Material Balance Equations

To illustrate the simplest possible model we can have for analysis of reservoir behavior, we will start with derivation of so-called Material Balance Equations. This type of model excludes fluid flow inside the reservoir, and considers fluid and rock expansion/compression effects only, in addition, of course, to fluid injection and production. First, let us define the symbols used in the material balance equations:

Symbols used in material balance equations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_g$</td>
<td>Formation volume factor for gas (res.vol./st.vol.)</td>
</tr>
<tr>
<td>$B_o$</td>
<td>Formation volume factor for oil (res.vol./st.vol.)</td>
</tr>
<tr>
<td>$B_w$</td>
<td>Formation volume factor for water (res.vol./st.vol.)</td>
</tr>
<tr>
<td>$C_r$</td>
<td>Pore compressibility (pressure$^{-1}$)</td>
</tr>
<tr>
<td>$C_w$</td>
<td>Water compressibility (pressure$^{-1}$)</td>
</tr>
<tr>
<td>$\Delta P$</td>
<td>Pressure change $P_2 - P_1$</td>
</tr>
<tr>
<td>$G_i$</td>
<td>Cumulative gas injected (st.vol.)</td>
</tr>
<tr>
<td>$G_p$</td>
<td>Cumulative gas produced (st.vol.)</td>
</tr>
<tr>
<td>$m$</td>
<td>Initial gas cap size (res.vol. of gas cap)/(res.vol. of oil zone)</td>
</tr>
<tr>
<td>$N$</td>
<td>Original oil in place (st.vol.)</td>
</tr>
<tr>
<td>$N_p$</td>
<td>Cumulative oil produced (st.vol.)</td>
</tr>
<tr>
<td>$P$</td>
<td>Pressure</td>
</tr>
<tr>
<td>$R_p$</td>
<td>Cumulative producing gas-oil ratio (st.vol./st.vol) = $G_p / N_p$</td>
</tr>
<tr>
<td>$R_{so}$</td>
<td>Solution gas-oil ratio (st.vol. gas/st.vol. oil)</td>
</tr>
<tr>
<td>$S_g$</td>
<td>Gas saturation</td>
</tr>
<tr>
<td>$S_o$</td>
<td>Oil saturation</td>
</tr>
<tr>
<td>$S_w$</td>
<td>Water saturation</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
</tr>
<tr>
<td>$V_b$</td>
<td>Bulk volume (res.vol.)</td>
</tr>
<tr>
<td>$V_p$</td>
<td>Pore volume (res.vol.)</td>
</tr>
<tr>
<td>$W_e$</td>
<td>Cumulative aquifer influx (st.vol.)</td>
</tr>
<tr>
<td>$W_i$</td>
<td>Cumulative water injected (st.vol.)</td>
</tr>
<tr>
<td>$W_p$</td>
<td>Cumulative water produced (st.vol.)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density (mass/vol.)</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Porosity</td>
</tr>
</tbody>
</table>

Then, the Black Oil fluid phase behavior is illustrated by the following figures:

Fluid phase behavior parameters (Black Oil)

Oil density: $\rho_o = \rho_o S + \rho_g S_{so} / B_o$
Water compressibility:
\[ C_w = -\left( \frac{1}{V_w} \frac{\partial V_w}{\partial P} \right)_T \]

Water volume change:
\[ B_{w2} = B_w e^{-c_w \Delta P} = B_w (I - c_w \Delta P) \]

Finally, we need to quantify the behavior of the pores during pressure change in the reservoir. The rock compressibility used in the following is the pore compressibility, and assumes that the bulk volume of the rock itself does not change.

**Pore volume behavior**

Rock compressibility:
\[ C_r = \left( \frac{1}{\phi} \frac{\partial \phi}{\partial P} \right)_T \]

Porosity change:
\[ \phi_{w2} = \phi_w e^{c_r \Delta P} = \phi_w (I + c_r \Delta P) \]

The material balance equations are based on simple mass balances of the fluids in the reservoir, and may in words be formulated as follows:

**Principle of material conservation**

\[
\begin{align*}
\text{Amount of fluids present} & \quad \text{Amount of fluids produced} \quad \text{Amount of fluids remaining} \\
\text{in the reservoir initially} & \quad \text{(st. vol.)} \quad \text{(st. vol.)} \quad \text{(st. vol.)}
\end{align*}
\]

We will define our reservoir system in terms of a simple block diagram, with an initial reservoir stage before production/injection starts, and a final stage at which time we would like to determine pressure and/or production.

**Block diagram of reservoir**
The two stages on the block diagram are reflected in the fluid phase behavior plots as follows:

**Initial and final fluid conditions**

![Block Diagram](image)

*Note: If a gas cap is present initially, then the initial pressure is equal to the bubble point pressure.*

Now, we will apply the above material balance equation to the three fluids involved, oil, gas and water:

**Equation 1: Oil material balance**

\[
\begin{align*}
\text{Oil present in the reservoir initially (st. vol.)} & - \text{Oil produced in the reservoir (st. vol.)} = \text{Oil remaining in the reservoir finally (st. vol.)} \\
N - N_p & = V_p S_o / B_o
\end{align*}
\]

or

\[
S_o = \frac{(N - N_p)B_o}{V_p}
\]

**Equation 2: Water material balance**

\[
\begin{align*}
\text{Water present in the reservoir initially (st. vol.)} & - \text{Water produced in the reservoir (st. vol.)} + \text{Water injected in the reservoir (st. vol.)} + \text{Aquifer influx in the reservoir (st. vol.)} = \text{Water remaining in the reservoir finally (st. vol.)} \\
V_{w1} S_{w1} / B_{w1} - W_p + W_i + W_e & = V_{w2} S_{w2} / B_{w2}
\end{align*}
\]

or

\[
S_{w2} = \left[ (1 + m) V B_o \left( \frac{S_{w1}}{1 - S_{w1}} \right) \left( \frac{I}{B_{w1}} \right) + \left( W_i + W_e - W_p \right) \right] B_{w2} / V_{w2}
\]
Equation 3: Gas material balance

\[
\begin{align*}
\begin{bmatrix}
\text{Solution gas} \\
\text{present in the reservoir initially (st. vol.)}
\end{bmatrix} & + \begin{bmatrix}
\text{Free gas} \\
\text{present in the reservoir initially (st. vol.)}
\end{bmatrix} - \begin{bmatrix}
\text{Gas produced} \\
\text{(st. vol.)}
\end{bmatrix} + \begin{bmatrix}
\text{Gas} \\
\text{injected} \\
\text{(st. vol.)}
\end{bmatrix} & = & \begin{bmatrix}
\text{Solution gas} \\
\text{present in the reservoir finally (st. vol.)}
\end{bmatrix} + \begin{bmatrix}
\text{Free gas} \\
\text{present in the reservoir finally (st. vol.)}
\end{bmatrix}
\end{align*}
\]

or

\[
NR_{s1}^{o} + mNB_{o1}^{g1} / B_{g1}^{o} - R_{p}^{o}N_{p}^{o} + G_{i} = (N - N_{p}^{o})R_{s2}^{o} + V_{p2}^{o}S_{g2}^{o} / B_{g2}^{o}
\]

yielding

\[
S_{g2}^{o} = \left\{ N\left[ (R_{s1}^{o} - R_{s2}^{o}) + m\frac{B_{o1}^{g1}}{B_{g1}^{o}} \right] - N_{p}^{o}(R_{p}^{o} - R_{s2}^{o}) + G_{i} \right\}\frac{B_{g2}^{o}}{V_{p2}^{o}}
\]

In addition to these three fluid balances, we have the following relationships for fluid saturations and pore volume change:

Equation 4: Sum of saturations

\[
S_{w} + S_{o} + S_{g} = 1.0
\]

Equation 5: Pore volume change

\[
V_{p2}^{o} = V_{p}^{o}(1 + c_{r} \Delta P)
\]
By combining the 5 equations above, and grouping terms, we obtain the material balance relationships, as shown below:

**THE COMPLETE BLACK OIL MATERIAL BALANCE EQUATION:**

\[
F = N \left( E_o + mE_g + E_{f,w} \right) + (W_f + W_o)B_{w2} + G_1B_{g2}
\]

where

*production terms are*

\[
F = N_p \left[ B_{o1} + \left( R_p - R_{so2} \right) B_{g2} \right] + W_p B_{w2}
\]

*oil and solution gas expansion terms are*

\[
E_o = \left( B_{o2} - B_{o1} \right) + \left( R_{so1} - R_{so2} \right) B_{g2}
\]

*gas cap expansion terms are*

\[
E_g = B_{o1} \left( \frac{B_{g2}}{B_{g1}} - 1 \right)
\]

*and rock and water compression/expansion terms are*

\[
E_{f,w} = -(1 + m)B_{w1} \frac{C_r + C_w S_{w,l} \Delta P}{1 - S_{w,l}}
\]
MATERIAL BALANCE EQUATION FOR A CLOSED GAS RESERVOIR

The material balance equation for a closed gas reservoir is very simple. Applying the mass balance principle to a closed reservoir with 100% gas, we may derive the general equation

\[ GB_{g1} = (G - G_p)B_{g2} \]

where \( G \) is gas initially in place, \( G_p \) is cumulative gas production, and \( B_g \) is the formation-volume-factor for gas. Since \( B_g \) is given by the real gas law

\[ B_g = \text{(constant)} \frac{Z}{P} \]

(here temperature is assumed to be constant)

the above material balance equation may be rewritten as

\[ G \left( \frac{Z}{P_1} \right) = (G - G_p) \left( \frac{Z}{P_2} \right) \]

or

\[ \frac{P_2}{Z_2} = \left(1 - \frac{G_p}{G}\right) \frac{P_1}{Z_1} \]

This equation represents a straight line relationship on a \( \left( \frac{P_2}{Z_2} \right) \) vs. \( G_p \) plot. The line passes through \( \left( \frac{P_1}{Z_1} \right) \) at \( G_p = 0 \), and through \( G \) at \( \left( \frac{P_1}{Z_1} \right) = 0 \). By making a best-fit straight line to measured data, and extrapolate, we may get an estimate of \( G \).

The straight-line relationship is very useful in estimating the initial volume of gas-in-place \( (G) \) from limited production history.