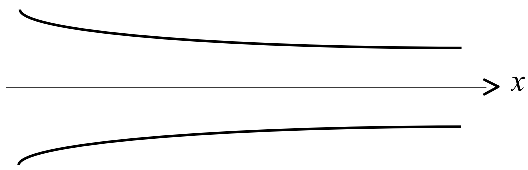
SYSTEMS OF VARIABLE FLOW AREA

We will look at two types of variable flow area. In linear coordinates, the thickness of the system may be variable, and in cylindrical coordinates, the radial coordinate itself causes the flow area to be a function of radius. For simplicity, we will again consider one-dimensional, horizontal, one phase flow.

Linear coordinates

If the thickness of the system is a function of position, x, such as in the following illustration:



flow area must be included in the continuity equation:

$$-\frac{\partial}{\partial x}(A\rho u)=\frac{\partial}{\partial t}(A\phi\rho).$$

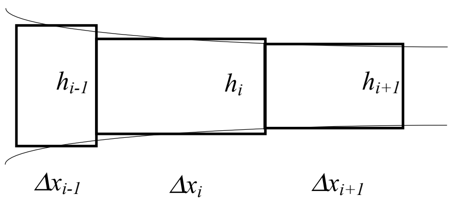
For one-dimensional flow only the layer height, h, is varying, so that the continuity equation becomes:

$$-\frac{\partial}{\partial x}(h\rho u)=\frac{\partial}{\partial t}(h\phi\rho),$$

and the one-dimensional, horizontal, single phase flow equation may be written:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu B}\frac{\partial P}{\partial x}\right)=\frac{\partial}{\partial t}\left(\frac{h\phi}{B}\right).$$

In discretization of the equation, our grid block system is modified to reflect the variable block height by using avarage values, as follows:



The difference form of the flow term in the partial differential equation above will be written in terms of transmissibilities just as before:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu B}\frac{\partial P}{\partial x}\right)_i \approx Tx_{i+1/2}(P_{i+1}-P_i)+Tx_{i-1/2}(P_{i-1}-P_i).$$

However, the definitions of the transmissibilities are now modified to include the block heights of the involved grid blocks:

$$Tx_{i+1/2}=\frac{2h_{i+1/2}}{\Delta x_i(\Delta x_{i+1}+\Delta x_i)}\left(\frac{k}{\mu B}\right)_{i+1/2}$$

$$Tx_{i-1/2}=\frac{2h_{i-1/2}}{\Delta x_i(\Delta x_{i-1}+\Delta x_i)}\left(\frac{k}{\mu B}\right)_{i-1/2},$$

where $h_{i+1/2}$ and $h_{i-1/2}$ are the flow heights in positive and negative directions, respectively. Using the positive direction as example, we shall define these heights by:

$$h_{i+1/2} = \frac{1}{(x_{i+1} - x_i)} \int_{x_i}^{x_{i+1}} h(x)\,dx \,,$$

which becomes:

$$h_{i+1/2} = \frac{\left(\Delta x_{i+1} h_{i+1} + \Delta x_i h_i\right)}{\left(\Delta x_{i+1} + \Delta x_i\right)} \,,$$

Correspondingly in the negative direction:

$$h_{i-1/2} = \frac{\left(\Delta x_{i-1} h_{i-1} + \Delta x_i h_i\right)}{\left(\Delta x_{i-1} + \Delta x_i\right)} \,.$$

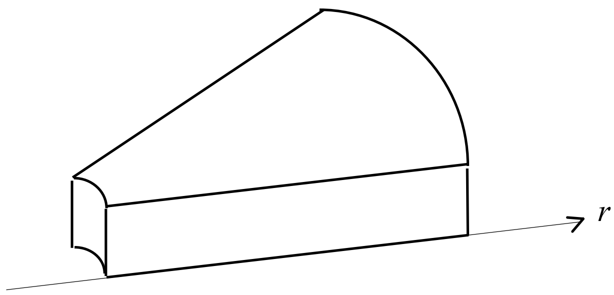
For the right hand side term, we only have to make a slight modification by multiplying by block height:

$$\frac{\partial}{\partial t}\left(\frac{\phi h}{B}\right) = \phi h \left[\frac{c_r}{B} + \frac{d(1/B)}{dP}\right]\frac{\partial P}{\partial t} \approx C_{Pi}(P_i - P_i^t),$$

where

$$C_{Pi} = \frac{h_i \phi_i}{\Delta t}\left[\frac{c_r}{B} + \frac{d(1/B)}{dP}\right]_i \,.$$

Cylindrical coordinates
Another type of variable flow area is one induced by the coordinate system used. The most common is the one occurring in radial flow as illustrated below:



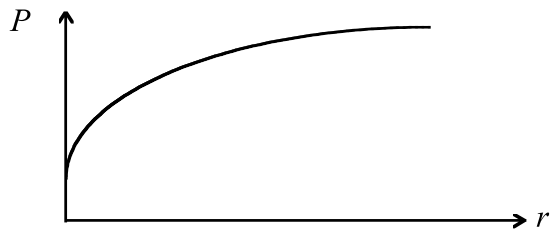
Here, even if the block height is constant, the flow area is a function of radius, and for a full cylinder (360 degrees) the area is:

$$A = 2\pi r h \,.$$

The continuity equation thus becomes, assuming constant height:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{k}{\mu B}\frac{\partial P}{\partial r}\right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B}\right).$$

In a radial system, the pressure distribution will be logarithmic of nature, with most of the pressure drop occurring close to the center, where the flow area is small:



Since our discretization formulas are more accurate the more linear the pressure distribution is, it is clear that if we discretize the radial flow term using the same approximations as for the linear equation, the error will be larger. Therefore, for the radial flow equation, we will first make the following transformation of the r-coordinate into a u-coordinate:

$$u=\ln(r)$$

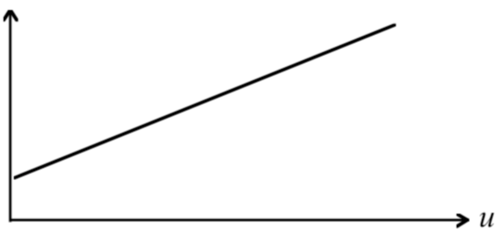
Thus,

$$\frac{du}{dr}=\frac{1}{r}$$

and

$$r=e^{u}$$

The effect of this transformation is that the logarithmic pressure distribution in the radial direction becomes linear along the u-coordinate:



Transformation of the radial flow equation by substitution for $u=\ln(r)$ yields:

$$e^{-u}\frac{\partial}{\partial u}\left(e^{u}\frac{k}{\mu B}\frac{\partial P}{\partial u}\frac{du}{dr}\right)\frac{du}{dr}=\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right),$$

or

$$e^{-2u}\frac{\partial}{\partial u}\left(\frac{k}{\mu B}\frac{\partial P}{\partial u}\right)=\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right).$$

This equation is more linear in u and except for the term e^{-2u} in front of the flow term, it is identical to the linear flow equation. We will therefore adopt the same approximation of the flow term in respect to u for the equation above as we used in respect to □ for the linear equation, with the modification for the e^{-2u} term:

$$Tu_{i+1/2}=\frac{2e^{-2u_i}}{\Delta u_i(\Delta u_{i+1}+\Delta u_i)}\left(\frac{k}{\mu B}\right)_{i+1/2}$$

$$Tu_{i-1/2}=\frac{2e^{-2u_i}}{\Delta u_i(\Delta u_{i-1}+\Delta u_i)}\left(\frac{k}{\mu B}\right)_{i-1/2}$$

Substituting back for $r=e^{u}$, we get the following expressions for the radial transmissibilities:

$$Tr_{i+1/2} = \frac{2r_i^{-2}}{\ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)\ln\left(\frac{r_{i+11/2}}{r_{i-1/2}}\right)}\left(\frac{k}{\mu B}\right)_{i+1/2}$$

$$Tr_{i-1/2} = \frac{2r_i^{-2}}{\ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)\ln\left(\frac{r_{i-1/2}}{r_{i-11/2}}\right)}\left(\frac{k}{\mu B}\right)_{i-1/2}$$

The harmonic averages for permeability in radial direction may be derived in a similar fashion from the linear formula:

$$k_{i+1/2} = \frac{\ln\left(\frac{r_{i+11/2}}{r_{i+1/2}}\right)}{\frac{1}{k_{i+1}}\ln\left(\frac{r_{i+11/2}}{r_{i+1/2}}\right)+\frac{1}{k_i}\ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)}$$

and

$$k_{i-1/2} = \frac{\ln\left(\frac{r_{i+1/2}}{r_{i-11/2}}\right)}{\frac{1}{k_{i-1}}\ln\left(\frac{r_{i-1/2}}{r_{i-11/2}}\right)+\frac{1}{k_i}\ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)}$$

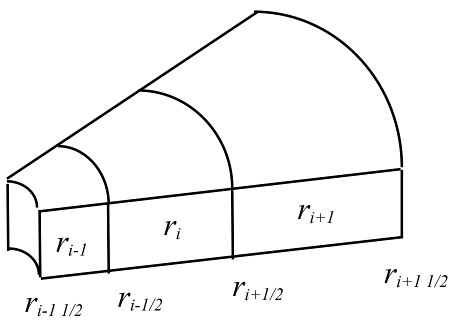
Expressions for average mobility terms become:

$$\lambda_{i+1/2} = \frac{\ln\left(\frac{r_{i+11/2}}{r_{i+1/2}}\right)\lambda_{i+1}+\ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)\lambda_i}{\ln\left(\frac{r_{i+11/2}}{r_{i-1/2}}\right)}$$

and

$$\lambda_{i-1/2} = \frac{\ln\left(\frac{r_{i-1/2}}{r_{i-11/2}}\right)\lambda_{i-1}+\ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)\lambda_i}{\ln\left(\frac{r_{i+1/2}}{r_{i-11/2}}\right)}$$

These formulas apply to the radial grid block system shown below:



The position of the grid block centers, relative to the block boundaries, may be computed using the midpoint between the u-coordinate boundaries:

$$u_i = (u_{i+1/2} + u_{i-1/2})/2\,,$$

or, in terms of radius:

$$r_i = \sqrt{r_{i+1/2} r_{i-1/2}} \ .$$

This is the geometric average of the block boundary radii.

Frequently in simulation of flow in the radial direction, the grid blocks sizes are chosen such that:

$$\Delta u_i = (u_{i+1/2} - u_{i-1/2}) = \text{constant}$$

or

$$\ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right) = \text{constant} \ ,$$

which for a system of N grid blocks and well and external radii of r_w and r_e, respectively, implies that

$$N \cdot \ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right) = \ln\left(\frac{r_e}{r_w}\right)$$

or

$$\frac{r_{i+1/2}}{r_{i-1/2}} = \left(\frac{r_e}{r_w}\right)^{1/N} = \text{constant} \ .$$

This is the formula for *logarithmic grid block sizes*, and is often used in reservoir simulation of well behavior. If grid sizes have to conform to other specifications, such as well damage radius, the above formula may still be useful as a guide to the block sizes.

For the ***right hand side*** of the difference equation, the above changes will have no effect provided that the height is constant. Thus, it will be identical to the one for the linear system.