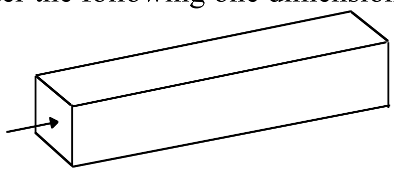
DERIVATION OF FLUID FLOW EQUATIONS

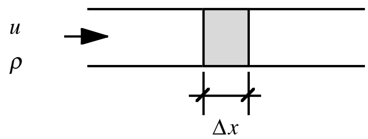
Review of basic steps
Generally speaking, flow equations for flow in porous materials are based on a set of mass, momentum and energy conservation equations, and constitutive equations for the fluids and the porous material involved. For simplicity, we will in the following assume isothermal conditions, so that we not have to involve an energy conservation equation. However, in cases of changing reservoir temperature, such as in the case of cold water injection into a warmer reservoir, this may be of importance.

Below, equations are initially described for single phase flow in linear, one-dimensional, horizontal systems, but are later on extended to multi-phase flow in two and three dimensions, and to other coordinate systems.

Conservation of mass
Consider the following one dimensional rod of porous material:



Mass conservation may be formulated across a control element of the slab, with one fluid of density ρ is flowing through it at a velocity u:



The mass balance for the control element is then written as:

$$\begin{Bmatrix}\text{Mass into the}\\ \text{element at } x\end{Bmatrix} - \begin{Bmatrix}\text{Mass out of the}\\ \text{element at } x+\Delta x\end{Bmatrix} = \begin{Bmatrix}\text{Rate of change of mass}\\ \text{inside the element}\end{Bmatrix},$$

or

$$\left\{u\rho A\right\}_x - \left\{u\rho A\right\}_{x+\Delta x} = \frac{\partial}{\partial t}\left\{\phi A \Delta x \rho\right\}.$$

Dividing by Δx, and taking the limit as Δx approaches zero, we get the conservation of mass, or continuity equation:

$$-\frac{\partial}{\partial x}\left(A\rho u\right) = \frac{\partial}{\partial t}\left(A\phi\rho\right).$$

For constant cross sectional area, the continuity equation simplifies to:

$$-\frac{\partial}{\partial x}\left(\rho u\right) = \frac{\partial}{\partial t}\left(\phi\rho\right).$$

Next, we need to replace the velocity term by an equation relating it to pressure gradient and fluid and rock properties, and the density and porosity terms by appropriate pressure dependent functions.

Conservation of momentum
Conservation of momentum is goverened by the Navier-Stokes equations, but is normally simplified for low velocity flow in porous materials to be described by the semi-empirical Darcy's equation, which for single phase, one dimensional, horizontal flow is:

$$u = -\frac{k}{\mu}\frac{\partial P}{\partial x}.$$

Alternative equations are the Forchheimer equation, for high velocity flow:

$$-\frac{\partial P}{\partial x} = u\frac{\mu}{k} + \beta u^n,$$

where n was proposed by Muscat to be 2, and the Brinkman equation, which applies to both porous and non-porous flow:

$$-\frac{\partial P}{\partial x} = u\frac{\mu}{k} - \mu\frac{\partial^2 u}{\partial x^2}.$$

Brinkman's equation reverts to Darcy's equation for flow in porous media, since the last term then normally is negligible, and to Stoke's equation for channel flow because the Darcy part of the equation then may be neglected.

In the following, we assume that Darcy's equation is valid for flow in porous media.

Constitutive equation for porous materials
To include pressure dependency in the porosity, we use the following definition of rock compressibility, which for constant temperature is written:

$$c_r = (\frac{1}{\phi})(\frac{\partial \phi}{\partial P})_T.$$

Normally, we may assume that the bulk volume of the porous material is constant, i.e. the bulk compressibility is zero. This is not always true, as witnessed by the subsidence in the Ekofisk area.

Constitutive equation for fluids
Recall the familiar fluid compressibility definition, which applies to any fluid at constant temperature:

$$c_f = -(\frac{1}{V})(\frac{\partial V}{\partial P})_T.$$

Equally familiar is the gas equation, which for an ideal gas is:

$$pV = nRT,$$

and for a real gas includes the deviation factor, Z:

$$pV = nZRT \, .$$

These descriptive equations for the fluids are frequently used in reservoir engineering applications. However, for more general purposes, such as in reservoir simulation models, we normally use either so-called *Black Oil* fluid description, or *compositional* fluid description. Below, we will review the Black Oil model.

The standard Black Oil model includes *Formation Volume Factor*, *B*, for each fluid, and *Solution Gas-Oil Ratio*, R_{so}, for the gas dissolved in oil, in addition to viscosity and density for each fluid. A modified model may also include oil dispersed in gas, r_s, and gas dissolved in water, R_{sw}. The definitions of formation volume factors and solution gas-oil ratio are:

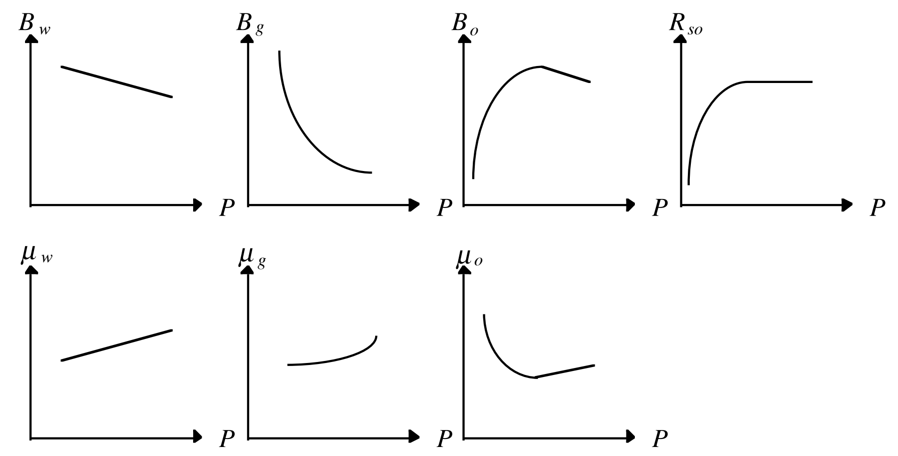
$$B = \frac{\text{volume at reservoir conditions}}{\text{volume at standard conditions}}$$

$$R_{so} = \frac{\text{volume of gas evolved from oil at standard conditions}}{\text{volume of oil at standard conditions}}$$

The density of oil at reservoir conditions is then, in terms of these parameters and the densities of oil and gas, defined as:

$$\rho_o = \frac{\rho_{oS} + \rho_{gs} R_{so}}{B_o} \, .$$

Typical pressure dependencies of the standard Black Oil parameters are:



Simple form of the flow equation and analytical solutions
In the following, we will briefly review the derivation of single phase, one dimensional, horizontal flow equation, based on continuity equation, Darcy's equation, and compressibility definitions for rock and fluid, assuming constant permeability and viscosity.

Let us substitute Darcy´s equation into the continuity equation derived above:

$$\frac{\partial}{\partial x}\left(\rho\frac{k}{\mu}\frac{\partial P}{\partial x}\right)=\frac{\partial}{\partial t}(\rho\phi)$$

The right hand side (RHS) of the equation may be expanded as:

$$\frac{\partial}{\partial t}(\rho\phi)=\rho\frac{\partial}{\partial t}(\phi)+\phi\frac{\partial}{\partial t}(\rho)$$

Since porosity and density both are functions of pressure only (assuming temperature to be constant), we may write:

$$\frac{\partial}{\partial t}(\phi)=\frac{d\phi}{dP}\frac{\partial P}{\partial t}$$

and

$$\frac{\partial}{\partial t}(\rho)=\frac{d\rho}{dP}\frac{\partial P}{\partial t}.$$

From the compressibility expressions we may obtain the following relationships:

$$\frac{d\rho}{dP}=\rho c_f \text{ and } \frac{d\phi}{dP}=\phi c_r.$$

By substituting these expressions into the equation, we obtain the following form of the right hand side of the flow equation:

$$\frac{\partial}{\partial t}(\rho\phi)=\phi\rho(c_f+c_r)\frac{\partial P}{\partial t}.$$

The left hand side of the flow equation may be expanded as follows:

$$\frac{\partial}{\partial x}\left(\rho\frac{k}{\mu}\frac{\partial P}{\partial x}\right)=\rho\frac{\partial}{\partial x}\left(\frac{k}{\mu}\frac{\partial P}{\partial x}\right)+\frac{k}{\mu}\frac{\partial P}{\partial x}\frac{\partial}{\partial x}(\rho)=\rho\frac{\partial}{\partial x}\left(\frac{k}{\mu}\frac{\partial P}{\partial x}\right)+\frac{k}{\mu}\frac{\partial P}{\partial x}\frac{d\rho}{dP}\frac{\partial P}{\partial x}$$

For now, let us assume that k=constant and μ=constant. Let us also substitute for $\frac{d\rho}{dP}=\rho c_f$. The LHS may now be written as:

$$\frac{\partial}{\partial x}\left(\rho\frac{k}{\mu}\frac{\partial P}{\partial x}\right)=\frac{\rho k}{\mu}\left[\frac{\partial^2 P}{\partial x^2}+c_f\left(\frac{\partial P}{\partial x}\right)^2\right].$$

Since c_f is small, at least for liquids, and the pressure gradient is small for the low velocity flow we normally have in reservoirs, we make the following assumption:

$$c_f\left(\frac{\partial P}{\partial x}\right)^2 << \frac{\partial^2 P}{\partial x^2}.$$

Then, our LHS simplifies to:

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$$\frac{\partial}{\partial x}\left(\rho \frac{k}{\mu}\frac{\partial P}{\partial x}\right) = \frac{\rho k}{\mu}\frac{\partial^2 P}{\partial x^2}.$$

The complete partial differential flow equation (PDE) for this simple rock-fluid system then becomes:

$$\frac{\partial^2 P}{\partial x^2} = (\frac{\phi \mu c}{k})\frac{\partial P}{\partial t},$$

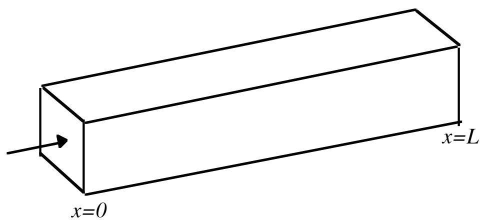
where c is the sum of the rock and fluid compressibilities.

Assumptions made in the derivation of the above PDE:

1. One dimensional flow
2. Linear flow
3. Horizontal flow
4. One phase flow
5. Darcy´s equation applies
6. Small fluid compressibility (liquid)
7. Permeability and viscosity are constants

Initial and boundary conditions

In order to solve the above equation, we need to specity one initial and two boundary conditions. The initial condition will normally specify a constant initial pressure, while the boundary conditions will either specify pressures or flow rates at two positions of the system. For our simple horizontal rod of porous material, these conditions may be specified as:



Initial condition (IC):

$$P(x, t=0) = P_i$$

Normally, the initial pressure of a horizontal system such as the one above is constant, but in principle it could be a function of position (x).

Boundary conditions (BC´s):

Pressure conditions (Dirichlet conditions) would typically be specified as:

$$P(x=0, t) = P_L$$
$$P(x=L, t) = P_R$$

The other commonly used BC´s are rate specifications (Neumann conditions). Using Darcy´s equation, flow rates would typically be specified as:

$$q_L = -\frac{kA}{\mu}\left(\frac{\partial P}{\partial x}\right)_{x=0}$$

$$q_R = -\frac{kA}{\mu}\left(\frac{\partial P}{\partial x}\right)_{x=L}$$

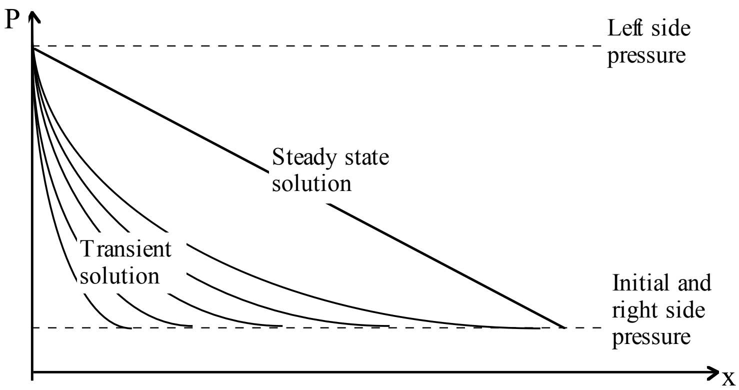
Analytical solution to the simple, linear PDE
Using the following set of initial and boundary conditions:

$$P(x,t=0) = P_i,\ P(x=0,t) = P_L \text{ and } P(x=L,t) = P_R,$$

we may obtain the following analytical solution of the transient pressure development in the porous rod above:

$$P(x,t) = P_L + (P_R - P_L)\left[\frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{n^2\pi^2}{L^2}\frac{k}{\phi\mu c}t\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

This solution is depicted graphically in the figure below.



Transient vs. steady state flow
The partial differential equation above includes time dependency through the right hand side term. Thus, it can describe transient, or time dependent flow. In the figure illustrating the solution, the system will first have a time dependent, or transient, period, where the pressure will gradually penetrate the porous material. Then, after some time, the flow reaches a state where it is no longer time dependent, and the pressure distribution is described by the straight line denoted steady state solution.

We could have reduced the partial differential equation directly to a steady state equation by setting the time dependent term on the right hand side to zero. Then the equation becomes an ordinary differential equation (ODE):

$$\frac{d^2P}{dx^2} = 0$$

By integrating this equation twice, and using the two boundary conditions to determine the integration constants, we obtain the steady state solution:

$$P(x,t) = P_L + (P_R - P_L)\frac{x}{L}.$$

which is a straight line connecting the two end pressures. As can be seen, the transient solution will reduce to this steady state expression as time becomes large.

General form of the one-phase, one-dimensional, horizontal PDE
Above we derived and solved the simplest forms of the PDE, using fluid compressibility definition as a constitutive fluid equation, and assuming constant viscosity and permeability. Generally, the Black Oil form of the fluid model is used, and the two parameters are not constants. Recall the Black Oil definition of oil density:

$$\rho_o = \frac{\rho_{oS} + \rho_{gs} R_{so}}{B_o}.$$

For undersaturated oil, the solution gas-oil ratio, R_{so}, is constant. Thus, the oil density may be written:

$$\rho_o = \frac{\text{constant}}{B_o}.$$

Similar expressions may be written for single phase gas and single phase water. Substitution of this fluid model into the continuity equation with Darcy´s equation yields a general Black Oil form of the single phase, one-dimensional, horizontal flow equation:

$$\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B}\right).$$

Multiphase flow
A continuity equation may be written for each fluid phase flowing:

$$-\frac{\partial}{\partial x}\left(\rho_l u_l\right) = \frac{\partial}{\partial t}\left(\phi \rho_l S_l\right), l = o, w, g,$$

and the corresponding Darcy equations for each phase are:

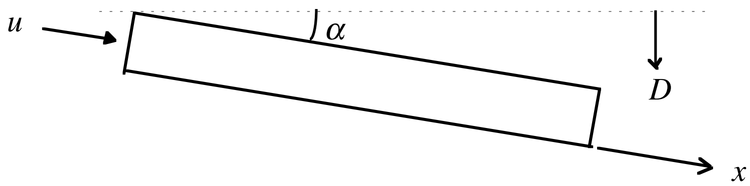
$$u_l = -\frac{k k_{rl}}{\mu_l}\frac{\partial P_l}{\partial x}, l = o, w, g,$$

where

$$P_{cow} = P_o - P_w$$
$$P_{cog} = P_g - P_o$$
$$\sum_{l=o,w,g} S_l = 1.$$

The continuity equation for gas has to be modified to include solution gas as well as free gas, and the one for oil to include dispersed oil in gas, if any.

Non-horizontal flow
For one-dimensional, inclined flow, as shown in the following figure:



the Darcy equation becomes:

$$u = -\frac{k}{\mu}\left(\frac{\partial P}{\partial x} - \rho g \frac{dD}{dx}\right),$$

or, in terms of dip angle, α, and hydrostatic gradient:

$$u = -\frac{k}{\mu}\left(\frac{\partial P}{\partial x} - \gamma \sin(\alpha)\right),$$

where $\gamma = \rho g$ is the hydrostatic gradient of the fluid.

Multidimensional flow

The continuity equation for one-phase, three-dimensional flow in cartesian coordinates, is:

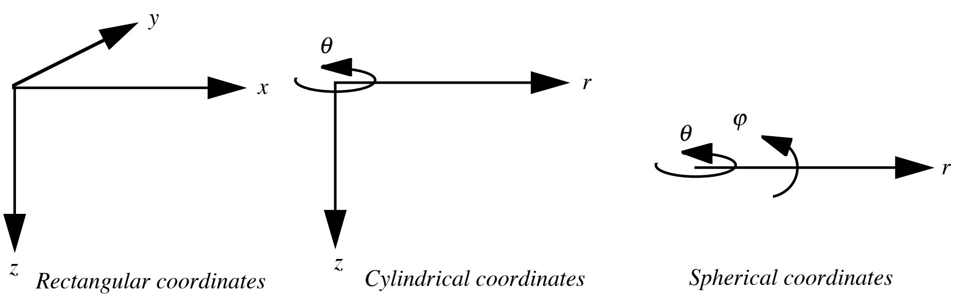
$$-\frac{\partial}{\partial x}(\rho u_x) - \frac{\partial}{\partial y}(\rho u_y) - \frac{\partial}{\partial z}(\rho u_z) = \frac{\partial}{\partial t}(\phi\rho),$$

and the corresponding Darcy equations are:

$$u_x = -\frac{k_x}{\mu}\left(\frac{\partial P}{\partial x} - \gamma \frac{dD}{dx}\right)$$
$$u_y = -\frac{k_y}{\mu}\left(\frac{\partial P}{\partial y} - \gamma \frac{\partial D}{\partial y}\right)$$
$$u_z = -\frac{k_z}{\mu}\left(\frac{\partial P}{\partial z} - \gamma \frac{\partial D}{\partial z}\right).$$

Coordinate systems

Normally, we use either a rectangular coordinate system, or a cylindrical coordinate system in reservoir engineering



Rectangular coordinates *Cylindrical coordinates* *Spherical coordinates*

In operator form, the continuity and the Darcy equations for one-phase flow may be written:

$$-\nabla \cdot (\rho \vec{u}) = \frac{\partial}{\partial t}(\phi \rho)$$

$$\vec{u} = -\frac{K}{\mu}(\nabla P - \gamma \nabla D),$$

where the operators are defined as:

<u>rectangular coordinates</u>

$$\nabla \cdot (\ ) = \frac{\partial}{\partial x}(\ ) + \frac{\partial}{\partial y}(\ ) + \frac{\partial}{\partial z}(\ ) \qquad \text{(divergence)}$$

$$\nabla(\ ) = \hat{i}\frac{\partial}{\partial x}(\ ) + \hat{j}\frac{\partial}{\partial y}(\ ) + \hat{k}\frac{\partial}{\partial z}(\ ) \qquad \text{(gradient)}$$

<u>cylindrical coordinates</u>

$$\nabla \cdot (\ ) = \frac{1}{r}\frac{\partial}{\partial r}(r(\ )) + \frac{1}{r}\frac{\partial}{\partial \theta}(\ ) + \frac{\partial}{\partial z}(\ )$$

$$\nabla(\ ) = \hat{i}\frac{\partial}{\partial r}(\ ) + \hat{j}\frac{\partial}{\partial \theta}(\ ) + \hat{k}\frac{\partial}{\partial z}(\ )$$

<u>spherical coordinates</u>

$$\nabla \cdot (\ ) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(\ )) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}((\ )\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\ )$$

$$\nabla(\ ) = \hat{i}\frac{\partial}{\partial r}(\ ) + \hat{j}\frac{\partial}{\partial \theta}(\ ) + \hat{k}\frac{\partial}{\partial \varphi}(\ )$$