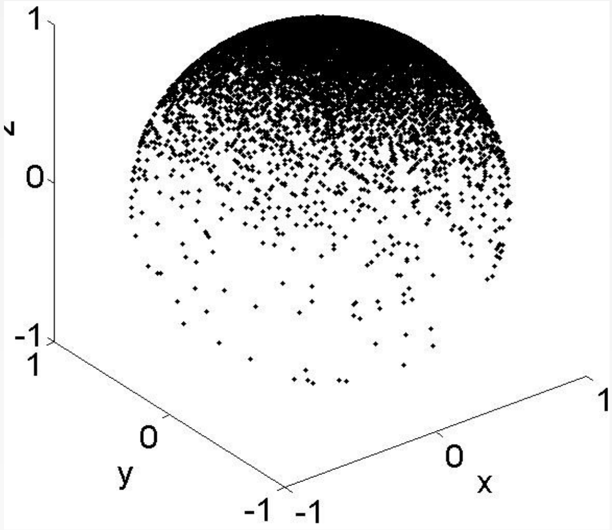


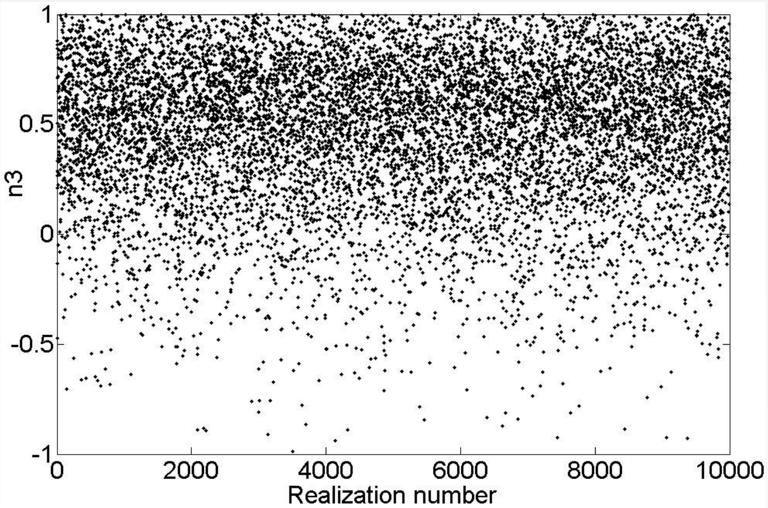


$$\alpha = \alpha_{3D} = 4$$

$$\alpha' = \alpha_{2D}$$

Non-uniform example

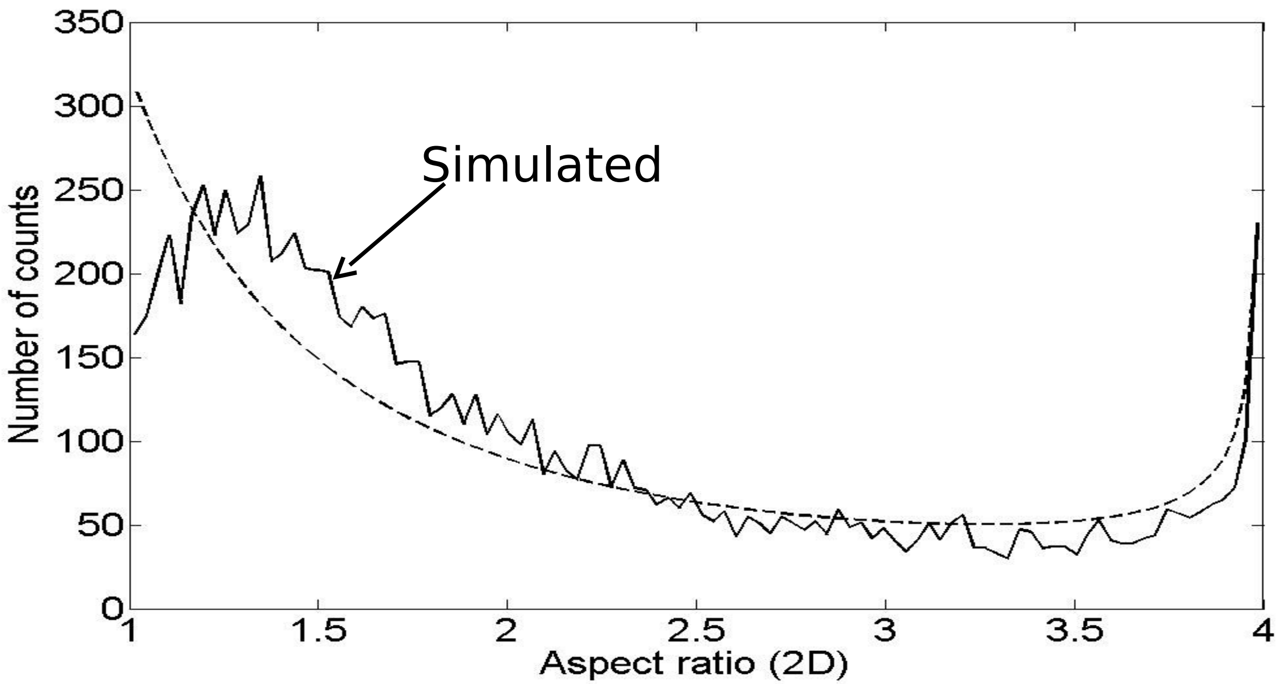




Normal distribution:

$$\langle n_1 \rangle = \langle n_2 \rangle = \langle n_3 \rangle = 0.47$$

$$\sigma = 0.33$$

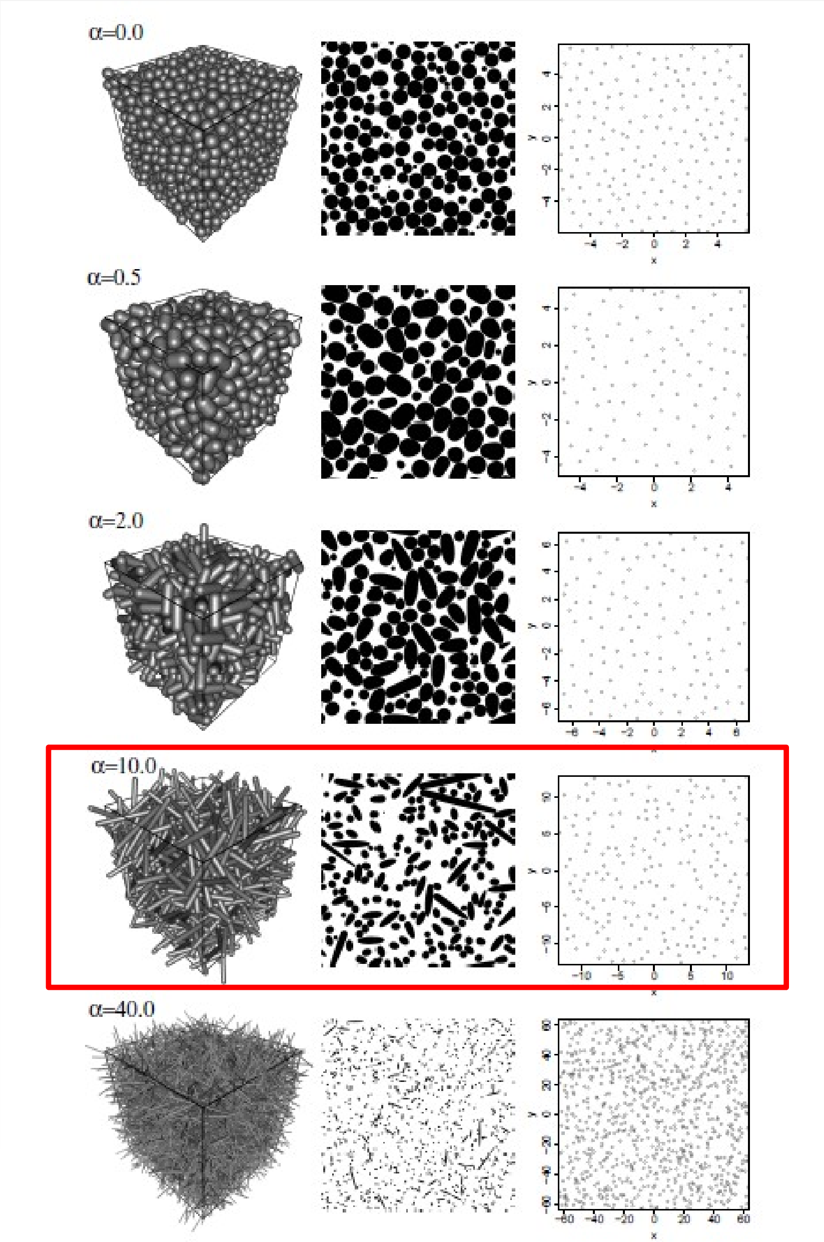


Decreasing the standard deviation yields a worse fit

$$h(\alpha') = \frac{N\alpha^2}{\alpha'^2 \sqrt{(\alpha^2 - 1)(\alpha^2 - \alpha'^2)}}$$

Note: This formula is general, not dependent on the assumed distribution –independent of n_3

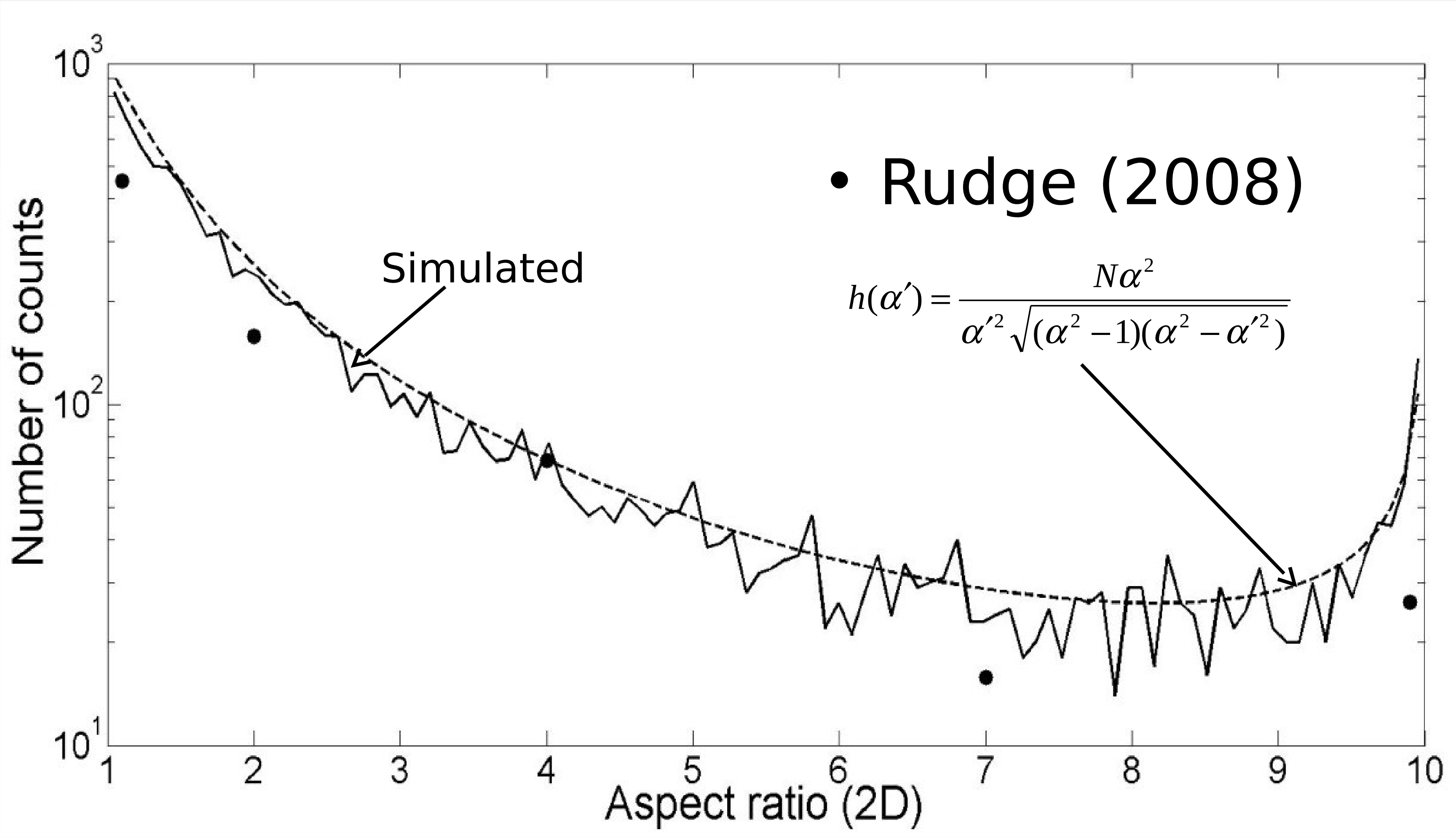
Rudge et al., 2008:



Rudge et al., 2008 use spherocylinders and use a random close packing – followed by one single 2D cut.

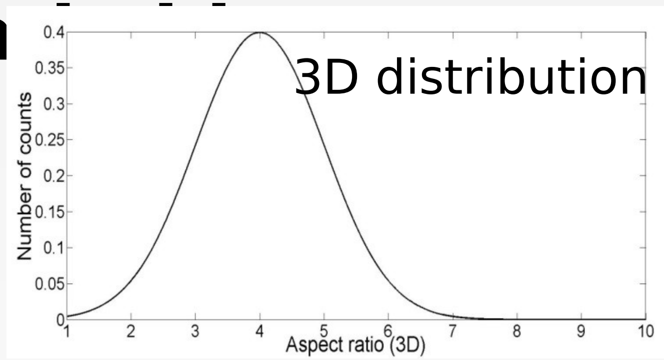
I computed the 2D aspect ratio from this figure and compared to my results =>

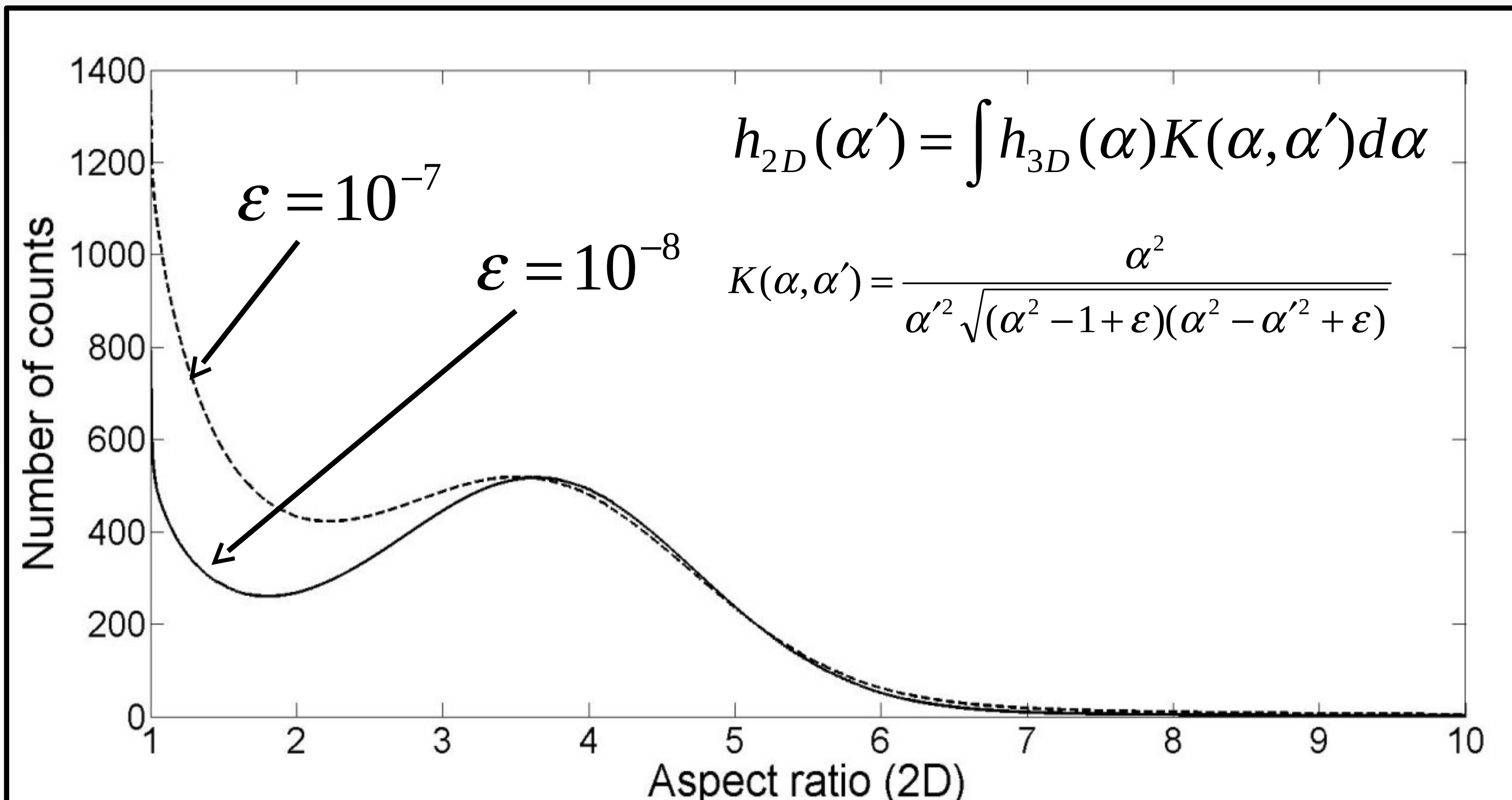
Comparison with Rudge et al., 2008



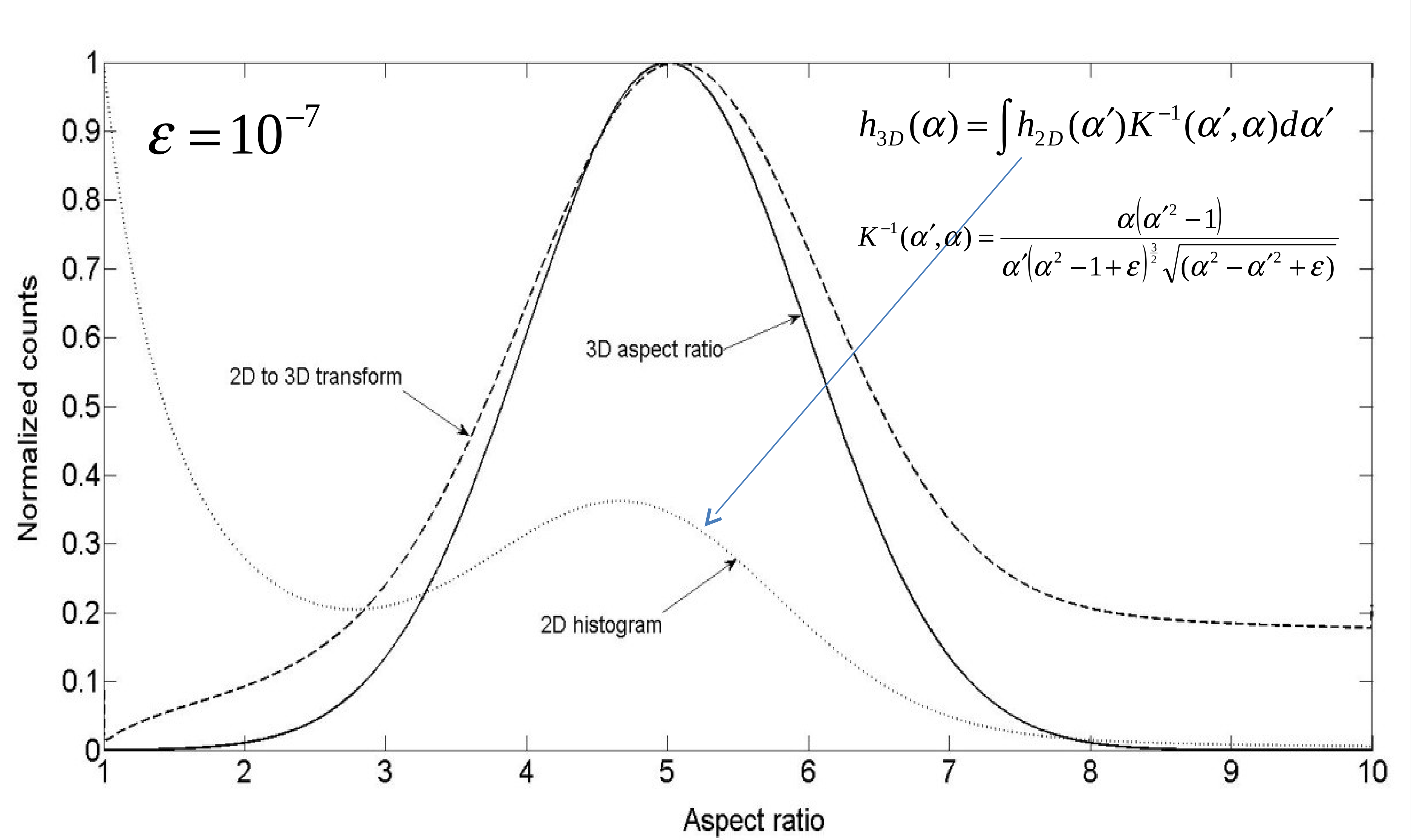
Rudge et al., 2008 use spherocylinders and use a random close packing – followed by one single 2D cut

A normal distribution of 3D aspect ratio –forward modeling of 2D aspect ra



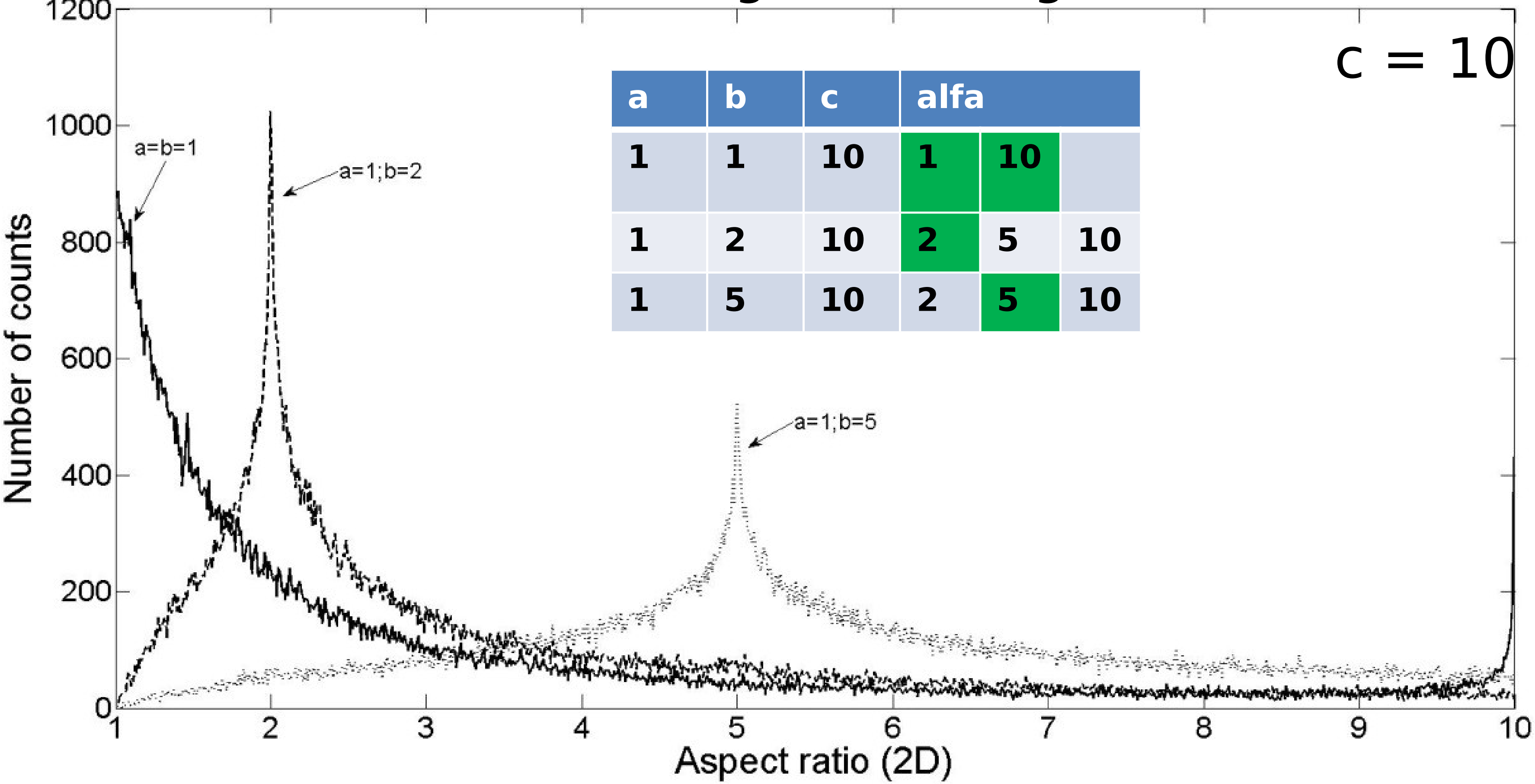


Testing the 2D to 3D transform



far, a=b; what if a , b and c are different?

Stochastic modeling of 2D histograms



a	b	c	alfa		
1	1	10	1	10	
1	2	10	2	5	10
1	5	10	2	5	10

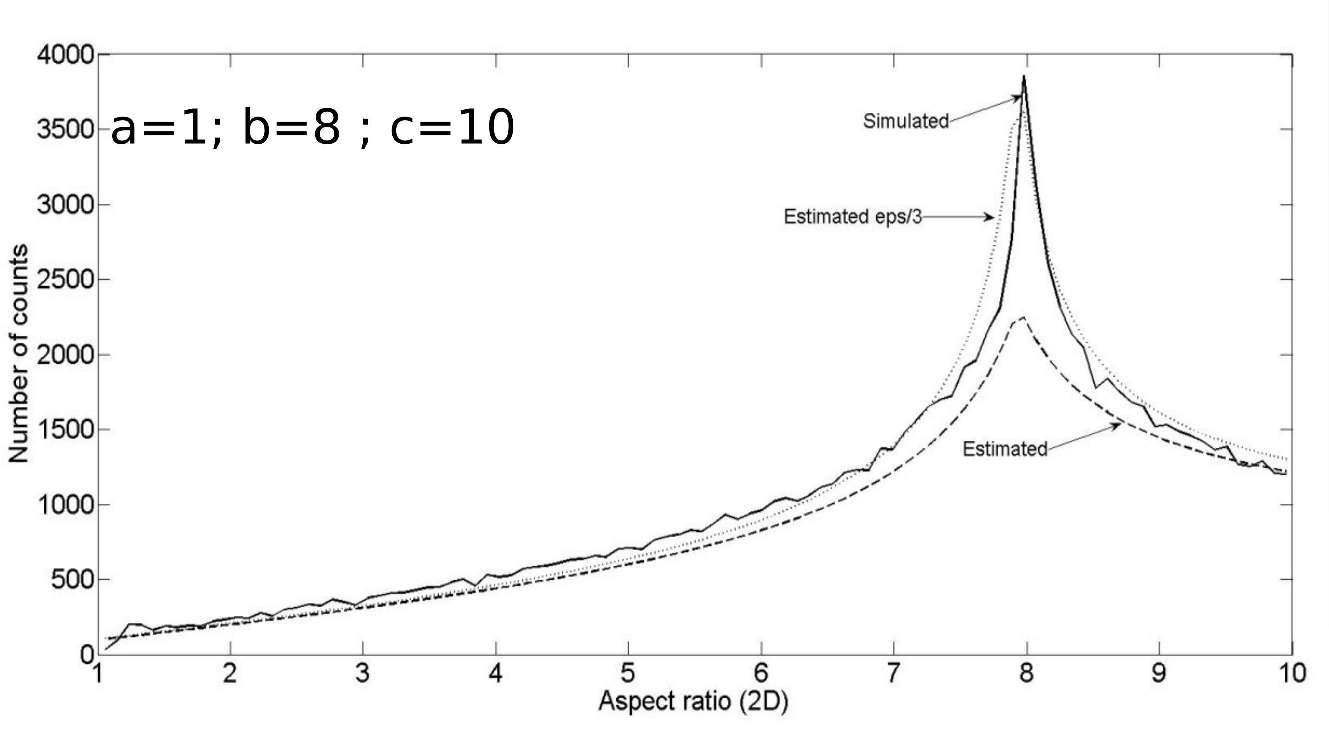
Shorter axes apsect ratios easier to detect

An attempt to find an equation for the general case (a=1, b and c different)

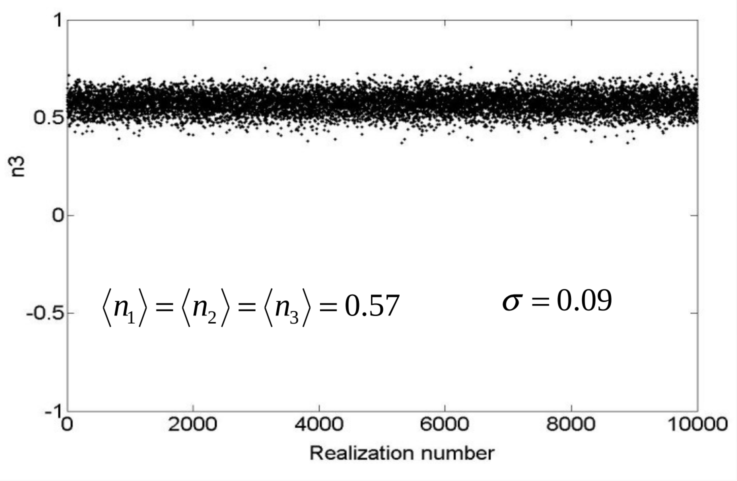
$$h(\alpha') \propto \frac{dn_1}{d\alpha'} = \frac{const \cdot \alpha'}{b\sqrt{b^2 - 1 - \alpha'^2 + \varepsilon_{bc} b^4}}$$

$$\varepsilon_{bc} = \frac{1}{3}\left(\frac{1}{b^2} - \frac{1}{c^2}\right)$$

Note: Approximate equation – far from exact!



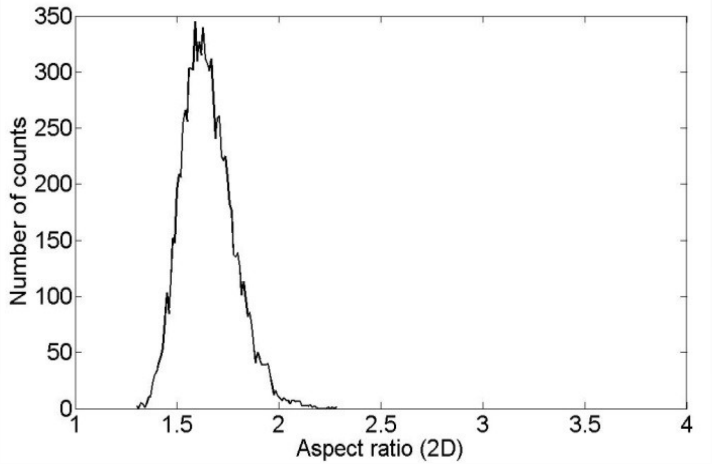
The importance of 3D selection of cuts to reveal the aspect ratio



$$\langle n_3^2 \rangle = \sigma^2 + \mu^2 = 0.09^2 + 0.57^2 = 0.33$$

$$\alpha' = \frac{\alpha}{\sqrt{1 + n_3^2(\alpha^2 - 1)}}$$

$$\alpha' = 4/\sqrt{1 + 0.33 \cdot 15} = 1.63$$



The true value of 4 is not observed on the 2D histogram, but the «wrong» value of 1.6

Coupling aspect ratio (3D) to rock physics

$$\frac{1}{K_\phi} = \frac{4(c/a)(1-\nu^2)}{3\pi K_0(1-2\nu)} = \frac{4\alpha(1-\nu^2)}{3\pi K_0(1-2\nu)}$$

Penny-shaped pore:
c >> a=b

Mavko et al., 2009

ν **Poisson ratio (solid mineral)**

K_0 **Bulk modulus (solid mineral)**

α **3D aspect ratio**

$$\frac{1}{K_{dry}} = \frac{1}{K_0} + \frac{\varphi}{K_\varphi}$$

K_ϕ **is the pore stiffness**

Another application: Anisotropy

Discussion and conclusions

- **Simple equation to derive 2D aspect ratio histograms from 3D ellipsoids is derived**
- **2D to 3D transform of aspect ratio is derived and tested by stochastic simulations**
- **Using thin sections to estimate 3D aspect ratio might be misleading**
- **Useful for rock physics applications?**

Acknowledgments

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