NTNU
Norwegian University of
Science and Technology

Department of Geoscience and Petroleum

Solution
Final Exam TPG4160 Reservoir Simulation

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Question 1 (12 points)

Use Taylor series to derive the following approximations (include error terms):

a) Backward approximation $\frac{\partial P}{\partial x}$ (constant Δx)

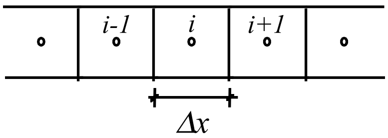
b) Central approximation of $\frac{\partial^2 P}{\partial x^2}$ (constant Δx)

c) Central approximation of $\frac{\partial^2 P}{\partial x^2}$ (variable Δx)

d) Central approximation of $\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right)$ (constant Δx)

Solution

a) Taylor expansion applied to the following grid:



$$P(x-\Delta x,t)=P(x,t)+\frac{(-\Delta x)}{1!}P'(x,t)+\frac{(-\Delta x)^2}{2!}P''(x,t)+\frac{(-\Delta x)^3}{3!}P'''(x,t)+.....$$

Solving for the derivative:

$$P'(x,t)=\frac{P(x,t)-P(x-\Delta x)}{\Delta x}+O(\Delta x)$$

b) Forward and backward expansions of pressure:

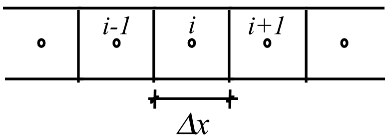
$$P(x+\Delta x,t)=P(x,t)+\frac{\Delta x}{1!}P'(x,t)+\frac{(\Delta x)^2}{2!}P''(x,t)+\frac{(\Delta x)^3}{3!}P'''(x,t)+.....$$

$$P(x-\Delta x,t)=P(x,t)+\frac{(-\Delta x)}{1!}P'(x,t)+\frac{(-\Delta x)^2}{2!}P''(x,t)+\frac{(-\Delta x)^3}{3!}P'''(x,t)+.....$$

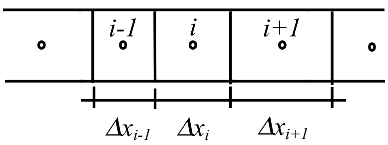
to yield (using sub- and superscripts)

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_i^t=\frac{P_{i+1}^t-2P_i^t+P_{i-1}^t}{(\Delta x)^2}+O(\Delta x^2),$$

which applies to the following grid system:



c) Variable grid size system:



Taylor expansions now become (dropping the time index for convenience):

$$P_{i+1}=P_i+\frac{(\Delta x_i+\Delta x_{i+1})/2}{1!}P_i'+\frac{[(\Delta x_i+\Delta x_{i+1})/2]^2}{2!}P_i''+\frac{[(\Delta x_i+\Delta x_{i+1})/2]^3}{3!}P_i'''.....$$

$$P_{i-1} = P_i + \frac{-(\Delta x_i + \Delta x_{i-1})/2}{1!} P_i' + \frac{\left[-(\Delta x_i + \Delta x_{i-1})/2\right]^2}{2!} P_i'' + \frac{\left[-(\Delta x_i + \Delta x_{i-1})/2\right]^3}{3!} P_i''' \ldots.$$

to yield

$$P_i'' = 4\frac{2\left(\dfrac{\Delta x_i + \Delta x_{i-1}}{2\Delta x_i + \Delta x_{i+1} + \Delta x_{i-1}}\right)P_{i+1} - 2P_i + 2\left(\dfrac{\Delta x_i + \Delta x_{i+1}}{2\Delta x_i + \Delta x_{i+1} + \Delta x_{i-1}}\right)P_{i-1}}{(\Delta x_i + \Delta x_{i+1})(\Delta x_i + \Delta x_{i-1})} + O(\Delta x).$$

d) First, let's rewrite $\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right)$ as $\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]$, where $f(x)$ includes permeability, mobility and flow area. Then we derive a central approximation for the first derivative: and apply it twice to this flow term.

$$\left[f(x)\frac{\partial P}{\partial x}\right]_{i+1/2} = \left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{\Delta x_i/2}{1!}\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{(\Delta x_i/2)^2}{1!}\frac{\partial^2}{\partial x^2}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \ldots.$$

and

$$\left[f(x)\frac{\partial P}{\partial x}\right]_{i-1/2} = \left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{-\Delta x_i/2}{1!}\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{(-\Delta x_i/2)^2}{1!}\frac{\partial^2}{\partial x^2}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \ldots.$$

which yields

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i = \frac{\left[f(x)\frac{\partial P}{\partial x}\right]_{i+1/2} - \left[f(x)\frac{\partial P}{\partial x}\right]_{i-1/2}}{\Delta x_i} + O(\Delta x^2).$$

Similarly, we may obtain the following expressions:

$$\left(\frac{\partial P}{\partial x}\right)_{i+1/2} = \frac{P_{i+1} - P_i}{(\Delta x_i + \Delta x_{i+1})/2} + O(\Delta x)$$

and

$$\left(\frac{\partial P}{\partial x}\right)_{i-1/2} = \frac{P_i - P_{i-1}}{(\Delta x_i + \Delta x_{i-1})/2} + O(\Delta x).$$

By inserting these expressions into the previous equation, we get the following approximation for the flow term:

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i = \frac{2f(x)_{i+1/2}\dfrac{(P_{i+1} - P_i)}{(\Delta x_{i+1} + \Delta x_i)} - 2f(x)_{i-1/2}\dfrac{(P_i - P_{i-1})}{(\Delta x_i + \Delta x_{i-1})}}{\Delta x_i} + O(\Delta x).$$

<u>Question 2</u> (10 points)

Use Taylor series and show all steps in the discretization of the following two equations (**you may refer to the derivations in Question 1**):

a) $$\frac{\partial^2 P}{\partial x^2} = \frac{\phi\mu c}{k}\frac{\partial P}{\partial t}$$

b) $$\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right) = \phi\left(\frac{c_r}{B} + \frac{d(1/B)}{dP}\right)\frac{\partial P}{\partial t}$$

Solution

a) Right side:

$$P(x,t) = P(x,t+\Delta t) + \frac{-\Delta t}{1!}P'(x,t+\Delta t) + \frac{(-\Delta t)^2}{2!}P''(x,t+\Delta t) + \frac{(-\Delta t)^3}{3!}P'''(x,t+\Delta t) + ...$$

Solving for the time derivative, we get:

$$(\frac{\partial P}{\partial t})_i^{t+\Delta t} = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t).$$

Left side

$$P(x+\Delta x,t+\Delta t) = P(x,t) + \frac{\Delta x}{1!}P'(x,t+\Delta t) + \frac{(\Delta x)^2}{2!}P''(x,t+\Delta t) + \frac{(\Delta x)^3}{3!}P'''(x,t+\Delta t) + ...$$

$$P(x-\Delta x,t+\Delta t) = P(x,t) + \frac{(-\Delta x)}{1!}P'(x,t+\Delta t) + \frac{(-\Delta x)^2}{2!}P''(x,t+\Delta t) + \frac{(-\Delta x)^3}{3!}P'''(x,t+\Delta t) + ...$$

By adding these two expressions, and solving for the second derivative, we get the following approximation:

$$(\frac{\partial^2 P}{\partial x^2})_i^{t+\Delta t} = \frac{P_{i+1}^{t+\Delta t} - 2P_i^{t+\Delta t} + P_{i-1}^{t+\Delta t}}{(\Delta x)^2} + O(\Delta x^2)$$

Substituting into the equation, we get:

$$\frac{P_{i+1}^{t+\Delta t} - 2P_i^{t+\Delta t} + P_{i-1}^{t+\Delta t}}{\Delta x^2} \approx (\frac{\phi\mu c}{k})\frac{P_i^{t+\Delta t} - P_i^t}{\Delta t}$$

b) Right side:

We use the same approximation for the pressure derivative as in a):

$$\left[\phi\left(\frac{c_r}{B} + \frac{d(1/B)}{dP}\right)(\frac{\partial P}{\partial t})\right]_i^{t+\Delta t} \approx \left[\phi\left(\frac{c_r}{B} + \frac{d(1/B)}{dP}\right)\right]_i^{t+\Delta t} \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t}$$

Left side:

$$\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_{i+1/2} = \left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i + \frac{\Delta x/2}{1!}\frac{\partial}{\partial x}\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i + \frac{(\Delta x/2)^2}{2!}\frac{\partial^2}{\partial x^2}\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i + ...$$

$$\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_{i-1/2} = \left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i + \frac{-\Delta x/2}{1!}\frac{\partial}{\partial x}\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i + \frac{(-\Delta x/2)^2}{2!}\frac{\partial^2}{\partial x^2}\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i + ...$$

Combining:

$$\frac{\partial}{\partial x}\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i = \frac{\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_{i+1/2} - \left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_{i-1/2}}{\Delta x} + O(\Delta x^2).$$

Using similar central difference approximations for the two pressure gradients:

$$\left(\frac{\partial P}{\partial x}\right)_{i+1/2} = \frac{P_{i+1} - P_i}{\Delta x} + O(\Delta x)$$

$$\left(\frac{\partial P}{\partial x}\right)_{i-1/2} = \frac{P_i - P_{i-1}}{\Delta x} + O(\Delta x).$$

the expression becomes:

$$\frac{\partial}{\partial x}\left[(\frac{k}{\mu B})\frac{\partial P}{\partial x}\right]_i \approx \frac{\left[(\frac{k}{\mu B})\frac{P_{i+1} - P_i}{\Delta x}\right]_{i+1/2} - \left[(\frac{k}{\mu B})\frac{P_i - P_{i-1}}{\Delta x}\right]_{i-1/2}}{\Delta x}$$

or

$$\frac{\partial}{\partial x}\left[\left(\frac{k}{\mu B}\right)\frac{\partial P}{\partial x}\right]_i \approx \left(\frac{k}{\mu B}\right)_{i+1/2}\frac{P_{i+1}-P_i}{\Delta x^2}-\left(\frac{k}{\mu B}\right)_{i-1/2}\frac{P_i-P_{i-1}}{\Delta x^2}$$

Thus, the difference equation becomes

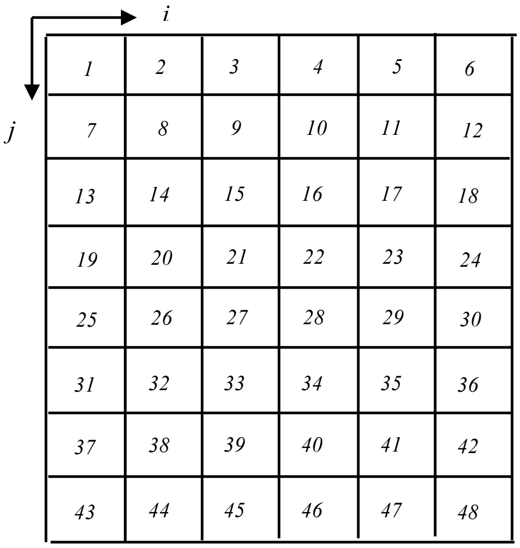
$$\left(\frac{k}{\mu B}\right)_{i+1/2}\frac{P_{i+1}-P_i}{\Delta x^2}-\left(\frac{k}{\mu B}\right)_{i-1/2}\frac{P_i-P_{i-1}}{\Delta x^2} \approx \left[\phi\left(\frac{c_r}{B}+\frac{d(1/B)}{dP}\right)\right]_i^{t+\Delta t}\frac{P_i^{t+\Delta t}-P_i^t}{\Delta t}$$

Question 3 (17 points)
Sketch the coefficient matrix for the grids below. Label diagonals (*a, b, c, d, e, f*). **It is sufficient to label diagonals as straight lines, and to mark individual points with an *x*.**

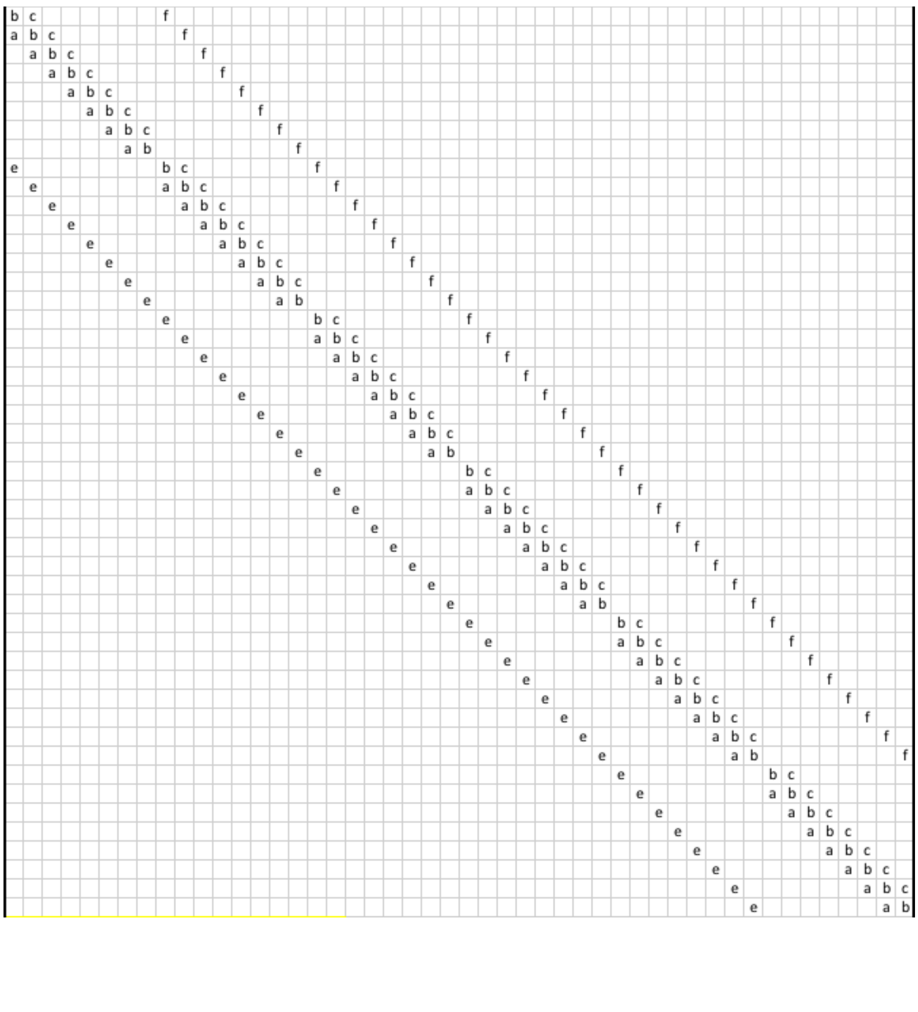
a) For two-dimensional (*x,y*), one phase flow, the pressure equation is:

$$e_{i,j}P_{i,j-1} + a_{i,j}P_{i-1,j} + b_{i,j}P_{i,j} + c_{i,j}P_{i+1,j} + f_{i,j}P_{i,j+1} = d_{i,j}, \qquad i = 1, \dots, N_x\,, j = 1, \dots, N_y$$

applicable to the following grid system, where numbering is in the shortest direction:



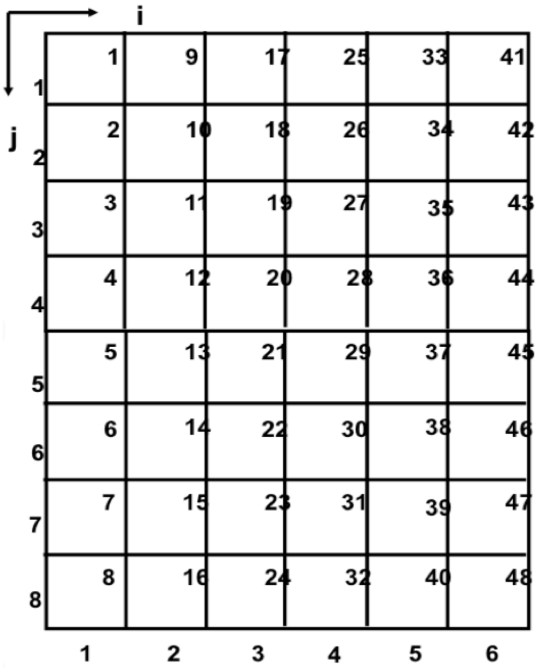
Solution



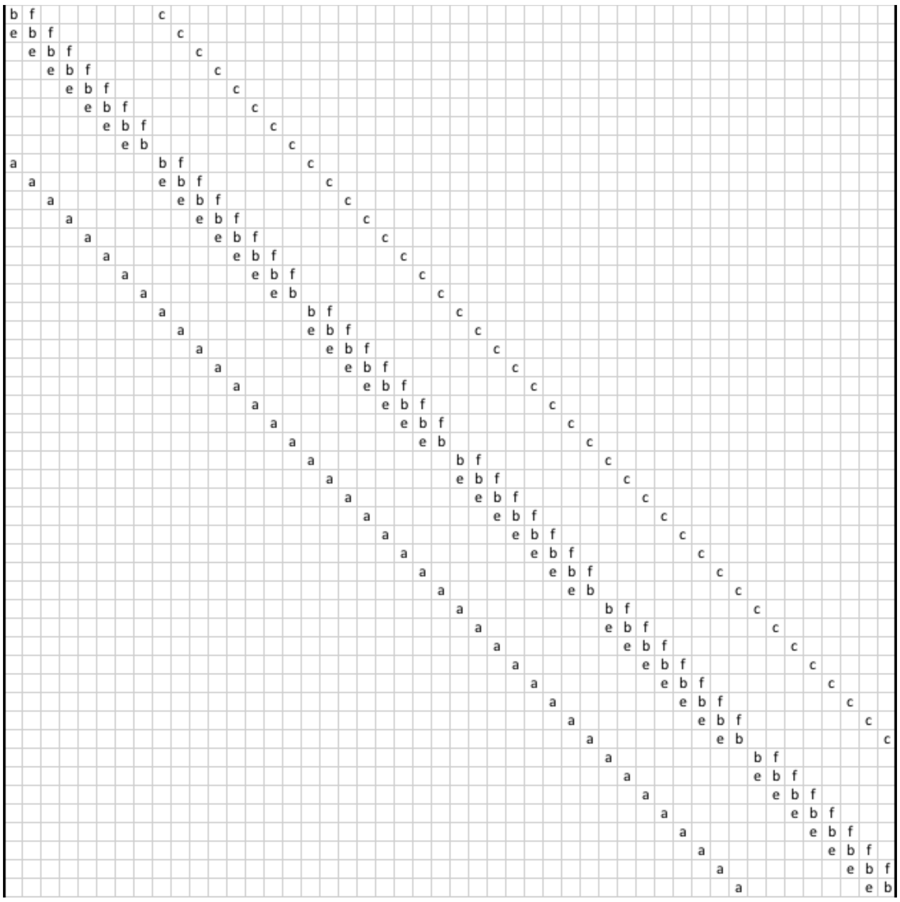
b) For two-dimensional (x,y), one-phase flow, with the pressure equation:

$$e_{i,j}P_{i,j-1} + a_{i,j}P_{i-1,j} + b_{i,j}P_{i,j} + c_{i,j}P_{i+1,j} + f_{i,j}P_{i,j+1} = d_{i,j}, \qquad i = 1, \dots, N_x\,, j = 1, \dots, N_y$$

applicable to the following grid system, where numbering is in the longest direction:



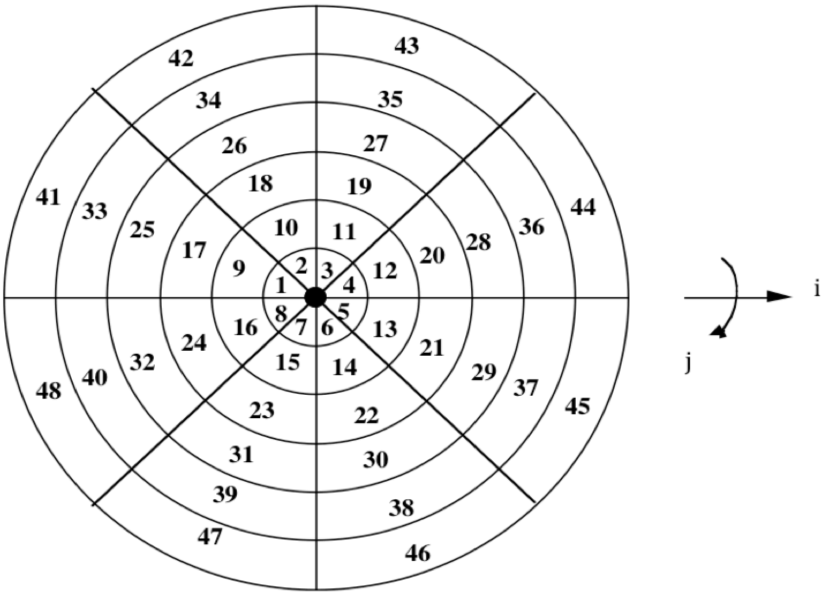
Solution



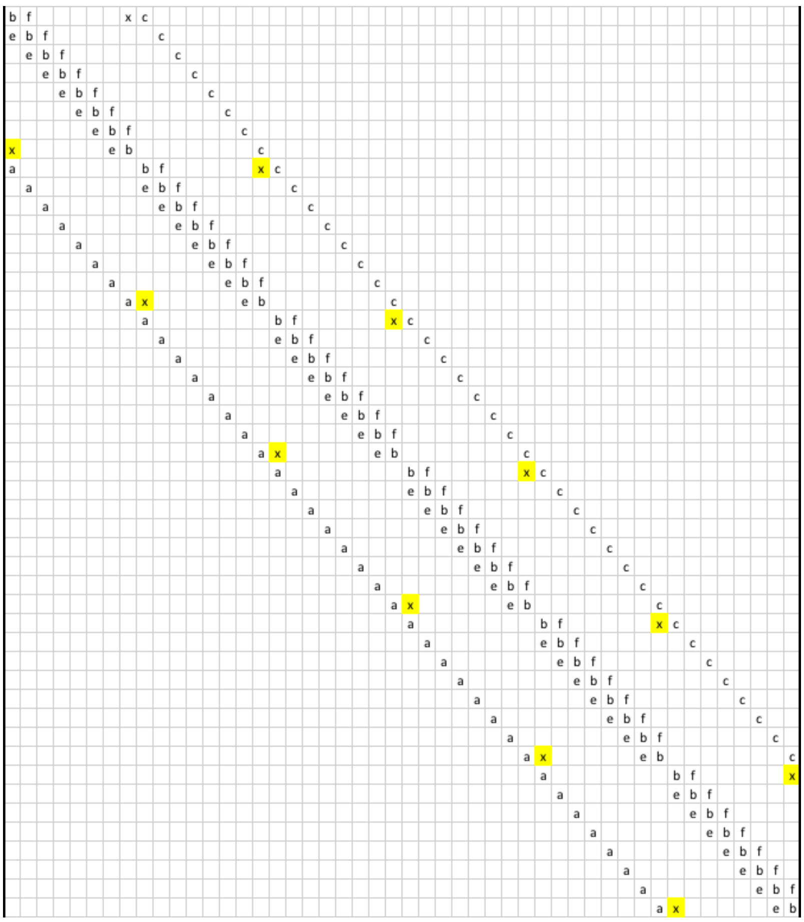
c) For two-dimensional (r, θ), one-phase flow, with the pressure equation:

$$e_{i,j}P_{i,j-1} + a_{i,j}P_{i-1,j} + b_{i,j}P_{i,j} + c_{i,j}P_{i+1,j} + f_{i,j}P_{i,j+1} = d_{i,j}, \qquad i = 1, \dots, N_r\,, j = 1, \dots, N_\theta$$

applicable to the cylindrical grid system below, where numbering is in the longest direction. Note that this grid is similar to the Cartesian grid under b) with one key exception.



Solution

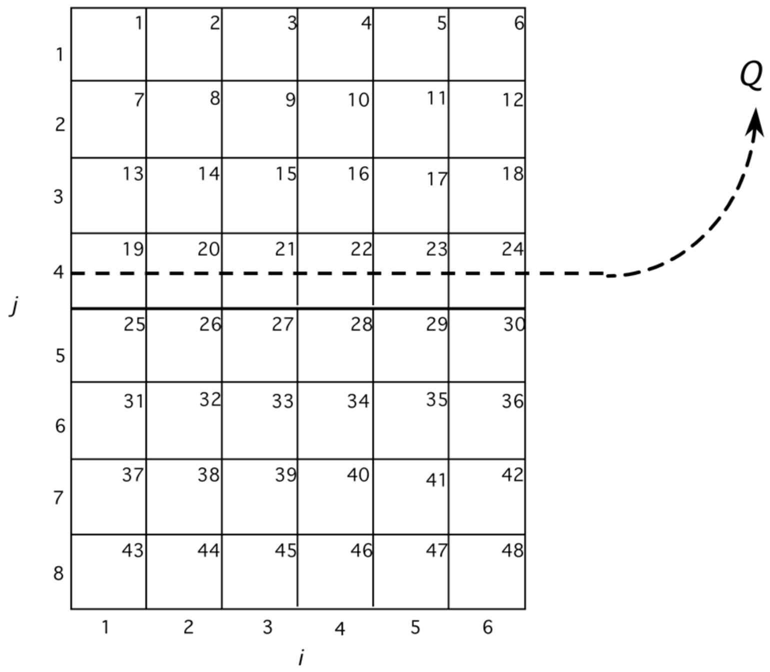


d) Explain the similarities and differences of b) and c)

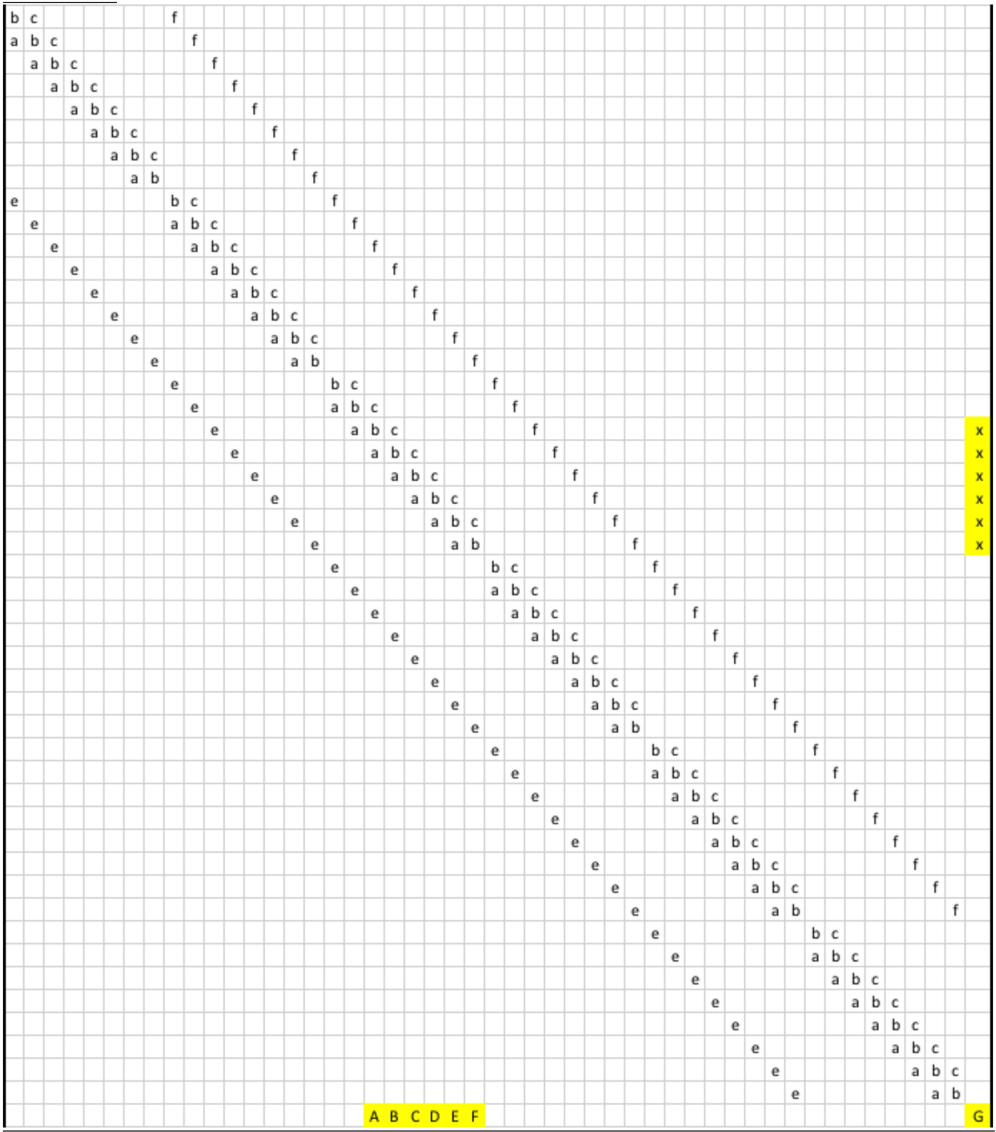
Solution

The additional coefficients for c), labeled in yellow above, are due to the closing of the circle, ie. the additional connections between blocks 1 & 8, 9 & 16, 17 & 24, 25 & 32, 33 & 40, 41 & 48. All the rest of the coefficients have the same structure as in b).

e) The next grid is the same as in a), but now a well producing at a rate Q is perforating blocks 19, 20, 21, 22, 23, 24. Redraw the coefficient matrix of a) and indicate with x's the added coefficients due to the well. Note that:
- Bottom-hole pressure (P_{bh}) is the same for all perforations and is an additional unknown
- The flow rate from a perforated grid block is $q_{i,j} = PI_{i,j}\left(P_{i,j} - P_{bh}\right)$
- Total production rate is $Q = q_{19} + q_{20} + q_{21} + q_{22} + q_{23} + q_{24}$



<u>Solution</u>



All the perforated blocks (19, 20, 21, 22, 23, 24) will be connected through the well, and we introduce one more unknown, the bottom hole pressure, which is represented by the added column at the right side above. Noe-zero coefficients will be in the locations of the perforated blocks. One additional equation is needed, namely the rate equation:

$$Q = q_{19} + q_{20} + q_{21} + q_{22} + q_{23} + q_{24}, \text{ where}$$

$$q_{i,j} = PI_{i,j}\left(P_{i,j} - P_{bh}\right)$$

yielding the coefficients A, B, C, D, E, F, G, as shown in the bottom row.

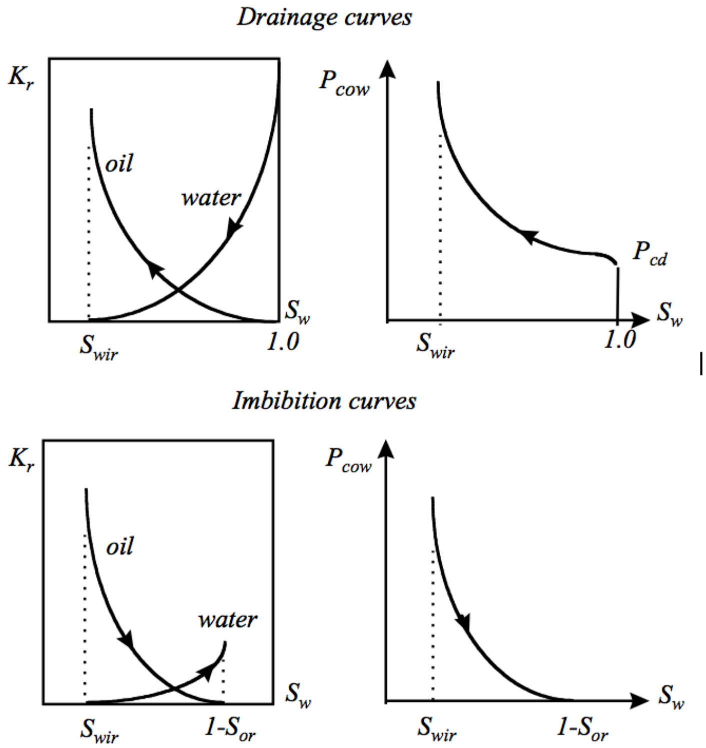
Question 4 (9 points)

For a completely water-wet system, make sketches of saturation functions (including labels for important points/areas)

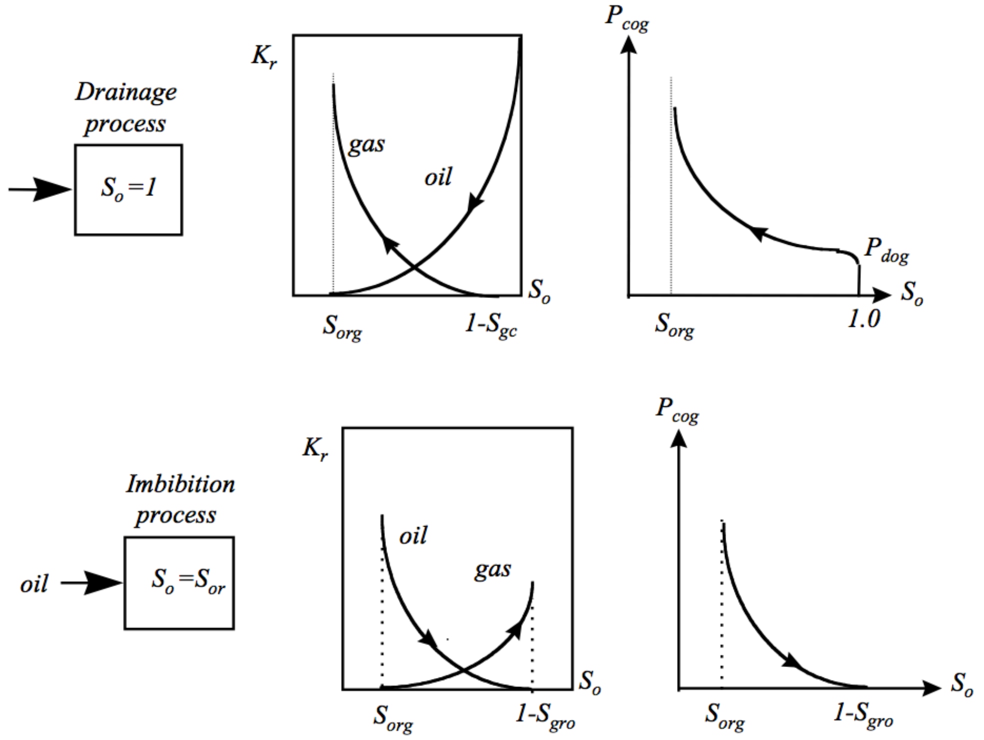
a) Oil-water system: imbibition and drainage k_{rw}, k_{row}, P_{cow} vs. S_w
b) Oil-gas system: imbibition and drainage k_{rg}, k_{rog}, P_{cog} vs. S_g
c) Typical contours of three-phase k_{ro} in a ternary (triangular) diagram (with axes S_o, S_w, S_g)

Solution

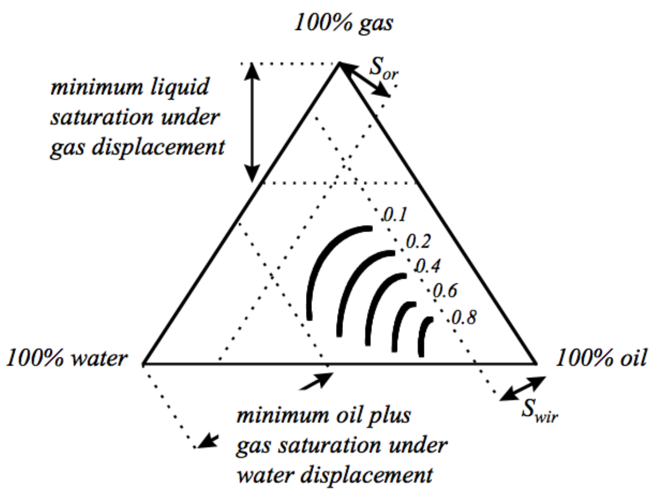
a)



b)


Drainage process
$S_o=1$
K_r
gas
oil
S_o
S_{org}
$1-S_{gc}$
P_{cog}
P_{dog}
S_{org}
1.0
S_o
Imbibition process
oil
$S_o=S_{or}$
K_r
oil
gas
S_o
S_{org}
$1-S_{gro}$
P_{cog}
S_{org}
$1-S_{gro}$
S_o

c)


100% gas
S_{or}
minimum liquid saturation under gas displacement
0.1
0.2
0.4
0.6
0.8
100% water
100% oil
S_{wir}
minimum oil plus gas saturation under water displacement

<u>Question 5</u> *(18 points)*

For a one-dimensional, horizontal, 3-phase oil, water, gas system, the general flow equations are (including well terms):

$$\frac{\partial}{\partial x}\left(\frac{kk_{ro}}{\mu_o B_o}\frac{\partial P_o}{\partial x}\right)-q_o'=\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right),$$

$$\frac{\partial}{\partial x}\left(\frac{kk_{rg}}{\mu_g B_g}\frac{\partial P_g}{\partial x}+R_{so}\frac{kk_{ro}}{\mu_o B_o}\frac{\partial P_o}{\partial x}\right)-q_g'-R_{so}q_o'=\frac{\partial}{\partial t}\left(\frac{\phi S_g}{B_g}+R_{so}\frac{\phi S_o}{B_o}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{kk_{rw}}{\mu_w B_w}\frac{\partial P_w}{\partial x}\right)-q_w'=\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right)$$

$$P_{cog}=P_g-P_o$$
$$P_{cow}=P_o-P_w$$
$$S_o+S_g+S_w=1$$

a) Write the three flow equations on discretized forms in terms of transmissibilities, storage coefficients and pressure differences (**no derivations needed**).

<u>Solution</u>

$$Tx_{o_{i+1/2}}\left(P_{o_{i+1}}-P_{o_i}\right)+Tx_{o_{i-1/2}}\left(P_{o_{i-1}}-P_{o_i}\right)-q'_{oi}$$
$$=C_{poo_i}\left(P_{o_i}-P_{o_i}^t\right)+C_{bpo_i}\left(P_{bp_i}-P_{bp_i}^t\right)+C_{swo_i}\left(S_{w_i}-S_{w_i}^t\right),\qquad i=1,N$$

$$\left(R_{so}Tx_o\right)_{i+1/2}\left(P_{o_{i+1}}-P_{o_i}\right)+\left(R_{so}Tx_o\right)_{i-1/2}\left(P_{o_{i-1}}-P_{o_i}\right)-\left(R_{so}q_o'\right)_i-q'_{gi}$$
$$=C_{pog_i}\left(P_{o_i}-P_{o_i}^t\right)+C_{bpg_i}\left(P_{bp_i}-P_{bp_i}^t\right)+C_{swg_i}\left(S_{w_i}-S_{w_i}^t\right),\qquad i=1,N$$

$$Tx_{w_{i+1/2}}\left[\left(P_{o_{i+1}}-P_{o_i}\right)-\left(P_{cow_{i+1}}-P_{cow_i}\right)\right]+Tx_{w_{i-1/2}}\left[\left(P_{o_{i-1}}-P_{o_i}\right)-\left(P_{cow_{i-1}}-P_{cow_i}\right)\right]-q'_{wi}$$
$$=C_{pow_i}\left(P_{o_i}-P_{o_i}^t\right)+C_{sww_i}\left(S_{w_i}-S_{w_i}^t\right),\qquad i=1,N$$

b) List the assumptions for an IMPES solution, and outline <u>briefly</u> how we solve for pressures and saturations

<u>Solution</u>

(Note: solution does not have to include all these details, it is sufficient that the student shows that he/she understands the principles)

IMPES solution:

Assumptions: all coefficients and parameters at time t, ie.

$$Tx_o^t, Tx_g^t, Tx_w^t$$
$$C_{poo}^t, C_{pog}^t, C_{pog}^t$$
$$C_{sgo}^t, C_{sgg}^t, C_{sgw}^t\ .$$
$$P_{cog}^t, P_{cow}^t, R_{so}^t$$

Having made these approximations, the discretized flow equations become:

$$
\begin{aligned}
& Txo_{i+1/2}^{t}\left(Po_{i+1}-Po_{i}\right)+Txo_{i-1/2}^{t}\left(Po_{i-1}-Po_{i}\right)-q'_{oi} \\
&\qquad = C^{t}{}_{poo_i}\left(Po_i-Po_i^{t}\right)+C^{t}{}_{sgo_i}\left(Sg_i-Sg_i^{t}\right)+C^{t}{}_{swo_i}\left(S_{wi}-S_{wi}^{t}\right), \qquad i=1,N \\
& Txg_{i+1/2}^{t}\left[\left(Po_{i+1}-Po_{i}\right)+\left(Pcog_{i+1}-Pcog_{i}\right)^{t}\right] \\
& +Txg_{i-1/2}^{t}\left[\left(Po_{i-1}-Po_{i}\right)+\left(Pcog_{i-1}-Pcog_{i}\right)^{t}\right]-q'_{gi} \\
& +\left(R_{so}Txo\right)_{i+1/2}^{t}\left(Po_{i+1}-Po_{i}\right)+\left(R_{so}Txo\right)_{i-1/2}^{t}\left(Po_{i-1}-Po_{i}\right)-\left(R_{so}{}^{t}q'_{o}\right)_{i} \\
&\qquad = C^{t}{}_{pog_i}\left(Po_i-Po_i^{t}\right)+C^{t}{}_{sgg_i}\left(Sg_i-Sg_i^{t}\right), \qquad i=1,N \\
& T^{t}{}_{xw_{i+1/2}}\left[\left(Po_{i+1}-Po_{i}\right)-\left(Pcow_{i+1}-Pcow_{i}\right)^{t}\right]+Txw_{i-1/2}\left[\left(Po_{i-1}-Po_{i}\right)-\left(Pcow_{i-1}-Pcow_{i}\right)^{t}\right]-q'_{wi} \\
&\qquad = C^{t}{}_{pow_i}\left(Po_i-Po_i^{t}\right)+C^{t}{}_{sww_i}\left(Sw_i-Sw_i^{t}\right), \qquad i=1,N
\end{aligned}
$$

IMPES pressure solution
By combining the three equations in order to eliminate the unknown saturations on the right hand sides of the equations, the pressure equation becomes:

$$a_i Po_{i-1}+b_i Po_i+c_i Po_{i+1}=d_i, \quad i=1,N$$

It may be solved for pressures using a number of solution methods, such as Gaussian elimination.

IMPES saturation solution
Having obtained the oil pressures above, we need to solve for gas and water saturations using either the oil equation or the gas equation. First using the water equation, we solve explicitly for water saturations

$$
Sw_i=Sw_i^{t}+\frac{1}{C^{t}{}_{sww}}\left[\begin{array}{l}
T^{t}{}_{xw_{i+1/2}}\left[\left(Po_{i+1}-Po_{i}\right)-\left(Pcow_{i+1}-Pcow_{i}\right)^{t}\right] \\
+Txw_{i-1/2}\left[\left(Po_{i-1}-Po_{i}\right)-\left(Pcow_{i-1}-Pcow_{i}\right)^{t}\right]-q'_{wi}-C^{t}{}_{pow_i}\left(Po_i-Po_i^{t}\right)
\end{array}\right], i=1,N
$$

Then, the gas saturations may be solved using the oil equation:

$$
Sg_i=Sg_i^{t}+\frac{1}{Csgo_i^{t}}\left[Txo_{i+1/2}^{t}\left(Po_{i+1}-Po_{i}\right)+Txo_{i-1/2}^{t}\left(Po_{i-1}-Po_{i}\right)-q'_{oi}-Cpoo_i^{t}\left(Po_i-Po_i^{t}\right)-C^{t}{}_{swo_i}\left(S_{wi}-S_{wi}^{t}\right)\right],
$$

$i=1,N$

c) Outline briefly how we can solve for pressures and saturations by Newtonian iteration (ie. fully implicit solution).

Solution

Newtonian iteration

Let us express the oil equation as F_{o_i} and the gas equation as F_{g_i}. Each equation will depend on pressures and saturations in blocks *i-1, i* and *i+1*., as indicated below.

$$F_{o_i}(P_{o_{i-1}},P_{o_i},P_{o_{i+1}},S_{g_{i-1}},S_{g_i},S_{g_{i+1}},S_{w_{i-1}},S_{w_i},S_{w_{i+1}})=0$$
$$F_{g_i}(P_{o_{i-1}},P_{o_i},P_{o_{i+1}},S_{g_{i-1}},S_{g_i},S_{g_{i+1}},S_{w_{i-1}},S_{w_i},S_{w_{i+1}})=0$$
$$F_{w_i}(P_{o_{i-1}},P_{o_i},P_{o_{i+1}},S_{w_{i-1}},S_{w_i},S_{w_{i+1}})=0\ , \qquad i=1,N$$

By first-order Taylor series expansions, we obtain the following expressions, where iteration level is given by *k*:

$$\begin{aligned}
F_{o_i}^{k+1} = F_{o_i}^{k} &+ \frac{\partial F_{o_i}}{\partial P_{o_{i-1}}}(P_{o_{i-1}}^{k+1}-P_{o_{i-1}}^{k}) + \frac{\partial F_{o_i}}{\partial P_{o_i}}(P_{o_i}^{k+1}-P_{o_i}^{k}) + \frac{\partial F_{o_i}}{\partial P_{o_{i+1}}}(P_{o_{i+1}}^{k+1}-P_{o_{i+1}}^{k}) \\
&+ \frac{\partial F_{o_i}}{\partial S_{g_{i-1}}}(S_{g_{i-1}}^{k+1}-S_{g_{i-1}}^{k}) + \frac{\partial F_{o_i}}{\partial S_{g_i}}(S_{g_i}^{k+1}-S_{g_i}^{k}) + \frac{\partial F_{o_i}}{\partial S_{g_{i+1}}}(S_{g_{i+1}}^{k+1}-S_{g_{i+1}}^{k}) \\
&+ \frac{\partial F_{o_i}}{\partial S_{w_{i-1}}}(S_{w_{i-1}}^{k+1}-S_{w_{i-1}}^{k}) + \frac{\partial F_{o_i}}{\partial S_{w_i}}(S_{w_i}^{k+1}-S_{w_i}^{k}) + \frac{\partial F_{o_i}}{\partial S_{w_{i+1}}}(S_{w_{i+1}}^{k+1}-S_{w_{i+1}}^{k}) \\
F_{g_i}^{k+1} = F_{g_i}^{k} &+ \frac{\partial F_{g_i}}{\partial P_{o_{i-1}}}(P_{o_{i-1}}^{k+1}-P_{o_{i-1}}^{k}) + \frac{\partial F_{g_i}}{\partial P_{o_i}}(P_{o_i}^{k+1}-P_{o_i}^{k}) + \frac{\partial F_{g_i}}{\partial P_{o_{i+1}}}(P_{o_{i+1}}^{k+1}-P_{o_{i+1}}^{k}) \\
&+ \frac{\partial F_{g_i}}{\partial S_{g_{i-1}}}(S_{g_{i-1}}^{k+1}-S_{g_{i-1}}^{k}) + \frac{\partial F_{g_i}}{\partial S_{g_i}}(S_{g_i}^{k+1}-S_{g_i}^{k}) + \frac{\partial F_{g_i}}{\partial S_{g_{i+1}}}(S_{g_{i+1}}^{k+1}-S_{g_{i+1}}^{k}) \\
&+ \frac{\partial F_{g_i}}{\partial S_{w_{i-1}}}(S_{w_{i-1}}^{k+1}-S_{w_{i-1}}^{k}) + \frac{\partial F_{g_i}}{\partial S_{w_i}}(S_{w_i}^{k+1}-S_{w_i}^{k}) + \frac{\partial F_{g_i}}{\partial S_{w_{i+1}}}(S_{w_{i+1}}^{k+1}-S_{w_{i+1}}^{k}) \\
F_{w_i}^{k+1} = F_{w_i}^{k} &+ \frac{\partial F_{w_i}}{\partial P_{o_{i-1}}}(P_{o_{i-1}}^{k+1}-P_{o_{i-1}}^{k}) + \frac{\partial F_{w_i}}{\partial P_{o_i}}(P_{o_i}^{k+1}-P_{o_i}^{k}) + \frac{\partial F_{w_i}}{\partial P_{o_{i+1}}}(P_{o_{i+1}}^{k+1}-P_{o_{i+1}}^{k}) \\
&+ \frac{\partial F_{w_i}}{\partial S_{w_{i-1}}}(S_{w_{i-1}}^{k+1}-S_{w_{i-1}}^{k}) + \frac{\partial F_{w_i}}{\partial S_{w_i}}(S_{w_i}^{k+1}-S_{w_i}^{k}) + \frac{\partial F_{w_i}}{\partial S_{w_{i+1}}}(S_{w_{i+1}}^{k+1}-S_{w_{i+1}}^{k}), \\
&i=1,\ldots,N
\end{aligned}$$

Thus, for a one-dimensional system we have *3N* equations and *3N* unknowns, and we can easily solve for estimates of oil pressures and gas and water saturations. By applying Newtonian iteration until we converge on a solution within some tolerance, we may obtain a solution to the equations. Our linear equations for iteration step *k+1* would then take the form:

$$a_{poo_i}P_{o_{i-1}}^{k+1} + b_{poo_i}P_{o_i}^{k+1} + c_{poo_i}P_{o_{i+1}}^{k+1} + a_{sgo_i}S_{g_{i-1}}^{k+1} + b_{sgo_i}S_{g_i}^{k+1} + c_{sgo_i}S_{g_{i+1}}^{k+1} + a_{swo_i}S_{w_{i-1}}^{k+1} + b_{swo_i}S_{w_i}^{k+1} + c_{swo_i}S_{w_{i+1}}^{k+1} = d_{o_i}$$

$$a_{pog_i}P_{o_{i-1}}^{k+1} + b_{pog_i}P_{o_i}^{k+1} + c_{pog_i}P_{o_{i+1}}^{k+1} + a_{sgg_i}S_{g_{i-1}}^{k+1} + b_{sgg_i}S_{g_i}^{k+1} + c_{sgg_i}S_{g_{i+1}}^{k+1} + a_{swg_i}S_{w_{i-1}}^{k+1} + b_{swg_i}S_{w_i}^{k+1} + c_{swg_i}S_{w_{i+1}}^{k+1} = d_{g_i}$$
$$a_{pow_i}P_{o_{i-1}}^{k+1} + b_{pow_i}P_{o_i}^{k+1} + c_{pow_i}P_{o_{i+1}}^{k+1} + a_{sww_i}S_{w_{i-1}}^{k+1} + b_{sww_i}S_{w_i}^{k+1} + c_{sww_i}S_{w_{i+1}}^{k+1} = d_{w_i}$$

$$i=1,\ N$$

The equations are solved for pressures and saturations iteratively, updating coefficients after each iteration.

Question 6 (12 points)

For three-phase flow (constant flow area) the right-hand side of the gas equation may be written (general case):

$$\frac{\partial}{\partial t}\left(\frac{\phi S_g}{B_g}+R_{so}\frac{\phi S_o}{B_o}\right)$$

a) Rewrite the expression above for under-saturated flow and state the pressure dependencies of B_o and R_{so} for both saturated and under-saturated systems.

b) For one-dimensional flow of saturated oil, gas and water, the discretized form of the right-hand side of the equation may be written:

$$Cpog_i\left(Po_i - Po_i^t\right)+Csgg_i\left(Sg_i - Sg_i^t\right)+Cswg_i\left(Sw_i - Sw_i^t\right)$$

Show the complete derivation the three storage coefficients ($Cpog_i, Csgg_i, Cswg_i$).

c) For one-dimensional flow of under-saturated oil and water, the discretized form of the right-hand side of the equation may be written:

$$Cpog_i\left(Po_i - Po_i^t\right)+Cpbg_i\left(Pbp_i - Pbp_i^t\right)+Cswg_i\left(Sw_i - Sw_i^t\right)$$

Show the complete derivations of the three storage coefficients ($Cpog_i, Cpbg_i, Cswg_i$).

Solution

a) Saturated: Since $P_o = P_{bp}$

$B_o(P_o)$

$R_{so}(P_o)$

Undersaturated: $P_o \ge P_{bp}$

$B_o(P_o, P_{bp})$

$R_{so}(P_{pb})$

b) The right hand side of the gas equation consists of a *free gas* term and a *solution gas* term:

$$\frac{\partial}{\partial t}\left(\frac{\phi S_g}{B_g}+\frac{\phi R_{so} S_o}{B_o}\right)=\frac{\partial}{\partial t}\left(\frac{\phi S_g}{B_g}\right)+\frac{\partial}{\partial t}\left(\frac{\phi R_{so} S_o}{B_o}\right)$$

The *free gas* term may be written:

$$\frac{\partial}{\partial t}\left(\frac{\phi S_g}{B_g}\right)=S_g\left(\frac{1}{B_g}\frac{d\phi}{dP_g}+\frac{d(1/B_g)}{dP_g}\right)\left[\frac{\partial P_o}{\partial t}+\left(\frac{dP_{cog}}{dS_g}\right)_i\frac{\partial S_g}{\partial t}\right]+\frac{\phi}{B_g}\frac{\partial S_g}{\partial t}$$

$$\approx \frac{\phi_i Sg_i}{\Delta t}\left(\frac{c_r}{B_g}+\frac{d(1/B_g)}{dP_g}\right)_i\left[(Po_i - Po_i^t)+\left(\frac{dP_{cog}}{dS_g}\right)_i(Sg_i - Sg_i^t)\right]+\frac{\phi_i}{Bg_i\Delta t}(Sg_i - Sg_i^t)$$

The *solution gas* term may be expanded into:

$$\frac{\partial}{\partial t}\left(\frac{\phi R_{so} S_o}{B_o}\right)=R_{so}\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)+\frac{\phi S_o}{B_o}\frac{\partial R_{so}}{\partial t}$$

The first term:

$$\left[R_{so}\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)\right]_i = R_{so}\left[S_o\frac{\partial}{\partial t}\left(\frac{\phi}{B_o}\right)+\frac{\phi}{B_o}\frac{\partial S_o}{\partial t}\right] = R_{so}S_o\left[\frac{1}{B_o}\frac{d\phi}{dP}+\phi\frac{d(1/B_o)}{dP}\right]\frac{\partial P_o}{\partial t}+\frac{R_{so}\phi}{B_o}\frac{\partial S_o}{\partial t}$$

$$\approx\frac{\phi_i R_{so_i}S_{o_i}}{\Delta t}\left[\frac{c_r}{B_o}+\frac{d(1/B_o)}{dP}\right]_i(P_{o_i}-P_{o_i}^t)+\left(\frac{R_{so}\phi}{B_o}\right)_i\left[-(S_{g_i}-S_{g_i}^t)-(S_{w_i}-S_{w_i}^t)\right]$$

The second term:

$$\left(\frac{\phi S_o}{B_o}\frac{\partial R_{so}}{\partial t}\right)_i=\left(\frac{\phi S_o}{B_o}\frac{dR_{so}}{dP_o}\frac{\partial P_o}{\partial t}\right)_i\approx\frac{1}{\Delta t}\left(\frac{\phi S_o}{B_o}\frac{dR_{so}}{dP_o}\right)_i\left(P_{o_i}-P_{o_i}^t\right)$$

Then, collecting the terms:

$$C_{pog_i}=\frac{\phi_i}{\Delta t}\left[S_{g_i}\left(\frac{c_r}{B_g}+\frac{d(1/B_g)}{dP}\right)+R_{soi}(1-S_{w_i}-S_{g_i})\left(\frac{c_r}{B_o}+\frac{d(1/B_o)}{dP_o}\right)_i+\frac{(1-S_{w_i}-S_{g_i})}{B_{oi}}\left(\frac{dR_{so}}{dP_o}\right)_i\right]$$

$$C_{sgg_i}=\frac{\phi_i}{\Delta t}\left[S_g\left(\frac{c_r}{B_g}+\frac{d(1/B_g)}{dP_g}\right)\frac{dP_{cog}}{dS_g}-\frac{R_{so}}{B_o}+\frac{1}{B_g}\right]_i$$

$$C_{swg_i}=-\frac{\phi_i R_{soi}}{\Delta t B_{oi}}$$

c) We expand the right hand side of the gas equation as follows:

$$\frac{\partial}{\partial t}\left(R_{so}\frac{\phi S_o}{B_o}\right)=R_{so}\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)+\frac{\phi S_o}{B_o}\frac{dR_{so}}{dP_{bp}}\frac{\partial P_{bp}}{\partial t}=$$

$$R_{so}\left[\left(\frac{\phi}{B_o}\right)\frac{\partial S_o}{\partial t}+S_o\left(\frac{1}{B_o}\frac{d\phi}{dP_o}\frac{\partial P_o}{\partial t}+\phi\frac{d(1/B_o)}{dP_o}\frac{\partial P_o}{\partial t}+\phi\frac{d(1/B_o)}{dP_{bp}}\frac{\partial P_{bp}}{\partial t}\right)\right]+\frac{\phi S_o}{B_o}\frac{dR_{so}}{dP_{bp}}\frac{\partial P_{bp}}{\partial t}$$

$$=R_{so}S_o\left(\frac{1}{B_o}\frac{d\phi}{dP_o}+\phi\frac{d(1/B_o)}{dP_o}\right)\frac{\partial P_o}{\partial t}+S_o\phi\left(R_{so}\frac{d(1/B_o)}{dP_{bp}}+\frac{1}{B_o}\frac{dR_{so}}{dP_{bp}}\right)\frac{\partial P_{bp}}{\partial t}-\frac{R_{so}\phi}{B_o}\frac{\partial S_w}{\partial t}$$

$$=R_{so}S_o\phi\left(\frac{c_r}{B_o}+\frac{d(1/B_o)}{dP_o}\right)\frac{\partial P_o}{\partial t}+S_o\phi\left(R_{so}\frac{d(1/B_o)}{dP_{bp}}+\frac{1}{B_o}\frac{dR_{so}}{dP_{bp}}\right)\frac{\partial P_{bp}}{\partial t}-\frac{R_{so}\phi}{B_o}\frac{\partial S_w}{\partial t}$$

Using standard discretization formulas, we get the discrete form:

$$\frac{\partial}{\partial t}\left(R_{so}\frac{\phi S_o}{B_o}\right)\approx\frac{(R_{so}S_o\phi)_i}{\Delta t}\left(\frac{c_r}{B_o}+\frac{d(1/B_o)}{dP_o}\right)_i\left(P_{oi}-P_{oi}^t\right)$$

$$+\frac{(S_o\phi)_i}{\Delta t}\left(R_{so}\frac{d(1/B_o)}{dP_{bp}}+\frac{1}{B_o}\frac{dR_{so}}{dP_{bp}}\right)_i\left(P_{bpi}-P_{bpi}^t\right)-\left(\frac{R_{so}\phi}{B_o}\right)_i\frac{1}{\Delta t}\left(S_{wi}-S_{wi}^t\right)$$

Thus, the coefficients are:

$$C_{pog_i}=\frac{(R_{so}\phi)_i(1-S_{w_i})}{\Delta t}\left(\frac{c_r}{B_o}+\frac{\partial(1/B_o)}{\partial P_o}\right)_i$$

$$C_{bpg_i}=\frac{\phi_i(1-S_{w_i})}{\Delta t}\left[R_{so}\frac{\partial(1/B_o)}{\partial P_{bp}}+\frac{1}{B_o}\frac{dR_{so}}{dP_{bp}}\right]_i$$

$$C_{swg_i}=-\frac{\phi_i R_{so_i}}{B_{oi}\Delta t}$$

Question 7 (10 points)

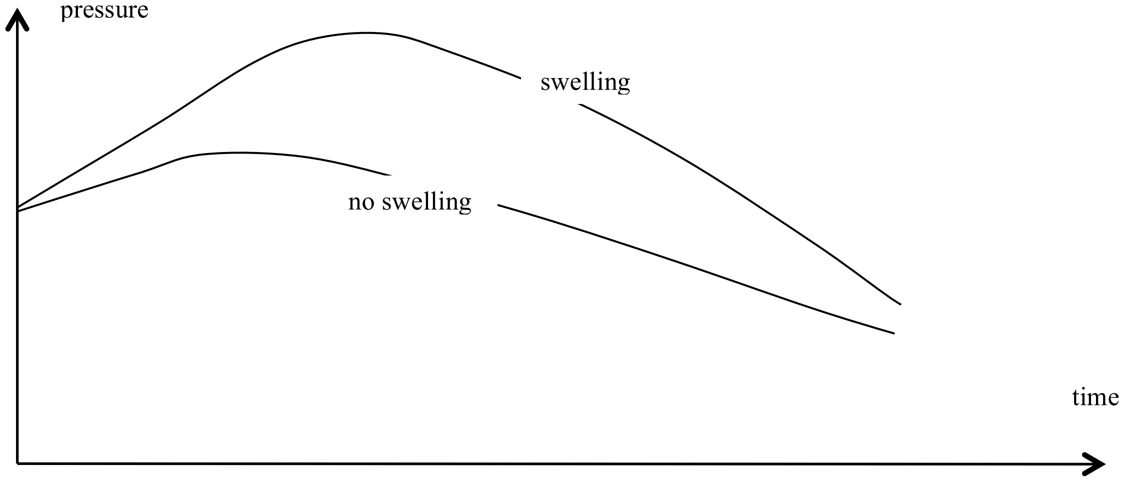
In Exercise 4 we injected gas into an under-saturated oil reservoir, and considered two cases: with and without swelling of the oil. **Make following qualitative plots (approximate):**

a) Pressure in the injection grid-block vs. time for the two cases (on the same figure)
b) Production rates (oil and gas) vs. time for the two cases (on the same figure)

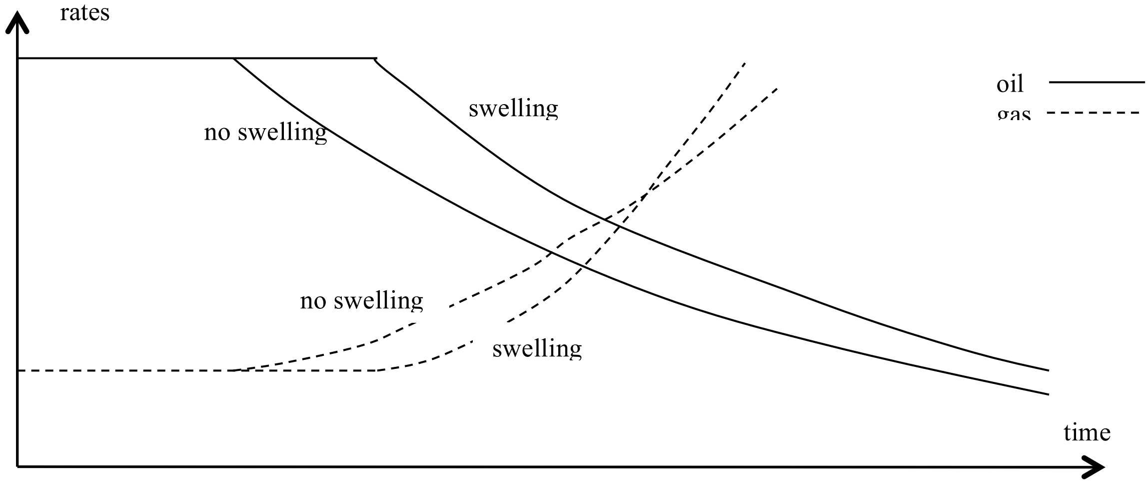
Explain why the curves differ based on the physical behavior of the two cases. Use plots of *Rso* vs. *Po* and *Bo* vs. *Po* in the explanation, and indicate typical pressure paths (for oil pressure and bubble point pressure) on these plots.

Solution

a)



b)



Because of the lower compressibility in the reservoir when oil is swelling (remaining liquid) compared to the case with free gas in the reservoir, pressure is increasing quicker. In the non-swelling case, free gas will move along the top of the reservoir, and gas break-through will occur quickly. The oil production will go off plateau quicker. The gas production will remain at initial solution gas ratio level until break-through (with a possible small increase in case of swelling). The increase in gas production will be slower in the non-swelling case, since there is the same amount of gas to be produced over time.

<u>Question 8</u> *(12 points)*

The discretized form of the oil equation may be written as

$$T_{xo_{i+1/2}}(P_{oi+1} - P_{oi}) + T_{xo_{i-1/2}}(P_{oi-1} - P_{oi}) - q'_{oi} = C_{poi}(P_{oi} - P^t_{oi}) + C_{soi}(S_{wi} - S^t_{wi})$$

a) What is the physical significance of each of the 5 terms in the equation?

<u>Solution</u>

$T_{xo_{i+1/2}}(P_{oi+1} - P_{oi}) =$ flow between grid blocks i and i+1
$T_{xo_{i-1/2}}(P_{oi-1} - P_{oi}) =$ flow between grid blocks i and i - 1
$q'_{oi} =$ production term
$C_{poi}(P_{oi} - P^t_{oi}) =$ fluids compression/expansion term
$C_{soi}(S_{wi} - S^t_{wi}) =$ volume change due to saturations

Using the following transmissibility as example,

$$T_{xo_{i-1/2}} = \frac{2k_{i-1/2}\lambda_{o_{i-1/2}}}{\Delta x_i\left(\Delta x_i + \Delta x_{i-1}\right)}$$

b) What type of averaging method is normally applied to absolute permeability between grid blocks? Why? Write the expression for average permeability between grid blocks (*i*-1) and (*i*).

<u>Solution</u>

Harmonic average is used, based on a derivation of average permeability of series flow, assuming steady flow and Darcy´s equation

$$\bar{k}_{i-1/2} = \frac{\Delta x_{i-1} + \Delta x_i}{\dfrac{\Delta x_{i-1}}{k_{i-1}} + \dfrac{\Delta x_i}{k_i}}$$

c) Write an expression for the selection of the conventional <u>*upstream mobility term*</u> for use in the transmissibility term of the oil equation above for flow between the grid blocks (*i*-1) and (*i*).

<u>Solution</u>

$$\lambda_{o_{i-1/2}} = \begin{cases} \lambda_{o_{i-1}} & if \quad P_{o_{i-1}} \geq P_{o_i} \\ \lambda_{o_i} & if \quad P_{o_{i-1}} < P_{o_i} \end{cases}$$

d) Make a sketch of a typical Buckley-Leverett saturation profile resulting from the displacement of oil by water (ie. analytical solution). Then, show how the corresponding profile, if calculated in a numerical simulation model, typically is influenced by the choice of mobilities between the grid blocks (sketch curves for saturations computed with upstream or average mobility terms, respectively).

Solution