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COMMENTS ON THE STATE VARIABLE METHOD

Technical Note

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SUMMARY

The variable state method is a numerical technique for analyzing, predicting and modeling dynamic systems. It is widely used in electrical engineering applications, especially for the resolution of electrical circuits. This is mainly because it offers advantages such as simplicity and suitability for programming when compared with other traditional methods like differential equations resolution and operational transforms.

In the present document, a brief overview of the method is exposed, explaining its basics, the procedure to follow when solving a problem using it, and a short example as a case of application.

1. INTRODUCTION – APPLICATION SCOPE OF THE STATE VARIABLE METHOD

In order to analyze the dynamic behavior of a physical system, traditionally two well known methods have been employed: solving of differential equations, and solving of operational transforms.

The differential equations approach entrains the physical laws of the system, it is solved in the time-space domain and can be easily extended to non linear or time varying systems. However, the solving methods sometimes comprise very heavy calculus, and a specific number of initial conditions are required depending on the order of the differential equation.

The operational transforms, such like the Laplace transform avoid working with complicated high order derivatives using functions in the complex domain, and do not require initial conditions. However, they are sometimes difficult to implement for computational purposes and the inversion procedure can be challenging for particular cases.

The state variable method constitutes an alternative to analyze the dynamic behavior of a system that combines the advantages of both methods presented before. It is based in the resolution of first order differential equations, therefore it maintains the basic physics of the problem. Initial values are required at a certain point of time and location, but higher order initial conditions are not needed and it is convenient for computer implementation.

2. METHOD FUNDAMENTALS

Consider a continuous in time dynamic system which has several inputs defined by the vector u with a dimension of "k", and several outputs, defined by the vector Y of dimension "N" (fig.1.). The state of the system is described by a set of "n" dynamically independent variables called state variables that constitute the vector X.

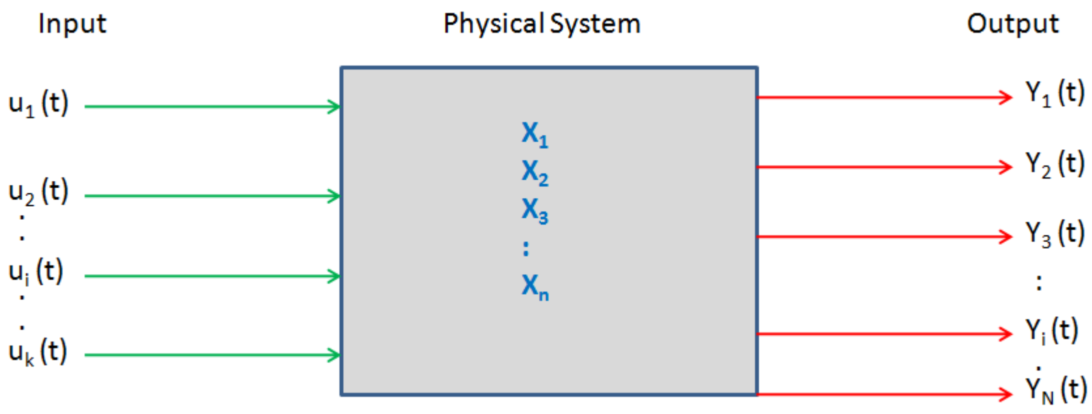

Fig.1. Block diagram of a physical system

The input represents the variables monitored, controlled, known. The state variables represent a set of equations that dictate an output once an input is given, conserving the physical laws applicable to the specific case. The output represents the variables which its prediction is desired.

The state information is necessary to solve the problem because it summarizes the essential information about the past of the system to predict the future of the system. Therefore in order to determine the dynamic behavior of the system, first it is important to solve the state of the system, which is represented by the following system of equations:

$$\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_i \\ \vdots \\ \dot{X}_n \end{Bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,i} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,i} & \cdots & A_{2,n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{i,1} & A_{i,2} & \cdots & A_{i,i} & \cdots & A_{i,n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,3} & \cdots & A_{n,n} \end{bmatrix} \cdot \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{Bmatrix} + \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,i} & \cdots & B_{1,k} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,i} & \cdots & B_{2,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ B_{i,1} & B_{i,2} & \cdots & B_{i,i} & \cdots & B_{i,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & B_{n,3} & \cdots & B_{n,k} \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_k \end{Bmatrix}$$

....(Eq. 1)

This system of equations is constituted by linear first order coupled derivative equations that can be solved for all the times "t" if the initial values for the state variables are known.

Once the state of the system is solved for the desired time frame, the output of the system can be calculated from the following algebraic equation system:

$$\begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_i \\ \vdots \\ Y_N \end{Bmatrix} = \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,i} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,i} & \cdots & C_{2,n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{i,1} & C_{i,2} & \cdots & C_{i,i} & \cdots & C_{i,n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{N,1} & C_{N,2} & \cdots & C_{N,3} & \cdots & C_{N,n} \end{bmatrix} \cdot \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{Bmatrix} + \begin{bmatrix} D_{1,1} & D_{1,2} & \cdots & D_{1,i} & \cdots & D_{1,k} \\ D_{2,1} & D_{2,2} & \cdots & D_{2,i} & \cdots & D_{2,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ D_{i,1} & D_{i,2} & \cdots & D_{i,i} & \cdots & D_{i,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ D_{N,1} & D_{N,2} & \cdots & D_{N,3} & \cdots & D_{N,k} \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_k \end{Bmatrix}$$

....(Eq. 2)

The two equations can be written in a more compact manner

$$\dot{X} = [A]_{n\,x\,n} \cdot X + [B]_{n\,x\,k} \cdot u \quad \text{(Eq. 3)}$$

$$Y = [C]_{N\,x\,n} \cdot X + [D]_{N\,x\,k} \cdot u \quad \text{(Eq. 4)}$$

It is important to note that the matrixes A, B, C, D are constant in time.

The basic solving procedure in the state variable method is to calculate the state variables behavior as a function of time and then to use that information to calculate the output variables trough a purely algebraic process.

Some of the advantages of this method are the following:

-Abundant literature in mathematics for solutions of coupled first order differential equations.

-Easy programming on computers

-Leads to generalizations for nonlinear/time varying systems

-Provides insights of the system behavior

Choosing the state variables

The state of a physical system is defined as the information about a system at a point of time that needs to be known for finding the output originated by an excitation at that exact point of time. In general their number must be chosen following the next guidelines:

-Number of independent initial conditions that can be arbitrarily prescribed

-Degree of the characteristic equation

-Number of natural frequencies in the system

3. APPLICATION EXAMPLE: ELECTRICAL CIRCUIT

In the following section the application of the state variable method is illustrated solving an electrical circuit example. Consider the RLC circuit presented in fig. 2. Initially (t = 0 s) it is under the influence of a source of magnitude "e" and the inductor current (i_L) and the capacitor voltage (V_C) are known. It is necessary to find out the voltage difference across the resistance R1 (V_{R1}), and the current passing through resistance R2 (i_{R2}) for a time t= 2 s.

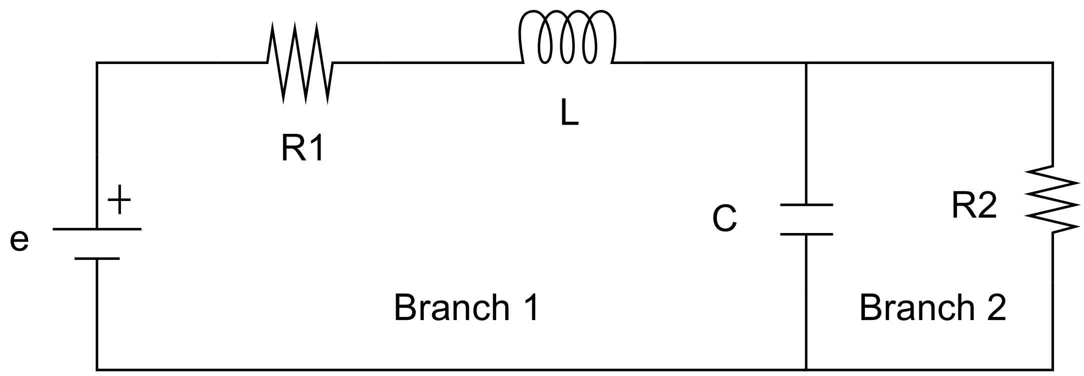

Fig.2. Electrical Network

For the applications of the state variable method, V_{R1} and i_{R2} will be defined as the desired output of the system, and "e" as the excitation (input) on the system. The state variables methods as was mentioned in the previous sections must contain information of the system at a certain point of time. For the particular case there are only two initial conditions, so the variables i_L and V_C are chosen as state variables.

The next step is to find the relationship between the input variables, the state variables and the output variables in the following form (eq. 5 and 6):

$$\dot{X} = A \cdot X + B \cdot u \qquad \text{(Eq. 5)}$$

$$Y = C \cdot X + D \cdot u \qquad \text{(Eq. 6)}$$

When substituting for the particular case:

$$\begin{Bmatrix} \dfrac{dV_C}{dt} \\ \dfrac{di_L}{dt} \end{Bmatrix} = A \cdot \begin{Bmatrix} V_C \\ i_L \end{Bmatrix} + B \cdot e \qquad \text{(Eq.7)}$$

$$\begin{Bmatrix} V_{R1} \\ i_{R2} \end{Bmatrix} = C \cdot \begin{Bmatrix} V_C \\ i_L \end{Bmatrix} + D \cdot e \qquad \text{(Eq.8)}$$

In order to obtain these equations, it is necessary to write the Kirchhoff voltage law on the branch 1 and 2, trying to express everything in terms of the input, output and state variables:

Kirchhoff voltage law for branch 1:

$$e = i_L \cdot R1 + L \cdot \frac{di_L}{dt} + V_C \qquad \text{(Eq. 9)}$$

Kirchhoff voltage law for branch 2:

$$V_C = \left(i_L - C \cdot \frac{dV_C}{dt} \right) \cdot R2 \qquad \text{(Eq. 10)}$$

Algebraic equations for deriving V_{R1} and i_{R2}:

$$\begin{aligned} V_{R1} &= i_L \cdot R1 \\ i_{R2} &= \frac{V_C}{R2} \end{aligned} \qquad \text{(Eq. 11)}$$

The state variables equations are the following:

$$\begin{Bmatrix} \dfrac{dV_C}{dt} \\ \dfrac{di_L}{dt} \end{Bmatrix} = \begin{bmatrix} -\dfrac{1}{C \cdot R2} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R1}{L} \end{bmatrix} \cdot \begin{Bmatrix} V_C \\ i_L \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dfrac{1}{L} \end{Bmatrix} \cdot e \qquad \text{(Eq.12)}$$

$$\begin{Bmatrix} V_{R1} \\ i_{R2} \end{Bmatrix} = \begin{bmatrix} 0 & R1 \\ \dfrac{1}{R2} & 0 \end{bmatrix} \cdot \cdot \begin{Bmatrix} V_C \\ i_L \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \cdot e \qquad \text{(Eq.13)}$$

The final procedure to obtain the solution for any time is to solve equation 12 to obtain V_C and i_L for the given time (t = 2 s) and then substitute in eq. 13 to find the two required outputs.

4. APPLICATION EXAMPLE: HYDRAULIC SYSTEM

In the following section the application of the state variable method is illustrated solving the hydraulic system shown in Fig 3. It consists of three tanks, with two fluid inlets (q_1, q_2) and one fluid outlet (q_3) that change with time. For simplicity of the modeling, the flow through restrictions is assumed to follow the following linear relationship:

$$q(t) = \frac{\Delta h(t)}{R} \quad \text{(Eq. 14)}$$

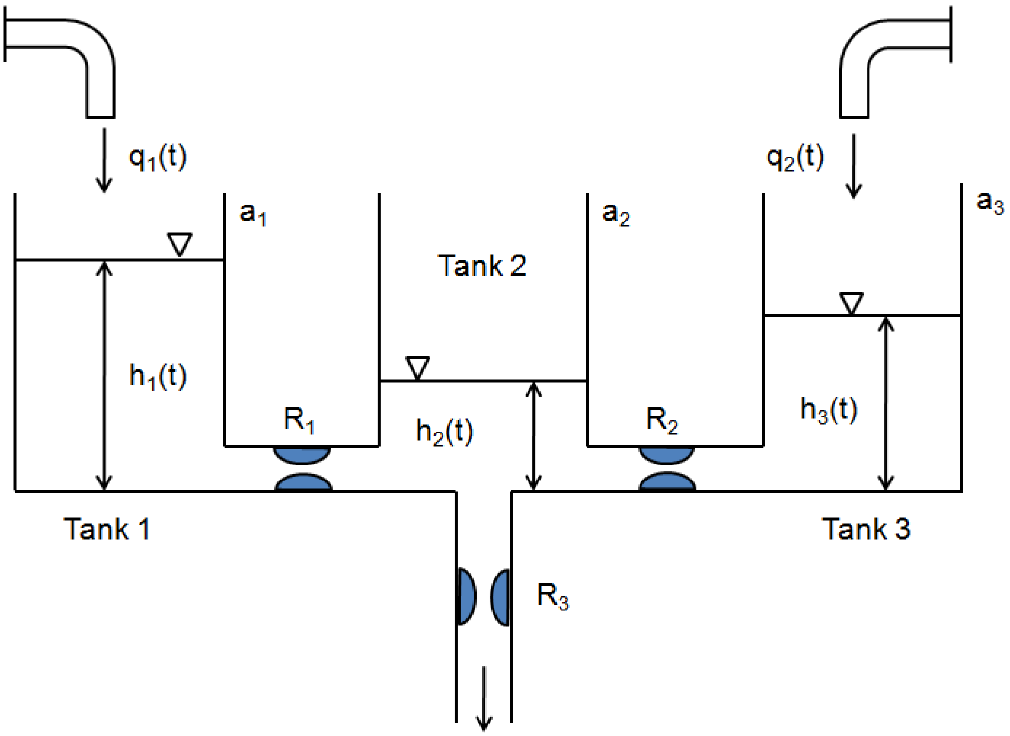

Fig.3. Multi-tank flow system

It is required to monitor at all times the height of the three tanks (h_1, h_2, h_3). In consequence, for the applications of the state variable method, they will be defined as the “desired” output of the system, and “q_1 ,q_2” as the excitation (input) on the system. In order to obtain the state variables and the matrixes of the system A, B, C and D a different approach from the previous case will be followed: the continuity equations for each one of the tanks are written:

$$a_1 \frac{dh_1(t)}{dt} = q_1(t) - \frac{1}{R_1}\left(h_1(t) - h_2(t)\right) \quad \text{(Eq. 15)}$$

$$a_2 \frac{dh_2(t)}{dt} = \frac{1}{R_1}\left(h_1(t) - h_2(t)\right) - \frac{1}{R_2}\left(h_2(t) - h_3(t)\right) - \frac{1}{R_3} h_2(t) \quad \text{(Eq. 16)}$$

$$a_3 \frac{dh_3(t)}{dt} = \frac{1}{R_2}\left(h_2(t) - h_3(t)\right) + q_2(t) \quad \text{(Eq. 17)}$$

Where a_1, a_2, a_3 are the transversal areas of the respective tanks.

From these equations it is a straightforward procedure to obtain the state variable method equations, namely:

$$\dot{X} = A \cdot X + B \cdot u \qquad \text{(Eq. 18)}$$

$$Y = C \cdot X + D \cdot u \qquad \text{(Eq. 19)}$$

Rearranging eq. 15, 16, 17:

$$\dot{h}_1(t) = -\frac{1}{R_1 \cdot a_1}\left(h_1(t) - h_2(t)\right) + \frac{q_1(t)}{a_1} \qquad \text{(Eq. 20)}$$

$$\dot{h}_2(t) = \frac{h_1(t)}{R_1 \cdot a_2} - \frac{h_2(t) \cdot \left(\cdot R_2 \cdot R_3 + \cdot R_1 \cdot R_3 + R_1 \cdot R_2\right)}{R_1 \cdot R_2 \cdot R_3 \cdot a_2} + \frac{h_3(t)}{R_2 \cdot a_2} \qquad \text{(Eq. 21)}$$

$$\dot{h}_3(t) = \frac{1}{R_2 \cdot a_3}\left(h_2(t) - h_3(t)\right) - \frac{q_2(t)}{a_3} \qquad \text{(Eq. 22)}$$

The state variables equations are the following:

$$\begin{Bmatrix} \dfrac{dh_1(t)}{dt} \\ \dfrac{dh_2(t)}{dt} \\ \dfrac{dh_3(t)}{dt} \end{Bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1 \cdot a_1} & \dfrac{1}{R_1 \cdot a_1} & 0 \\ \dfrac{1}{R_1 \cdot a_2} & -\dfrac{\left(R_2 \cdot R_3 + \cdot R_1 \cdot R_3 + R_1 \cdot R_2\right)}{R_1 \cdot R_2 \cdot R_3 \cdot a_2} & \dfrac{1}{R_2 \cdot a_2} \\ 0 & \dfrac{1}{R_2 \cdot a_3} & -\dfrac{1}{R_2 \cdot a_3} \end{bmatrix} \cdot \begin{Bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{Bmatrix} + \qquad \text{(Eq. 23)}$$

$$\begin{bmatrix} \dfrac{1}{a_1} & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{a_3} \end{bmatrix} \cdot \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$$

In this particular case it is possible to see that the state variables are the same as the output variables, so the algebraic equations system is the following:

$$\begin{Bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \cdot \begin{Bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \cdot \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \qquad \text{(Eq. 24)}$$

5. APPLICATION EXAMPLE: MUD SYSTEM FOR DRILLING OIL WELLS

For controlling the inflow of formation fluids into the well bore while drilling an oil well, usually a hydraulic system like the one presented in fig 4 is used. It consists of two tanks, with one fluid inlet (q_1) and two fluid outlets (q_2 ,q_3) that change with time. Tank 1 represents the mud column inside the drill string and Tank 2 represents the mud column inside the well bore (annular space). q_1 represents the mud flow being injected to the system, q_2 represents the mud losses into the formation, and q_3 represents the mud flow being drained from the well bore.

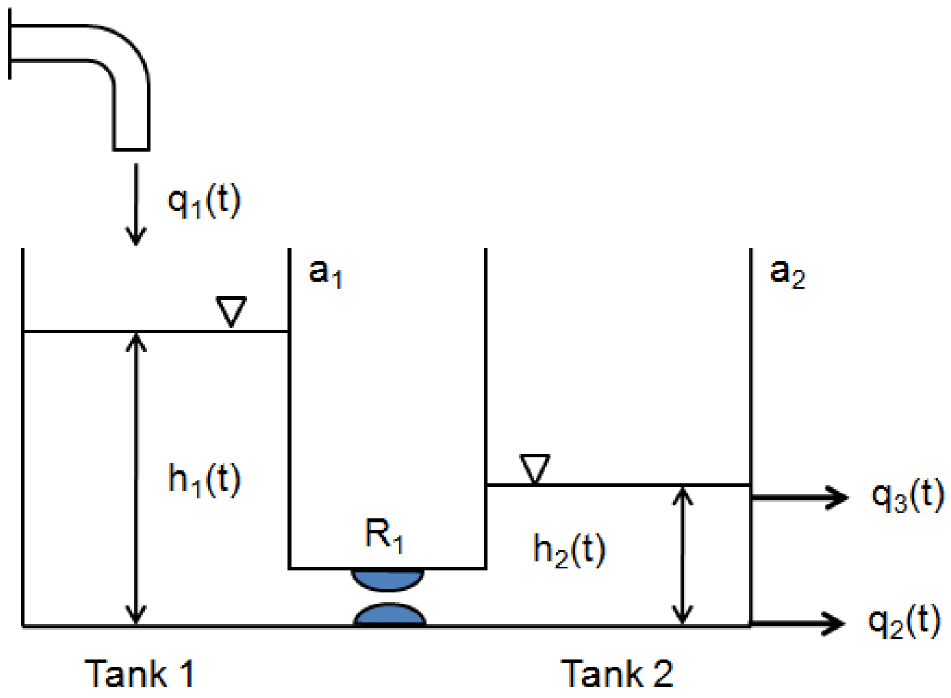

Fig.4. Mud height control flow system

For simplicity of the modeling, the flow through restrictions is assumed to follow the following linear relationship:

$$q(t) = \frac{\Delta h(t)}{R_1} \qquad \text{(Eq. 25)}$$

It is important to state that this equation neglects the kinematic and viscous losses effect in each mud column.

And the mud losses into the formation are assumed to follow a linear relationship:

$$q_2(t) = \frac{h_2(t) - C}{R_2} \qquad \text{(Eq. 26)}$$

As for the previous problem, it is required to monitor at all times the height of the two tanks (h_1, h_2). In consequence, for the applications of the state variable method, they will be defined as the "desired" output of the system, and "q_1 ,q_3" as the excitation (input) on the system. In order to obtain the state variables and the matrixes of the system A, B, C and D the same approach from the previous case will be followed: the continuity equations for each one of the tanks are written:

$$a_1 \frac{dh_1(t)}{dt} = q_1(t) - \frac{1}{R_1}\left(h_1(t) - h_2(t)\right) \qquad \text{(Eq. 27)}$$

$$a_2 \frac{dh_2(t)}{dt} = \frac{1}{R_1}\left(h_1(t) - h_2(t)\right) - \frac{\left(h_2(t) - C\right)}{R_2} - q_3(t) \qquad \text{(Eq. 28)}$$

Where a_1, a_2 are the transversal areas of the respective tanks.

From these equations it is a straightforward procedure to obtain the state variable method equations, namely:

$$\dot{X} = A \cdot X + B \cdot u \qquad \text{(Eq. 29)}$$

$$Y = C \cdot X + D \cdot u \qquad \text{(Eq. 30)}$$

Rearranging eq. 15, 16, 17:

$$\dot{h}_1(t) = -\frac{1}{R_1 \cdot a_1}\left(h_1(t) - h_2(t)\right) + \frac{q_1(t)}{a_1} \qquad \text{(Eq. 31)}$$

$$\dot{h}_2(t) = \frac{h_1(t)}{R_1 \cdot a_2} - \frac{h_2(t) \cdot (R_1 + R_2)}{R_2 \cdot a_2 \cdot R_1} + \frac{q_3(t)}{a_2} + \frac{C}{R_2} \qquad \text{(Eq. 32)}$$

The state variables equations are the following:

$$\begin{Bmatrix} \dfrac{dh_1(t)}{dt} \\ \dfrac{dh_2(t)}{dt} \end{Bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1 \cdot a_1} & \dfrac{1}{R_1 \cdot a_1} \\ \dfrac{1}{R_1 \cdot a_2} & -\dfrac{(R_1 + R_2)}{R_2 \cdot a_2 \cdot R_1} \end{bmatrix} \begin{Bmatrix} h_1(t) \\ h_2(t) \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{a_1} & 0 \\ 0 & -\dfrac{1}{a_2} \end{bmatrix} \cdot \begin{Bmatrix} q_1(t) \\ q_3(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dfrac{C}{R_2} \end{Bmatrix} \qquad \text{(Eq. 33)}$$

In this particular case, again it is possible to see that the state variables are the same as the output variables, so the algebraic equations system is the following:

$$\begin{Bmatrix} h_1(t) \\ h_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \cdot \begin{Bmatrix} h_1(t) \\ h_2(t) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \cdot \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \qquad \text{(Eq. 24)}$$

5.1. Resolution for a particular case:

To illustrate the resolution of the system, the following data will be used:

Physical properties:

$\rho_{mud} = 2500\, kg/m^3$

$g = 9.81\, m/s^2$

Geometrical and well data:

Hole size: 6.5 in

Pipe OD: 4 in

Pipe ID: 3.34 in

It is necessary to calculate the areas a_1 and a_2:

$$a_1 = \frac{\pi \cdot \phi_1^{\,2}}{4} = \frac{\pi \cdot \left(3.34\, in \cdot \frac{2.54}{100} \frac{m}{in}\right)^2}{4} = 0.00565263\, m^2$$

$$a_2 = \frac{\pi \cdot \left(\phi_{2OD}^{\,2} - \phi_{2ID}^{\,2}\right)}{4} = \frac{\pi \cdot \left[\left(6.5\, in \cdot \frac{2.54}{100} \frac{m}{in}\right)^2 - \left(4\, in \cdot \frac{2.54}{100} \frac{m}{in}\right)^2\right]}{4} = 0.0133011\, m^2$$

Flow data:

For a first aproximation, the inlet and outlet flows are assumed to be zero:

$$q_1(t) = q_3(t) = 0\ gpm = 0\ \frac{\mathrm{m}^3}{s}$$

Assuming that for a flow of 220 gpm the pressure drop at the drill nozzles is 3933 psi (2.711708e+001 MPa):

$$R_1 = \frac{\Delta h}{q} = \frac{\Delta p}{q \cdot \rho \cdot g} = \frac{2.711708\mathrm{e}+001\,\mathrm{MPa}}{0.01387984\ \frac{\mathrm{m}^3}{s} \cdot 2500\ kg/m^3 \cdot 9.81 \cdot m/s^2} = 79661.6752\ m/(m^3/s)$$

The formation constants will be imposed arbitrarily so that the leakage to the formation is minimum:

C = 1 m

R_2 = 2E15 m/m^3/s

Initial conditions

At t = 0 s:

$$h_1(0) = 2000\,\mathrm{m}$$

$$h_2(0) = 1900\,\mathrm{m}$$

The system of equations to solve, once substituted the physical variables, for each time is the following:

$$\begin{Bmatrix} \dfrac{dh_1(t)}{dt} \\ \dfrac{dh_2(t)}{dt} \end{Bmatrix} = \begin{bmatrix} -2.221E-3 & 2.221E-3 \\ 9.438E-4 & -9.438E-4 \end{bmatrix} \begin{Bmatrix} h_1(t) \\ h_2(t) \end{Bmatrix} + \begin{bmatrix} 176.909 & 0 \\ 0 & -75.182 \end{bmatrix} \cdot \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1E-15 \end{Bmatrix}$$

The solving procedure is the following: The equation system is solved for each time step, and the heights for the next time step are calculated with:

$$\frac{h_{NEW} - h_{OLD}}{\Delta t} = \frac{dh(t)}{dt}$$

For the particular case, a time step of 2 s was used.

The solution is shown in fig 6:

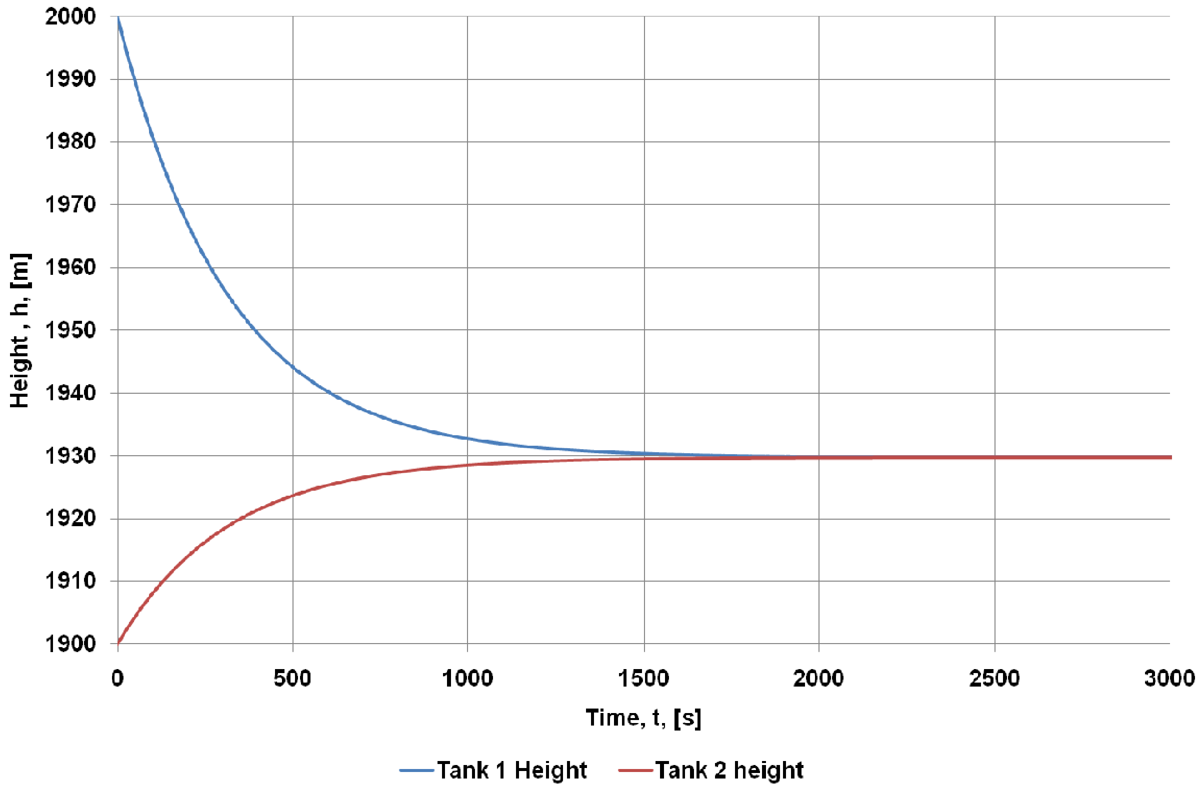

Fig.6. Evolution of the height with time for two tanks

The solution indicates that the system reaches an equilibrium after 2000 s, when the two tanks reach the same height: 1930 m

6. REFERENCES

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Appendix I: Calculation of the resistance of the formation

Assuming that the following linear relationship is followed:

$$q_{form} = \frac{\Delta P}{R_{Form}}$$

And that the pressure difference can be expressed as:

$$\Delta P = \left(P_{atm} + \rho \cdot g \cdot h\right) - P_{formation}$$

Substituting:

$$q_{form} = \frac{\left(P_{atm} + \rho \cdot g \cdot h\right) - P_{formation}}{R_{Form}}$$

Rearranging:

$$q_{form} = \frac{h - \dfrac{\left(P_{formation} - P_{atm}\right)}{\rho \cdot g}}{\dfrac{R_{Form}}{\rho \cdot g}}$$

$$q_{form} = \frac{h - C}{R'_{Form}}$$