DISCRETIZATION OF THE FLOW EQUATIONS

As we already have seen, finite difference approximations of the partial derivatives appearing in the flow equations may be obtained from Taylor series expansions. We shall now proceed to derive approximations for all terms needed in reservoir simulation.

Spatial discretization

Constant grid block sizes

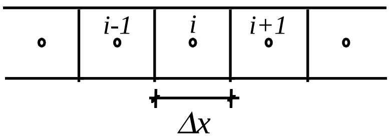
We showed that the approximation of the second derivative of pressure may be obtained by forward and backward expansions of pressure:

$$P(x+\Delta x,t)=P(x,t)+\frac{\Delta x}{1!}P'(x,t)+\frac{(\Delta x)^2}{2!}P''(x,t)+\frac{(\Delta x)^3}{3!}P'''(x,t)+\ldots..$$
$$P(x-\Delta x,t)=P(x,t)+\frac{(-\Delta x)}{1!}P'(x,t)+\frac{(-\Delta x)^2}{2!}P''(x,t)+\frac{(-\Delta x)^3}{3!}P'''(x,t)+\ldots..$$

to yield

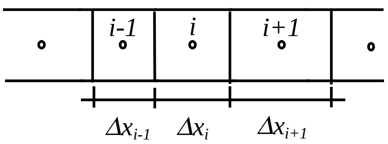
$$(\frac{\partial^2 P}{\partial x^2})_i^t=\frac{P_{i+1}^t-2P_i^t+P_{i-1}^t}{(\Delta x)^2}+O(\Delta x^2)\,,$$

which applies to the following grid system:



Variable grid block sizes

A more realistic grid system is one of variable block lengths, which will be the case in most simulations. Such a grid would enable finer description of geometry, and better accuracy in areas of rapid changes in pressures and saturations, such as in the neighborhood of production and injection wells. For the simple one-dimensional system, a variable grid system would be:



the Taylor expansions become (dropping the time index):

$$P_{i+1}=P_i+\frac{(\Delta x_i+\Delta x_{i+1})/2}{1!}P_i'+\frac{\left[(\Delta x_i+\Delta x_{i+1})/2\right]^2}{2!}P_i''+\frac{\left[(\Delta x_i+\Delta x_{i+1})/2\right]^3}{3!}P_i'''\ldots..$$

$$P_{i-1}=P_i+\frac{-(\Delta x_i+\Delta x_{i-1})/2}{1!}P_i'+\frac{\left[-(\Delta x_i+\Delta x_{i-1})/2\right]^2}{2!}P_i''+\frac{\left[-(\Delta x_i+\Delta x_{i-1})/2\right]^3}{3!}P_i'''\ldots..$$

to yield

$$P_i''=4\,\frac{2\left(\dfrac{\Delta x_i+\Delta x_{i-1}}{2\Delta x_i+\Delta x_{i+1}+\Delta x_{i-1}}\right)P_{i+1}-2P_i+2\left(\dfrac{\Delta x_i+\Delta x_{i+1}}{2\Delta x_i+\Delta x_{i+1}+\Delta x_{i-1}}\right)P_{i-1}}{(\Delta x_i+\Delta x_{i+1})(\Delta x_i+\Delta x_{i-1})}+O(\Delta x)\,.$$

An important difference is now that the error term is of only first order, due to the different block sizes.

However, normally the flow terms in our simulation equations will be of the type $\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]$, where $f(x)$ includes permeability, mobility and flow area. Therefore, we will instead derive a central approximation for the first derivative, and apply it twice to this flow term.

$$\left[f(x)\frac{\partial P}{\partial x}\right]_{i+1/2} = \left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{\Delta x_i/2}{1!}\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{(\Delta x_i/2)^2}{1!}\frac{\partial^2}{\partial x^2}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \ldots$$

and

$$\left[f(x)\frac{\partial P}{\partial x}\right]_{i-1/2} = \left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{-\Delta x_i/2}{1!}\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \frac{(-\Delta x_i/2)^2}{1!}\frac{\partial^2}{\partial x^2}\left[f(x)\frac{\partial P}{\partial x}\right]_i + \ldots$$

which yields

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i = \frac{\left[f(x)\frac{\partial P}{\partial x}\right]_{i+1/2} - \left[f(x)\frac{\partial P}{\partial x}\right]_{i-1/2}}{\Delta x_i} + O(\Delta x^2).$$

Similarly, we may obtain the following expressions:

$$\left(\frac{\partial P}{\partial x}\right)_{i+1/2} = \frac{P_{i+1} - P_i}{(\Delta x_i + \Delta x_{i+1})/2} + O(\Delta x)$$

and

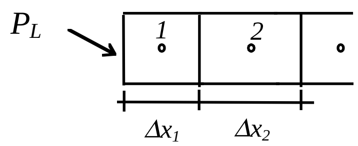
$$\left(\frac{\partial P}{\partial x}\right)_{i-1/2} = \frac{P_i - P_{i-1}}{(\Delta x_i + \Delta x_{i-1})/2} + O(\Delta x)\,.$$

As we can see, due to the different block sizes, the error terms for the last two approximations are again of first order only. By inserting these expressions into the previous equation, we get the following approximation for the flow term:

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i = \frac{2f(x)_{i+1/2}\dfrac{(P_{i+1} - P_i)}{(\Delta x_{i+1} + \Delta x_i)} - 2f(x)_{i-1/2}\dfrac{(P_i - P_{i-1})}{(\Delta x_i + \Delta x_{i-1})}}{\Delta x_i} + O(\Delta x)\,.$$

Boundary conditions

We have seen earlier that we have two types of boundary conditions, *Dirichlet*, or pressure condition, and *Neumann*, or rate condition. If we first consider a pressure condition at the left side of our slab, as follows:



then we will have to modify our approximation of the first derivative at the left face, $i = 1/2$, to become a forward difference instead of a central difference:

$$\left(\frac{\partial P}{\partial x}\right)_{1/2} = \frac{P_1 - P_L}{(\Delta x_1)/2} + O(\Delta x)\,,$$

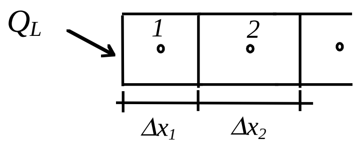
and the flow term approximation thus becomes:

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_1 = \frac{2f(x)_{1\,1/2}\dfrac{(P_2-P_1)}{(\Delta x_2+\Delta x_1)} - 2f(x)_{1/2}\dfrac{(P_1-P_L)}{(\Delta x_1)}}{\Delta x_1} + O(\Delta x)\,.$$

With a pressure P_R specified at the right hand face, we get a similar approximation for block N:

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_N = \frac{2f(x)_{N+1/2}\dfrac{(P_R-P_N)}{(\Delta x_N)} - 2f(x)_{N-1/2}\dfrac{(P_N-P_{N-1})}{(\Delta x_N+\Delta x_{N-1})}}{\Delta x_N} + O(\Delta x)\,.$$

For a flow rate specified at the left side (injection/production),



we make use of Darcy's equation:

$$Q_L = -\frac{kA}{\mu B}\left(\frac{\partial P}{\partial x}\right)_{1/2}$$

or

$$\left(\frac{\partial P}{\partial x}\right)_{1/2} = -Q_L\frac{\mu B}{kA}\,.$$

Then, by substituting into the approximation, we get:

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_1 = \frac{2f(x)_{1\,1/2}\dfrac{(P_2-P_1)}{(\Delta x_2+\Delta x_1)} + Q_L\dfrac{\mu B}{kA}}{\Delta x_1} + O(\Delta x)\,.$$

With a rate Q_R specified at the right hand face, we get a similar approximation for block N:

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_N = \frac{-Q_R\dfrac{\mu B}{kA} - 2f(x)_{N-1/2}\dfrac{(P_N-P_{N-1})}{(\Delta x_N+\Delta x_{N-1})}}{\Delta x_N} + O(\Delta x)\,.$$

For the case of a no-flow boundary between blocks i and $i+1$, the flow terms for the two blocks become:

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i = -2f(x)_{i-1/2}\frac{(P_i-P_{i-1})}{\Delta x_i(\Delta x_i+\Delta x_{i-1})} + O(\Delta x)$$

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_{i+1} = 2f(x)_{i+1\,1/2}\frac{(P_{i+2}-P_{i+1})}{\Delta x_{i+1}(\Delta x_{i+2}+\Delta x_{i+1})} + O(\Delta x)$$

Time discretization

We showed earlier that by expansion backward in time:

$$P(x,t) = P(x,t+\Delta t) + \frac{-\Delta t}{1!}P'(x,t+\Delta t) + \frac{(-\Delta t)^2}{2!}P''(x,t+\Delta t) + \frac{(-\Delta t)^3}{3!}P'''(x,t+\Delta t) + \ldots..$$

the following backward difference approximation with first order error term is obtained:

$$(\frac{\partial P}{\partial t})_i^{t+\Delta t} = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t) .$$

An expansion forward in time:

$$P(x,t+\Delta t) = P(x,t) + \frac{\Delta t}{1!} P'(x,t) + \frac{(\Delta t)^2}{2!} P''(x,t) + \frac{(\Delta t)^3}{3!} P'''(x,t) + ...$$

yields a forward approximation, again with first order error term:

$$(\frac{\partial P}{\partial t})_i^t = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t) .$$

Finally, expanding in both directions:

$$P(x,t+\Delta t) = P(x,t+\tfrac{\Delta t}{2}) + \frac{\frac{\Delta t}{2}}{1!} P'(x,t+\tfrac{\Delta t}{2}) + \frac{(\frac{\Delta t}{2})^2}{2!} P''(x,t+\tfrac{\Delta t}{2}) + \frac{(\frac{\Delta t}{2})^3}{3!} P'''(x,t+\tfrac{\Delta t}{2}) + ...$$

$$P(x,t) = P(x,t+\tfrac{\Delta t}{2}) + \frac{-\frac{\Delta t}{2}}{1!} P'(x,t+\tfrac{\Delta t}{2}) + \frac{(-\frac{\Delta t}{2})^2}{2!} P''(x,t+\tfrac{\Delta t}{2}) + \frac{(-\frac{\Delta t}{2})^3}{3!} P'''(x,t+\tfrac{\Delta t}{2}) + ...$$

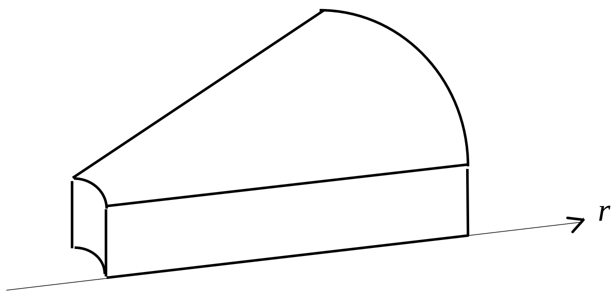
we get a central approximation, with a second order error term:

$$(\frac{\partial P}{\partial t})_i^{t+\frac{\Delta t}{2}} = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t)$$

The time approximation used as great influence on the solutions of the equations. Using the simple case of the flow equation and constant grid size as example, we may write the difference form of the equation for the three cases above.

Cylindrical coordinates

Another type of variable flow area is one induced by the coordinate system used. The most common is the one occurring in radial flow as illustrated below:



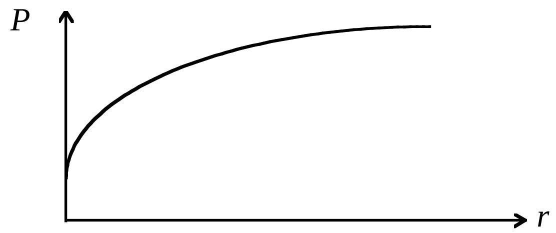
Here, even if the block height is constant, the flow area is a function of radius, and for a full cylinder (360 degrees) the area is:

$$A = 2\pi rh .$$

The continuity equation thus becomes, assuming constant height:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{k}{\mu B}\frac{\partial P}{\partial r}\right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B}\right).$$

In a radial system, the pressure distribution will be logarithmic of nature, with most of the pressure drop occurring close to the center, where the flow area is small:



Since our discretization formulas are more accurate the more linear the pressure distribution is, it is clear that if we discretize the radial flow term using the same approximations as for the linear equation, the error will be larger. Therefore, for the radial flow equation, we will first make the following transformation of the r-coordinate into a u-coordinate:

$$u = ln(r).$$

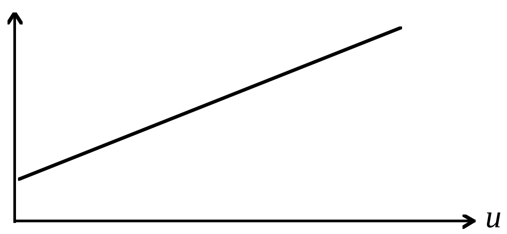
Thus,

$$\frac{du}{dr} = \frac{1}{r}$$

and

$$r = e^u.$$

The effect of this transformation is that the logarithmic pressure distribution in the radial direction becomes linear along the u-coordinate:



Transformation of the radial flow equation by substitution for $u = ln(r)$ yields:

$$e^{-u} \frac{\partial}{\partial u}\left(e^u \frac{k}{\mu B} \frac{\partial P}{\partial u} \frac{du}{dr} \right) \frac{du}{dr} = \frac{\partial}{\partial t}\left(\frac{\phi}{B} \right),$$

or

$$e^{-2u} \frac{\partial}{\partial u}\left(\frac{k}{\mu B} \frac{\partial P}{\partial u} \right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B} \right).$$

This equation is more linear in u, and except for the term e^{-2u} in front of the flow term, it is identical to the linear flow equation. We will therefore adopt the same approximation of the flow term in respect to u for the equation above as we used in respect to x for the linear equation, with the modification for the e^{-2u} term:

$$Tu_{i+1/2} = \frac{2e^{-2u_i}}{\Delta u_i(\Delta u_{i+1} + \Delta u_i)} \left(\frac{k}{\mu B} \right)_{i+1/2}$$

$$Tu_{i-1/2} = \frac{2e^{-2u_i}}{\Delta u_i(\Delta u_{i-1} + \Delta u_i)} \left(\frac{k}{\mu B} \right)_{i-1/2}.$$

Substituting back for $r = e^u$, we get the following expressions for the radial transmissibilities:

$$Tr_{i+1/2} = \frac{2r_i^{-2}}{ln\left(\frac{r_{i+1/2}}{r_{i-1/2}} \right) ln\left(\frac{r_{i+11/2}}{r_{i-1/2}} \right)} \left(\frac{k}{\mu B} \right)_{i+1/2}$$

$$Tr_{i-1/2} = \frac{2 r_i^{-2}}{ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right) ln\left(\frac{r_{i-1/2}}{r_{i-11/2}}\right)} \left(\frac{k}{\mu B}\right)_{i-1/2}.$$

The harmonic averages for permeability in radial direction may be derived in a similar fashion from the linear formula:

$$k_{i+1/2} = \frac{ln\left(\frac{r_{i+11/2}}{r_{i+1/2}}\right)}{\frac{1}{k_{i+1}} ln\left(\frac{r_{i+11/2}}{r_{i+1/2}}\right) + \frac{1}{k_i} ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)}$$

and

$$k_{i-1/2} = \frac{ln\left(\frac{r_{i+1/2}}{r_{i-11/2}}\right)}{\frac{1}{k_{i-1}} ln\left(\frac{r_{i-1/2}}{r_{i-11/2}}\right) + \frac{1}{k_i} ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)}.$$

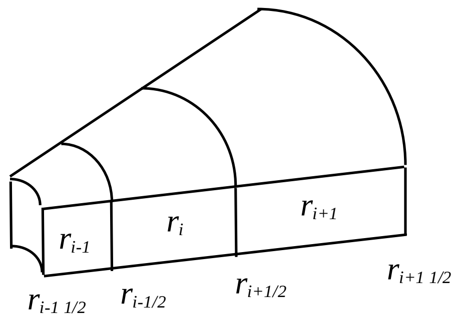
Expressions for average mobility terms become:

$$\lambda_{i+1/2} = \frac{ln\left(\frac{r_{i+11/2}}{r_{i+1/2}}\right)\lambda_{i+1} + ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)\lambda_i}{ln\left(\frac{r_{i+11/2}}{r_{i-1/2}}\right)}$$

and

$$\lambda_{i-1/2} = \frac{ln\left(\frac{r_{i-1/2}}{r_{i-11/2}}\right)\lambda_{i-1} + ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right)\lambda_i}{ln\left(\frac{r_{i+1/2}}{r_{i-11/2}}\right)}.$$

These formulas apply to the radial grid block system shown below:



The position of the grid block centers, relative to the block boundaries, may be computed using the midpoint between the *u*-coordinate boundaries:

$$u_i = (u_{i+1/2} + u_{i-1/2})/2,$$

or, in terms of radius:

$$r_i = \sqrt{r_{i+1/2} r_{i-1/2}}.$$

This is the geometric average of the block boundary radii.

Frequently in simulation of flow in the radial direction, the grid blocks sizes are chosen such that:

$$\Delta u_i = (u_{i+1/2} - u_{i-1/2}) = constant$$

or

$$ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right) = constant\ ,$$

which for a system of N grid blocks and well and external radii of r_w and r_e, respectively, implies that

$$N \cdot ln\left(\frac{r_{i+1/2}}{r_{i-1/2}}\right) = ln\left(\frac{r_e}{r_w}\right)$$

or

$$\frac{r_{i+1/2}}{r_{i-1/2}} = \left(\frac{r_e}{r_w}\right)^{1/N} = constant\ .$$

This is the formula for *logarithmic grid block sizes*, and is often used in reservoir simulation of well behavior. If grid sizes have to conform to other specifications, such as well damage radius, the above formula may still be useful as a guide to the block sizes.

For the ***right hand side*** of the difference equation, the above changes will have no effect provided that the height is constant. Thus, it will be identical to the one for the linear system.