SOLUTION OF NON-LINEAR EQUATIONS

Considering the one-dimensional saturated three phase equations:

$$Txo_{i+1/2}\left(Po_{i+1} - Po_i\right) + Txo_{i-1/2}\left(Po_{i-1} - Po_i\right) - q'_{oi}$$
$$= Cpoo_i\left(Po_i - Po_i^t\right) + Csgo_i\left(Sg_i - Sg_i^t\right) + Cswo_i\left(Sw_i - Sw_i^t\right), \qquad i = 1, N$$

$$Txg_{i+1/2}\left[\left(Po_{i+1} - Po_i\right) + \left(Pcog_{i+1} - Pcog_i\right)\right] + Txg_{i-1/2}\left[\left(Po_{i-1} - Po_i\right) + \left(Pcog_{i-1} - Pcog_i\right)\right] - q'_{gi}$$
$$+\left(R_{so}Txo\right)_{i+1/2}\left(Po_{i+1} - Po_i\right) + \left(R_{so}Txo\right)_{i-1/2}\left(Po_{i-1} - Po_i\right) - \left(R_{so}q'_o\right)_i$$
$$= Cpog_i\left(Po_i - Po_i^t\right) + Csgg_i\left(Sg_i - Sg_i^t\right) + Cswg_i\left(Sw_i - Sw_i^t\right), \qquad i = 1, N$$

$$Txw_{i+1/2}\left[\left(Po_{i+1} - Po_i\right) - \left(Pcow_{i+1} - Pcow_i\right)\right] + Txo_{i-1/2}\left[\left(Po_{i-1} - Po_i\right) - \left(Pcow_{i-1} - Pcow_i\right)\right] - q'_{wi}$$
$$= Cpow_i\left(Po_i - Po_i^t\right) + Csgw_i\left(Sg_i - Sg_i^t\right) + Csww_i\left(Sw_i - Sw_i^t\right), \qquad i = 1, N$$

All coefficients and caillary pressures in the equations above are functions of the solution, ie. the new saturations and pressures.

IMPES method

We have so far only discussed the ***IMPES method*** for solution of the non-linear equations arising in reservoir simulation. In this method we made the assumption of all coefficients and capillary pressures are evaluated at old time step level, and then eliminated unknown saturations by combining the three equations and obtained a pressure equation. Having thus solved for pressures, saturations could be solved for explicitly from one of the above equations.

Simultaneous solution method

Obviously, another alternative for this type of simplified solution, is to keep all three phase pressures as unknowns in the equations, and eliminate saturations by the following operations:

$$Sw_i - Sw_i^t = \left(\frac{dP_{cow}}{dS_w}\right)_i^{-1}\left[(Po_i - Po_i^t) - (Pw_i - Pw_i^t)\right]$$

$$Sg_i - Sg_i^t = \left(\frac{dP_{cog}}{dS_g}\right)_i^{-1}\left[(Pg_i - Pg_i^t) - (Po_i - Po_i^t)\right].$$

Except for the capillary pressures, we now make identical assumptions as for the IMPES method. Then, we may solve for the three phase pressures. The coefficients would then have the form shown on the next page.

By this method, the ***simultaneous solution method***, we have achieved to keep capillary pressures implicit, while the other coefficients are explicit. The set of linear equations has tripled in size, however.

$$
\begin{bmatrix}
& c &&&&&&&&& cg &&&&&&&&&& cw \\
a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
& a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
&& a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
&&& a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
&&&& a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
&&&&& a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
&&&&&& a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
&&&&&&& a & b & c &&&&&&&&& cg &&&&&&&&&& cw \\
&&&&&&&& a & b &&&&&&&&&& cg &&&&&&&&&& cw \\
co &&&&&&&&&& b & c &&&&&&&&& cw \\
& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&&& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&&&& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&&&&& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&&&&&& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&&&&&&& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&&&&&&&& co &&&&&&&&& a & b & c &&&&&&&&& cw \\
&&&&&&&&& co &&&&&&&&& a & b &&&&&&&&&& cw \\
cg &&&&&&&&&& cg &&&&&&&&&& b & c \\
& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&&& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&&&& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&&&&& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&&&&&& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&&&&&&& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&&&&&&&& cg &&&&&&&&&& cg &&&&&&&&& a & b & c \\
&&&&&&&&& cg &&&&&&&&&& cg &&&&&&&&& a & b
\end{bmatrix}
\begin{bmatrix}
Po \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
Pg \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
Pw \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
\end{bmatrix}
$$

Having thus obtained the three phase pressures, saturations may be interpolated from the capillary pressure-saturation relationships.

Iterative methods

Today, the IMPES method is still being used for field scale simulation, provided changes in pressures and saturation are reasonably slow. However, generally we need to solve the flow equations implicitly, and in order to do so, we apply iterative techniques.

For the purpose of illustration of ***Newtonian iteration***, recall the Newton-Raphson method for solving a non-linear equation with one unknown. Having a nonlinear equation of the form:

$$F(x) = 0.,$$

for instance:

$$x^5 - 3x^3 + 2x - 4 = 0,$$

we can, by using Taylor expansion, derive an iterative formula:

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}.$$

By iteration, provided that the first guess of the solution is sufficiently close to the correct solution, rapidly converge to the solution.

If we have two simultaneous non-linear equations:

$$F(x, y) = 0$$

$$G(x, y) = 0,$$

the iterative formula becomes:

$$x_{k+1}=x_k-\frac{F(x_k,y_k)\frac{\partial G}{\partial y}(x_k,y_k)-G(x_k,y_k)\frac{\partial F}{\partial y}(x_k,y_k)}{J(x_k,y_k)}$$

$$y_{k+1}=y_k-\frac{F(x_k,y_k)\frac{\partial G}{\partial x}(x_k,y_k)-G(x_k,y_k)\frac{\partial F}{\partial x}(x_k,y_k)}{J(x_k,y_k)}$$

where the *Jacobian* is

$$J(x_k,y_k)=\frac{\partial F}{\partial x}(x_k,y_k)\frac{\partial G}{\partial y}(x_k,y_k)-\frac{\partial F}{\partial y}(x_k,y_k)\frac{\partial G}{\partial x}(x_k,y_k).$$

Our three phase flow equations may be written on a similar form as the above equations. Generally, in one dimensional flow, we have $3N$ unknowns. However, due to the structure of the finite difference method used, each node equation will have only 9 unknowns, so that our system of equations becomes:

$$F_{o_i}(P_{o_{i-1}},P_{o_i},P_{o_{i+1}},S_{g_{i-1}},S_{g_i},S_{g_{i+1}},S_{w_{i-1}},S_{w_i},S_{w_{i+1}})=0$$

$$F_{g_i}(P_{o_{i-1}},P_{o_i},P_{o_{i+1}},S_{g_{i-1}},S_{g_i},S_{g_{i+1}},S_{w_{i-1}},S_{w_i},S_{w_{i+1}})=0$$

$$F_{w_i}(P_{o_{i-1}},P_{o_i},P_{o_{i+1}},S_{g_{i-1}},S_{g_i},S_{g_{i+1}},S_{w_{i-1}},S_{w_i},S_{w_{i+1}})=0, \qquad i=1,N$$

By first-order Taylor series expansions, we obtain the following expressions:

$$\begin{aligned}F_{o_i}^{k+1}=F_{o_i}^{k}&+\frac{\partial F_{o_i}}{\partial P_{o_{i-1}}}(P_{o_{i-1}}^{k+1}-P_{o_{i-1}}^{k})+\frac{\partial F_{o_i}}{\partial P_{o_i}}(P_{o_i}^{k+1}-P_{o_i}^{k})+\frac{\partial F_{o_i}}{\partial P_{o_{i+1}}}(P_{o_{i+1}}^{k+1}-P_{o_{i+1}}^{k})\\&+\frac{\partial F_{o_i}}{\partial S_{g_{i-1}}}(S_{g_{i-1}}^{k+1}-S_{g_{i-1}}^{k})+\frac{\partial F_{o_i}}{\partial S_{g_i}}(S_{g_i}^{k+1}-S_{g_i}^{k})+\frac{\partial F_{o_i}}{\partial S_{g_{i+1}}}(S_{g_{i+1}}^{k+1}-S_{g_{i+1}}^{k})\\&+\frac{\partial F_{o_i}}{\partial S_{w_{i-1}}}(S_{w_{i-1}}^{k+1}-S_{w_{i-1}}^{k})+\frac{\partial F_{o_i}}{\partial S_{w_i}}(S_{w_i}^{k+1}-S_{w_i}^{k})+\frac{\partial F_{o_i}}{\partial S_{w_{i+1}}}(S_{w_{i+1}}^{k+1}-S_{w_{i+1}}^{k})\end{aligned}$$

$$\begin{aligned}F_{g_i}^{k+1}=F_{g_i}^{k}&+\frac{\partial F_{g_i}}{\partial P_{o_{i-1}}}(P_{o_{i-1}}^{k+1}-P_{o_{i-1}}^{k})+\frac{\partial F_{g_i}}{\partial P_{o_i}}(P_{o_i}^{k+1}-P_{o_i}^{k})+\frac{\partial F_{g_i}}{\partial P_{o_{i+1}}}(P_{o_{i+1}}^{k+1}-P_{o_{i+1}}^{k})\\&+\frac{\partial F_{g_i}}{\partial S_{g_{i-1}}}(S_{g_{i-1}}^{k+1}-S_{g_{i-1}}^{k})+\frac{\partial F_{g_i}}{\partial S_{g_i}}(S_{g_i}^{k+1}-S_{g_i}^{k})+\frac{\partial F_{g_i}}{\partial S_{g_{i+1}}}(S_{g_{i+1}}^{k+1}-S_{g_{i+1}}^{k})\\&+\frac{\partial F_{g_i}}{\partial S_{w_{i-1}}}(S_{w_{i-1}}^{k+1}-S_{w_{i-1}}^{k})+\frac{\partial F_{g_i}}{\partial S_{w_i}}(S_{w_i}^{k+1}-S_{w_i}^{k})+\frac{\partial F_{g_i}}{\partial S_{w_{i+1}}}(S_{w_{i+1}}^{k+1}-S_{w_{i+1}}^{k})\end{aligned}$$

$$\begin{aligned}F_{w_i}^{k+1}=F_{w_i}^{k}&+\frac{\partial F_{w_i}}{\partial P_{o_{i-1}}}(P_{o_{i-1}}^{k+1}-P_{o_{i-1}}^{k})+\frac{\partial F_{w_i}}{\partial P_{o_i}}(P_{o_i}^{k+1}-P_{o_i}^{k})+\frac{\partial F_{w_i}}{\partial P_{o_{i+1}}}(P_{o_{i+1}}^{k+1}-P_{o_{i+1}}^{k})\\&+\frac{\partial F_{w_i}}{\partial S_{g_{i-1}}}(S_{g_{i-1}}^{k+1}-S_{g_{i-1}}^{k})+\frac{\partial F_{w_i}}{\partial S_{g_i}}(S_{g_i}^{k+1}-S_{g_i}^{k})+\frac{\partial F_{w_i}}{\partial S_{g_{i+1}}}(S_{g_{i+1}}^{k+1}-S_{g_{i+1}}^{k})\\&+\frac{\partial F_{w_i}}{\partial S_{w_{i-1}}}(S_{w_{i-1}}^{k+1}-S_{w_{i-1}}^{k})+\frac{\partial F_{w_i}}{\partial S_{w_i}}(S_{w_i}^{k+1}-S_{w_i}^{k})+\frac{\partial F_{w_i}}{\partial S_{w_{i+1}}}(S_{w_{i+1}}^{k+1}-S_{w_{i+1}}^{k})\end{aligned}$$

$$i=1,N$$

Thus, for a one-dimensional system we have *3N* equations and *3N* unknowns, and we can easily solve for estimates of oil pressures and gas and water saturations. By applying Newtonian iteration until we converge on a solution within some tolerance, we may obtain a solution to the equations. Our linear equations for iteration step *k+1* would then take the form:

$$apoo_i Po_{i-1}^{k+1} + bpoo_i Po_i^{k+1} + cpoo_i Po_{i+1}^{k+1} + asgo_i Sg_{i-1}^{k+1} + bsgo_i Sg_i^{k+1} + csgo_i Sg_{i+1}^{k+1}$$
$$+ aswo_i Sw_{i-1}^{k+1} + bswo_i Sw_i^{k+1} + cswo_i Sw_{i+1}^{k+1} = do_i$$

$$apog_i Po_{i-1}^{k+1} + bpog_i Po_i^{k+1} + cpog_i Po_{i+1}^{k+1} + asgg_i Sg_{i-1}^{k+1} + bsgg_i Sg_i^{k+1} + csgg_i Sg_{i+1}^{k+1}$$
$$+ aswg_i Sw_{i-1}^{k+1} + bswg_i Sw_i^{k+1} + cswg_i Sw_{i+1}^{k+1} = dg_i$$

$$apow_i Po_{i-1}^{k+1} + bpow_i Po_i^{k+1} + cpow_i Po_{i+1}^{k+1} + asgw_i Sg_{i-1}^{k+1} + bsgw_i Sg_i^{k+1} + csgw_i Sg_{i+1}^{k+1}$$
$$+ asww_i Sw_{i-1}^{k+1} + bsww_i Sw_i^{k+1} + csww_i Sw_{i+1}^{k+1} = dw_i$$

$$i = 1,\ N$$

or, on a compact form

$$\hat{a}_i \vec{X}_{i-1}^{k+1} + \hat{b}_i \vec{X}_i^{k+1} + \hat{c}_i \vec{X}_{i+1}^{k+1} = \vec{d}_i\ , \quad i = 1, N$$

where

$$\hat{a}_i = \begin{Vmatrix} apoo_i & asgo_i & aswo_i \\ apog_i & asgg_i & aswg_i \\ apow_i & asgw_i & asww_i \end{Vmatrix} \qquad \hat{b}_i = \begin{vmatrix} bpoo_i & bsgo_i & bswo_i \\ bpog_i & bsgg_i & bswg_i \\ bpow_i & bsgw_i & bsww_i \end{vmatrix} \qquad \hat{c}_i = \begin{Vmatrix} cpoo_i & csgo_i & cswo_i \\ cpog_i & csgg_i & cswg_i \\ cpow_i & csgw_i & csww_i \end{Vmatrix}$$

$$\vec{X}_i^{k+1} = \begin{vmatrix} Po_i^{k+1} \\ Sg_i^{k+1} \\ Sw_i^{k+1} \end{vmatrix} \qquad \vec{d}_i = \begin{vmatrix} do_i \\ dg_i \\ dw_i \end{vmatrix}$$

A simplified, non-iterative method that was extensively used in the past, uses estimated chord-slopes instead of the derivatives above, and accepts the solution obtained after the first iteration. This method, called ***semi-implicit method***, is hardly used today.