DIRECT SOLUTION OF LINEAR SETS OF EQUATIONS

As an illustration of solution of linear equations, consider the following set of 3 equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \tag{1}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2 \tag{2}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3 \tag{3}$$

The ***Gauss elimination method*** starts by multiplying Eqn. (1) by $-a_{21}/a_{11}$, and then adds the resulting equation to Eqn. (2). The new Eqn. (2) becomes:

$$a'_{22}x_2 + a'_{23}x_3 = d'_2$$

Next step is to multiply Eqn. (1) by $-a_{31}/a_{11}$ and then add the resulting equation to Eqn. (3). The new Eqn. (3) becomes:

$$a'_{32}x_2 + a'_{33}x_3 = d'_3$$

The set of equations has now become:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \tag{4}$$

$$a'_{22}x_2 + a'_{23}x_3 = d'_2 \tag{5}$$

$$a'_{32}x_2 + a'_{33}x_3 = d'_3 \tag{6}$$

The next step is to multiply Eqn. (5) by $-a'_{32}/a'_{22}$ and then add the resulting equation to Eqn. (6). Our set of equations is now:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \tag{7}$$

$$a'_{22}x_2 + a'_{23}x_3 = d'_2 \tag{8}$$

$$a''_{33}x_3 = d''_3 \tag{9}$$

The above elimination process is called *forward elimination*. Eqn. (9) can now be solved directly for x_3:

$$x_3 = d''_3/a''_{33} \tag{10}$$

We shall now perform a *backward substitution*. This simply means that as each unknown is computed, it is substituted into the equation above, and a additional unknown can be found. For Eqs. (7) and (8), this is done as follows:

$$x_2 = (d'_2 - a'_{23}x_3)/a'_{22} \tag{11}$$

$$x_1 = (d_1 - a_{12}x_2 - a_{13}x_3)/a_{11} \tag{12}$$

Based on the procedure above, a general formula for solving a set of equations consisting of *n* equations and *n* unknowns using ***Gaussian elimination method*** may be derived:

1) Forward elimination

$$a_{ij} = a_{ij} + a_{kj}\left(-a_{ik}/a_{kk}\right), \qquad \{[(j=k+1,n), \quad i=k+1,n], \quad k=1,n\text{-}1\}$$

$$d_i = d_i + d_k\left(-a_{ik}/a_{kk}\right), \qquad [(i=k+1,n), \quad k=1,n-1]$$

2) Backward substitution

$$x_i = \left(d_i - \sum_{j=i+1}^{n} a_{ij}x_j\right)/a_{ii}, \quad i=n,\ldots,1$$

Banded coefficient matrix for regular grid systems

The ***one-dimensional*** finite difference equations derived for the following grid system

1			*i-1*	*i*	*i+1*			*N*
•	•	•	•	•	•	•	•	•

we generally solve the 3-diagonal pressure equation

$$a_i Po_{i-1} + b_i Po_i + c_i Po_{i+1} = d_i, \quad i=1,N$$

Graphically, the coefficient matrix may be presented as

$$\begin{vmatrix}
b & c & & & & & & & & & & & & & \\
a & b & c & & & & & & & & & & & & \\
& a & b & c & & & & & & & & & & & \\
& & a & b & c & & & & & & & & & & \\
& & & a & b & c & & & & & & & & & \\
& & & & a & b & c & & & & & & & & \\
& & & & & a & b & c & & & & & & & \\
& & & & & & a & b & c & & & & & & \\
& & & & & & & a & b & c & & & & & \\
& & & & & & & & a & b & c & & & & \\
& & & & & & & & & a & b & c & & & \\
& & & & & & & & & & a & b & c & & \\
& & & & & & & & & & & a & b & c & \\
& & & & & & & & & & & & a & b & c \\
& & & & & & & & & & & & & a & b
\end{vmatrix}$$

As can be seen, the compact band of this coefficient matrix only consists of 3 non-zero diagonals. Thus, our Gaussian elimination algorithm may be simplified to operate only on the band itself, as shown in the following:

Forward elimination, i=2,N

$$b_i = b_i - c_{i-1}\left(a_i/b_{i-1}\right)$$

$$d_i = d_i - d_{i-1}\left(a_i/b_{i-1}\right)$$

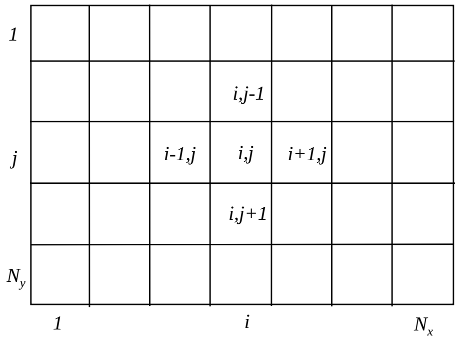
Computation of P_N

$$P_N = d_N / b_N$$

Backward substitution, i=N-1,1

$$P_i = d_i - c_i P_{i+1} / b_i \, .$$

For a ***two-dimensional*** grid system



the set of linear equations to be solved for pressures is

$$e_{i,j}P_{i,j-1} + a_{i,j}P_{i-1,j} + b_{i,j}P_{i,j} + c_{i,j}P_{i+1,j} + f_{i,j}P_{i,j+1} = d_{i,j} \qquad i = 1, N_x, j = 1, N_y$$

Again, we have a banded matrix, but now with 5 non-zero diagonals. The band is not compact, as the two outer diagonals are positioned apart from the rest. For numbering along the *i*-direction, we get the coefficient matrix at right below:

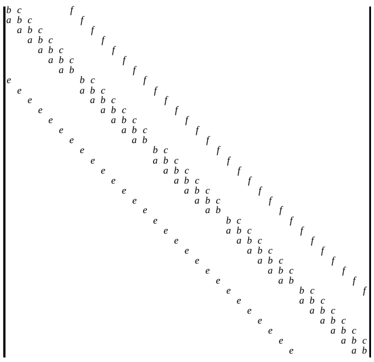
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35

The bandwidth of the above system may be computed as

$$N_b = 2N_x + 1$$

If we had numered the grid blocks along the j-direction instead of along the i-direction as we did above, the coefficients *e* and *a*, and *f* and *c*, would have changed places:

1	6	11	16	21	26	31
2	7	12	17	22	27	32
3	8	13	18	23	28	33
4	9	14	19	24	29	34
5	10	15	20	25	30	35



and the bandwidth would have become

$$N_b = 2N_y + 1$$

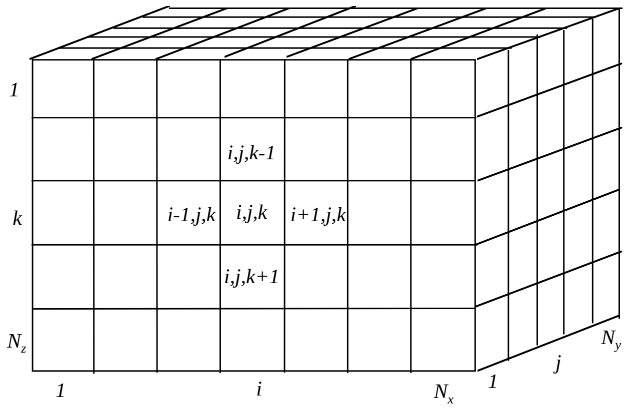
Since the number of operations involved in a Gaussian elimination solutions is approximately

$$no.\ of\, operations \approx N_b^2 N_x N_y,$$

it is important to number the grid system along the shortest direction. A check of this may be included in the Gaussian elimination algorithm.

Even though the matrix contains only 5 non-zero diagonals, the zeros between the diagonals will quickly be filled during the forward elimination process. Therefore, we need to let the elimination process include the entire band. Banded elimination routines are readily available[1], and will not be discussed more here.

For a ***three-dimensional*** grid system

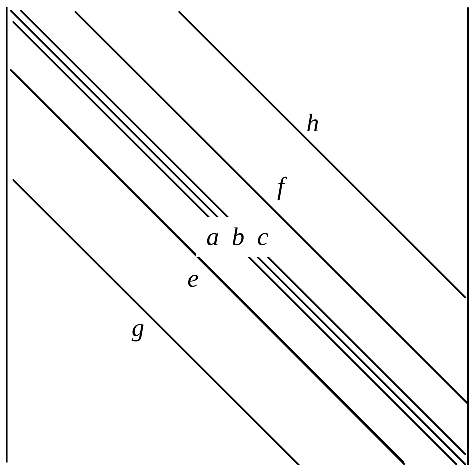


the set of linear equations to be solved for pressures has 7 non-zero diagonals on the left hand side:

$$g_{i,j,k}P_{i,j,k-1} + e_{i,j,k}P_{i,j-1,k} + a_{i,j,k}P_{i-1,j,k} + b_{i,j,k}P_{i,j,k}$$

$$+ c_{i,j,k}P_{i+1,j,k} + f_{i,j,k}P_{i,j+1,k} + h_{i,j,k}P_{i,j,k+1} = d_{i,j,k} \qquad i = 1, N_x, j = 1, N_y, k = 1, N_z$$

resulting in the following coefficient structure:



The bandwidth of the three-dimensional system, if numbered along the x- and z-directions first, may be computed as

$$N_b = 2N_xN_z + 1$$

[1] Press, W. H. et al.: Numerical Receipes in Fortran, 2nd. Ed., Cambridge, N.Y., 1992, p.22

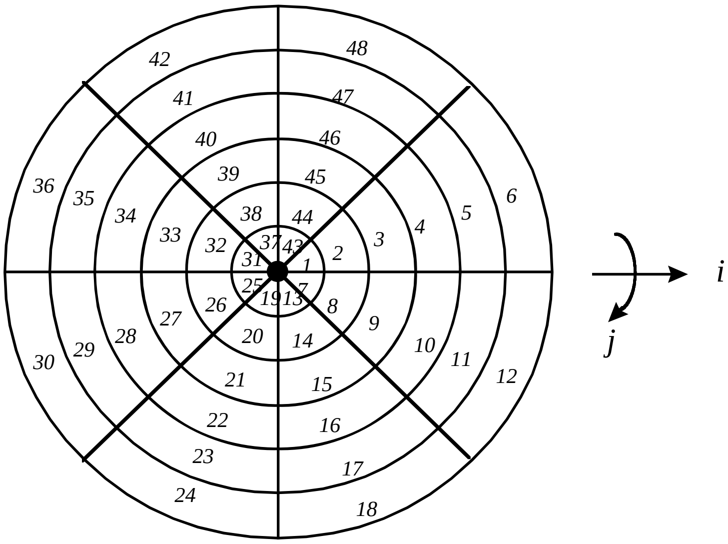
Again, in order to reduce the bandwidth, and thus the number of operations involved in the solution, we should number the smallest plane first. For this system, the number of operations is approximately

$$no.\ of\ operations \approx 4N_x^3 N_y N_z^3 \, .$$

Thus, even for small systems in three dimensions, the number of operations becomes large. For instance, for a small 1000 block system where $N_x = N_y = N_z = 10$, the number of operations for a single solution is around 40 millions. Therefore, direct solution is normally limited to small systems. For large systems, iterative methods are required.

Coefficient matrix for circular grids

A two-dimensional grid that requires special attention is the $r-\theta$ system shown below.



The set of linear equations for this system is, of course, identical to the one for the two-dimensional, rectangular coordinates system:

$$e_{i,j}P_{i,j-1} + a_{i,j}P_{i-1,j} + b_{i,j}P_{i,j} + c_{i,j}P_{i+1,j} + f_{i,j}P_{i,j+1} = d_{i,j} \qquad i = 1, N_x, \, j = 1, N_y$$

The numbering sequence used in the example above is similar to the one used for the rectangular system below:

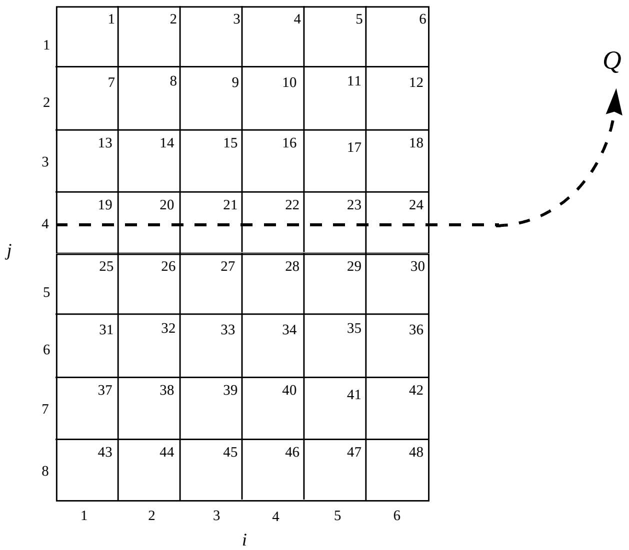
j \ *i*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	7	8	9	10	11	12
3	13	14	15	16	17	18
4	19	20	21	22	23	24
5	25	26	27	28	29	30
6	31	32	33	34	35	36
7	37	38	39	40	41	42
8	43	44	45	46	47	48

However, there is a significant difference in the coefficient matrix structures of these two grids. While all *f*-coefficients are zero for row 8 in the case of the rectangular grid, the equivalent "row" in the cylindrical case is connected to "row" 1. Thus, the structure of the coefficient matrix for the cylindrical grid becomes:

Although the required modifications to the banded Gaussian elimination routine are minor, most standard routines do not have provisions for this type of structure.

Effects of wells on coefficient matrix

Using the rectangular grid above as example, and single phase flow, let us add a well to the grid, with perforations in all grid blocks of row *J:*



The coefficient matrix of this system will of course be affected by the well. In case the production rate is constant, the bottom hole pressure will be an additional unknown that must be included in the solution. We add a term to the linear equations:

$$e_{i,j}P_{i,j-1} + a_{i,j}P_{i-1,j} + b_{i,j}P_{i,j} + c_{i,j}P_{i+1,j} + f_{i,j}P_{i,j+1} + w_{i,j}P_J^{bh} = d_{i,j} \qquad i = 1, N_x, j = 1, N_y$$

where

$w_{i,j} \neq 0$ for perforated grid blocks,

and

$$w_{i,j} = 0 \quad \text{for non-perforated grid blocks,}$$

In addition, we add a well constraint equation to the system of equations:

$$Q_J = \sum_{perf} q_{i,J}$$

where

$$q_{i,J} = WC_{i,J}\lambda_{i,J}(P_{i,J} - P_J^{bh})$$

The well term will of course also alter the *b*-term of the linear equation.

With the well term included in the discrete flow equations, and adding a well constraint equation, the system of linear equations become:

Again, the required modifications to the banded Gaussian elimination routine are minor, at least for the simple case above, but most standard routines do not have provisions for this type of structure.