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IMPES Stability: The Stable Step

K.H. Coats, SPE, Coats Engineering, Inc.

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Abstract

An Impes stability criterion is derived for multidimensional three-phase flow for black oil and compositional models. The grid may be structured or unstructured. Tensor considerations are neglected. The criterion can be used to set the time steps in an Impes formulation or as a switching criterion in an adaptive implicit model.

The criterion extends previous work by accounting for three-phase flow, including capillary, gravity and viscous forces, with all the possible cocurrent and countercurrent flow configurations in a general grid. The criterion derivation uses stability theory, to the limits of its applicability, augmented by numerical experimentation, including extensive one-dimensional tests and numerous field study datasets.

Introduction

A reservoir simulation model consists of $N*N_c$ nonlinear difference equations which express conservation of mass of N_c components in each of N grid blocks comprising the reservoir. The form of each of these equations, for a given block, is

$$M_{I,n+1} - M_{I,n} = \Delta t * \{\text{sum (interblock flow rates) - sum (well rates)}\} \quad \ldots\ldots\ldots\ldots (1)$$

where $M_{I,n}$ is the mass of component I in the block at time level n. The first sum is over all block neighbors; the second sum is over all wells completed in the block. Well terms are assumed to be treated fully implicitly and are dropped from consideration.

The Impes formulation[1,2] treats the interblock flow rates implicitly in pressure, but explicitly in saturations and compositions. This explicit treatment gives rise to a conditional stability for Impes,

$$\frac{F_i \Delta t}{V_{pi}} < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where Δt is maximum stable timestep and F_i is some function of rates and reservoir and fluid properties. This paper derives the function F_i for compositional and black oil models, accounting for viscous, gravity, and capillary pressure forces in cocurrent and countercurrent 3-phase flow. The grid may be unstructured, or structured (e.g. Cartesian) with or without non-neighbor connections. The flow may be one-dimensional (1D), two-dimensional (2D) or three-dimensional (3D). Tensor considerations are neglected. The Condition 2 gives a different stable step value for each block. In an Impes model, the time step used is the minimum of all blocks' stable step values. In an AIM (adaptive implicit) model,[3] each block's stable step size can be used to determine if the block needs implicit treatment. The effects on stable step size of (a) interphase mass transfer and (b) the pressure and composition dependence of fluid properties are assumed small and neglected.

Two functions F_i are derived for use in the Condition 2. The first relates to effects of explicit treatment of the saturation-dependent terms (relative permeability and capillary pressure) in the interblock flow rates. The second relates to the explicit treatment of compositions in the interblock flow rates.

Derivations of the function F_i are lengthy and, at various points, tedious. This tends to obscure the simplicity and low cpu expense associated with the final results. Therefore, a Summary section gives the final results, followed by sections describing the derivations.

Summary

For the unstructured grid case, the subscript i denotes a grid block and the subscript j denotes one of its neigbors. Derivations using a Cartesian grid use subscipts i, j, k as the grid block indices in the x, y, and z directions, respectively. In all equations throughout the paper, each phase mobility and its derivatives are evaluated at the upstream block for the phase.

Maximum Stable Step Due to Explicit k_r and P_c. For three-phase multidimensional flow, for each grid block i, the maximum stable time step is limited by the Condition 2 with

$$F_i = 1/2\ |f11_i + f22_i + \sqrt{(f11_i + f22_i)^2 - 4\det(F_i)}\ | \quad (3)$$

where each f is a sum of J_i terms, one for each of the J_i neighbors of block i, $j = 1,2,...,J_i$. For example, $f11_i = \sum_{j=1}^{J_i} f11_{ij}$.

$$f11_{ij} = T_{ij}\,[(\lambda_o+\lambda_g)\lambda'_{ww}\,|\Delta\Phi_w| - \lambda_w\lambda'_{ow}\,|\Delta\Phi_o| - \lambda_w(\lambda_o+\lambda_g)(P'_{cwoi}+P'_{cwoj})]/\lambda_t \quad \ldots\ldots\ldots (4a)$$

$$f12_{ij} = -T_{ij}\,[\lambda_w\lambda'_{og}\,|\Delta\Phi_o| + \lambda_w\lambda'_g\,|\Delta\Phi_g| - (\lambda_o+\lambda_g)\lambda'_{wg}\,|\Delta\Phi_w| + \lambda_w\lambda_g(P'_{cgoi}+P'_{cgoj})]/\lambda_t \quad \ldots\ldots\ldots\ldots (4b)$$

$$f21_{ij} = -T_{ij}\,[\lambda_g\lambda'_{ww}\,|\Delta\Phi_w| + \lambda_g\lambda'_{ow}\,|\Delta\Phi_o| - \lambda_g\lambda_w(P'_{cgoi}+P'_{cgoj})]/\lambda_t \quad \ldots\ldots\ldots\ldots (4c)$$

$$f22_{ij} = T_{ij}\,[-\lambda_g\lambda'_{og}\,|\Delta\Phi_o| + (\lambda_w+\lambda_o)\lambda'_g\,|\Delta\Phi_g| - \lambda_g\lambda'_{wg}\,|\Delta\Phi_w| + \lambda_g(\lambda_o+\lambda_w)(P'_{cgoi}+P'_{cgoj})]/\lambda_t \quad (4d)$$

$$\det(F_i) = f11_i f22_i - f12_i f21_i \quad \ldots\ldots\ldots\ldots (4e)$$

The flow potential definitions are

$$\Delta\Phi_w = \Delta p - \gamma_w\Delta Z - \Delta P_{cwo} \quad \ldots\ldots\ldots\ldots\ldots (5a)$$

$$\Delta\Phi_o = \Delta p - \gamma_o\Delta Z \quad \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

$$\Delta\Phi_g = \Delta p - \gamma_g\Delta Z + \Delta P_{cgo} \quad \ldots\ldots\ldots\ldots\ldots (5c)$$

where ΔX is $X_j - X_i$, X being any of p, Z, P_{cwo} and P_{cgo}. For each neighbor j, the four frs_{ij} (rs=11,12,21,22) values are stored in the four frs arrays at the gas phase upstream block position, if λ_g is not zero. If λ_g is 0, they are stored at the water phase upstream block position. The frs_i satisfy $f11_i > 0$, $f22_i > 0$, and $\det(F_i) > 0$, which provide a check on coding errors.

The F_i given by Eq. 3 accounts for viscous, gravity, and capillary forces, and for all possible cocurrent and countercurrent flows of three phases between two blocks. The above equations collapse to simple forms for each of the three two-phase cases. For example, for 1D two-phase gas-oil flow, $\lambda_w = \lambda'_{wg} = \lambda'_{ow} = \lambda'_{ww} = 0$, which give $f11 = f12 = f21 = 0$ and

$$F = f22 = T\,[\lambda_o\lambda'_g\,|\Delta\Phi_g| - \lambda_g\lambda'_{og}\,|\Delta\Phi_o| + \lambda_g\lambda_o(P'_{cgoi}+P'_{cgoj})]/(\lambda_o+\lambda_g) \quad (6)$$

If P_{cgo} and gravity are neglected, this further reduces to $F = qf'_g$, giving the well known result

$$\frac{qf'_g\Delta t}{V_p} < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where f_g is fractional flow $\lambda_g/(\lambda_o+\lambda_g)$.

The coding requirements associated with Eq. 3 are low. All terms in Eqs. 3-5, except the derivatives of phase mobilities and capillary pressures, exist in the interblock flow subroutine of an Impes model. The above results do not depend upon choice of Impes variables or upon whether variable substitution is used.

Maximum Stable Step due to Explicit Compositions. The explicit treatment of compositions gives the Condition 2 with

$$F_i = Max(I)\,\frac{Q_o\rho_o x_I + Q_g\rho_g y_I}{S_o\rho_o x_I + S_g\rho_g y_I} \quad \ldots\ldots\ldots\ldots\ldots (8)$$

where $I = 1,2,...,N_c$. The rate Q_o is the sum of all oil outflow rates from block i to its neighbors; Q_g is similarly defined. The maximum stable step is the lesser of the two values given by Condition 2 when F_i is substituted from Eq. 3 and Eq. 8. The F_i of Eq. 8 is simple and inexpensive in cpu. The rates Q_o and Q_g are stored in arrays in the interblock flow subroutine.

A Comparison with Previous Work

Previous papers referenced elsewhere[4] give an Impes stability condition for multi-dimensional flow in a Cartesian grid. For cocurrent, hyperbolic, 1D gas-oil flow, that condition is

$$qf'_g\Delta t/V_{pi} < 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Previous work is unclear regarding use of this condition when the flow is counter-current. For 1D counter-current flow in a vertical column with uniform grid spacing Δz,

$$qf'_g = -T_z(\gamma_o-\gamma_g)\Delta z\Psi' \quad \ldots\ldots\ldots\ldots\ldots (10)$$

Substituting this in Condition 9 gives

$$-T_z(\gamma_o-\gamma_g)\Delta z\Psi'\Delta t/V_{pi} < 1 \quad \ldots\ldots\ldots\ldots (11)$$

or, expanding the derivative Ψ',

$$-\frac{\Delta t}{V_{pi}} T_z(\gamma_o - \gamma_g)\Delta z\,(\lambda_o^2\lambda_g' + \lambda_g^2\lambda_{og}')/\lambda_t^2 < 1 \quad \ldots\ldots\ldots (12)$$

The derivative Ψ' is positive for low to moderate S_g and negative for larger S_g. Condition 11 presents a dilemma since it gives unconditional stability and conditional stability for positive and negative Ψ', respectively. For this counter-current flow, Eq. 3 gives $F_i = f22_i$ and Condition 2 is

$$\frac{\Delta t}{V_{pi}} T_z(\gamma_o - \gamma_g)\Delta z\,(\lambda_o^2\lambda_g' - \lambda_g^2\lambda_{og}')/\lambda_t^2 < 1 \quad (13)\cdot$$

Numerous 1D vertical countercurrent flow simulations were performed, initializing the column with uniform saturation distributions or with the upper half containing the heavier phase and the lower half containing the lighter phase. In all cases, use of Δt from Condition 13, using = in place of <, gave non-oscillatory stability. In most cases, instability occurred for Δt values 10 to 20 % larger. In all cases, results exhibited significant instability using Δt from Condition 11 with the conservative choice of $|\Psi'|$ replacing $-\Psi'$. Conditions 12 and 13 give the data-dependent ratio

$$\frac{\Delta t\,(this\ work)}{\Delta t\,(previous\ work)} = \frac{Max(k)\,(|\lambda_o^2\lambda_g' + \lambda_g^2\lambda_{og}'|)}{Max(k)\,(\lambda_o^2\lambda_g' - \lambda_g^2\lambda_{og}')} < 1 \quad . (14)$$

where *Max(k)* denotes maximum over all grid blocks at the current time step. Numerous similar simulations with three phase flow gave stability using Eq. 3 and instability with Δt values 20 % larger.

Derivation of *F* for Effects of Explicit Treatment of Saturations

Methods of stability analysis generally apply to linear difference equations with constant coefficients. For the case of variable coefficients, mathematicians long ago suggested applying the analysis using constant coefficients equal to local values and then applying the resulting analysis criterion to each subregion separately.[5] That procedure is followed here, with the local region being a single block-pair interface.

The Two-Phase Flow Case. For the case of 1D incompressible gas-oil flow, the Darcy expressions for oil and gas flow rates from grid block *i* to block *i*+1 are

$$q_{oi} = -T_{i+1/2}\lambda_o(\Delta p - \gamma_o\Delta Z) \quad \ldots\ldots\ldots\ldots (15a)$$

$$q_{gi} = -T_{i+1/2}\lambda_g(\Delta p - \gamma_g\Delta Z + \Delta P_{cgo}) \quad \ldots\ldots\ldots (15b)$$

where ΔX for any quantity X is $X_{i+1} - X_i$. Eliminating Δp from these two equations gives gas and oil phase flow rates as

$$q_{gi} = \lambda_g/\lambda_t\,[q - T_{i+1/2}\lambda_o(g + \Delta P_{cgo})] \quad \ldots\ldots\ldots (16a)$$

$$q_{oi} = \lambda_o/\lambda_t\,[q + T_{i+1/2}\lambda_g(g + \Delta P_{cgo})] \quad \ldots\ldots\ldots (16b)$$

where g is the gravitational term $(\lambda_o - \lambda_g)\Delta Z$, q is $q_{oi} + q_{gi}$, and λ_t is $\lambda_o + \lambda_g$. Each of q_{gi}, q_{oi}, q, g, and ΔP_{cgo} may be positive or negative.

The equation expressing conservation of mass of gas in grid block *i* is

$$\frac{V_{pi}}{\Delta t}(S_{gi,n+1} - S_{gi,n}) = q_{gi-1}(S_{gi-1,n}, S_{gi,n}) - q_{gi}(S_{gi,n}, S_{gi+1,n}) \quad \ldots\ldots\ldots (17)$$

which emphasizes that q_{gi} is a single-valued function of S_{gi} and S_{gi+1}, depending upon the oil and gas flow rate directions. The right hand side interblock flow terms are explicitly dated in saturation in accordance with the Impes treatment.

Stability analysis requires a constant-coefficient, linear difference equation which is, in some sense, an approximation to Eq. 17. That equation is obtained by writing Eq. 17 twice, once as shown where the S_{gi} are the actual calculated values and again with the S_{gi} values replaced by S_{gi}^* which is the exact (error-free) solution. Subtracting the resulting two equations gives

$$\frac{V_{pi}}{\Delta t}(\varepsilon_{i,n+1} - \varepsilon_{i,n}) = \delta q_{gi-1} - \delta q_{gi} \quad \ldots\ldots (18)$$

where $\varepsilon_{i,n}$ is the error $S_{gi,n} - S_{gi,n}^*$ and

$$\delta q_{gi} = q_{gi}(S_{gi,n}, S_{gi+1,n}) - q_{gi}(S_{gi,n}^*, S_{gi+1,n}^*) \quad \ldots\ldots\ldots (19)$$

Using the first Taylor series terms gives

$$\delta q_{gi} = q_{gi,i}'\varepsilon_{i,n} + q_{gi,i+1}'\varepsilon_{i+1,n} \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots (20)$$

where $q_{gi,i}'$ is the partial derivative of q_{gi} with respect to S_{gi} and $q_{gi,i+1}'$ is the partial derivative of q_{gi} with respect to S_{gi+1}. The term δq_{gi-1} in Eq. 18 is expressed in similar fashion, with care to use locally constant derivatives (i.e. $q_{gi-1,i-1}' = q_{gi,i}'$ and $q_{gi-1,i}' = q_{gi,i+1}'$,

$$\delta q_{gi-1} = q_{gi,i}'\varepsilon_{i-1,n} + q_{gi,i+1}'\varepsilon_{i,n} \quad (21)$$

Substituting Eqs. 20 and 21 into Eq. 18 gives the error equation

$$\frac{V_{pi}}{\Delta t}(\varepsilon_{i,n+1}-\varepsilon_{i,n}) = a_i\varepsilon_{i+1,n} - b_i\varepsilon_{i,n} + c_i\varepsilon_{i-1,n} \quad \ldots\ldots (22)$$

where

$$a_i = -q'_{gi,i+1} \quad \ldots\ldots (23a)$$

$$b_i = q'_{gi,i} - q'_{gi,i+1} \quad \ldots\ldots (23b)$$

$$c_i = q'_{gi,i} \quad \ldots\ldots (23c)$$

Eqs. 23 show that $a_i + c_i = b_i$. Also, $q'_{gi,i} \geq 0$ and $q'_{gi,i+1} \leq 0$ for all cases of cocurrent and countercurrent flow, so that each of $a_i, b_i, c_i \geq 0$.

The Appendix describes application of stability analysis to Eq. 22 with the variable coefficients V_{pi}, a_i, b_i, c_i assumed to be constants independent of i and n. The resulting stability condition is

$$\frac{\Delta t}{V_{pi}}(a_i + b_i + c_i) < 2 \quad \ldots\ldots (24)$$

$$a_i + c_i < b_i \quad \ldots\ldots (25)$$

and using Eqs. 23, the stability condition is

$$\frac{\Delta t}{V_{pi}}(q'_{gi,i} - q'_{gi,i+1}) < 1 \quad \ldots\ldots (26)$$

The terms $q'_{gi,i}$ and $q'_{gi,i+1}$ are obtained by differentiating Eq. 16a with respect to S_{gi} and S_{gi+1}. It might appear that different expressions for $q'_{gi,i} - q'_{gi,i+1}$ would result for the four possible cocurrent and countercurrent flow configurations. However, that is not the case. For all four cases,

$$q'_{gi,i} - q'_{gi,i+1} = \frac{T_{i+1/2}\lambda_o\lambda_g}{\lambda_t}(P'_{cgoi} + P'_{cgoi+1}) + \frac{\lambda_o}{\lambda_t\lambda_g}|q_{gi}|\lambda'_g - \frac{\lambda_g}{\lambda_t\lambda_o}|q_{oi}|\lambda'_{og} \quad \ldots\ldots (27)$$

Note that the term $q'_{gi,i} - q'_{gi,i+1}$ is always positive. The last two (non-P_c) terms in Eq. 27 can be expressed equivalently as

$$\frac{T_{i+1/2}}{\lambda_t}[\lambda_o\lambda'_g|\Delta\Phi_g| - \lambda_g\lambda'_{og}|\Delta\Phi_o|] \quad \ldots\ldots (28)$$

These results (Eqs. 26-28) for 1D gas-oil flow are equivalent to Condition 2 with $F_i = f22_i$ in Eq. 3. For multidimensional flow there are more contributions to F_i, each of the same form as $f22_i$.

The Three-Phase Flow Case. For the case of 1D three-phase incompressible flow, the Darcy expressions for phase rates from grid block i to block i+1 are

$$q_{wi} = -T_{i+1/2}\lambda_w(\Delta p - \gamma_w\Delta Z - \Delta P_{cwo}) \quad \ldots\ldots (29a)$$

$$q_{oi} = -T_{i+1/2}\lambda_o(\Delta p - \gamma_o\Delta Z) \quad \ldots\ldots (29b)$$

$$q_{gi} = -T_{i+1/2}\lambda_g(\Delta p - \gamma_g\Delta Z + \Delta P_{cgo}) \quad \ldots\ldots (29c)$$

Eliminating Δp from these equations gives the phase rates as

$$q_{wi} = \lambda_w[q + (\lambda_o+\lambda_g)a_{wo} + \lambda_g a_{go}]/\lambda_t \quad \ldots\ldots (30a)$$

$$q_{gi} = \lambda_g[q - (\lambda_o+\lambda_w)a_{go} - \lambda_w a_{wo}]/\lambda_t \quad \ldots\ldots (30b)$$

$$q_{oi} = \lambda_o[q - \lambda_w a_{wo} + \lambda_g a_{go}]/\lambda_t \quad \ldots\ldots (30c)$$

where q is total rate $q_o+q_w+q_g$, λ_t is total mobility $\lambda_w+\lambda_o+\lambda_g$, and

$$a_{wo} = T_{i+1/2}[(\gamma_w-\gamma_o)\Delta Z + \Delta P_{cwo})] \quad \ldots\ldots (31a)$$

$$a_{go} = T_{i+1/2}[(\gamma_o-\gamma_g)\Delta Z + \Delta P_{cgo})] \quad \ldots\ldots (31b)$$

Since the fluids are assumed incompressible and saturations add to 1.0, q is a constant independent of i and there are only two independent conservation equations. Selecting the water and gas equations,

$$\frac{V_{pi}}{\Delta t}(S_{wi,n+1} - S_{wi,n}) = q_{wi-1,n} - q_{wi,n} \quad \ldots\ldots (32a)$$

$$\frac{V_{pi}}{\Delta t}(S_{gi,n+1} - S_{gi,n}) = q_{gi-1,n} - q_{gi,n} \quad \ldots\ldots (32b)$$

Following the same procedure as in the two phase case (Eqs. 17-22), we obtain two equations in the two errors $\varepsilon 1 = S_w - S_w^*$ and $\varepsilon 2 = S_g - S_g^*$,

$$\frac{V_{pi}}{\Delta t}(\varepsilon 1_{i,n+1} - \varepsilon 1_{i,n}) = - q'_{wwi,i+1}\,\varepsilon 1_{i+1,n} - (q'_{wwi,i} - q'_{wwi,i+1})\,\varepsilon 1_{i,n} + q'_{wwi,i}\,\varepsilon 1_{i-1,n} - q'_{wgi,i+1}\,\varepsilon 2_{i+1,n} - (q'_{wgi,i} - q'_{wgi,i+1})\,\varepsilon 2_{i,n} + q'_{wgi,i}\,\varepsilon 2_{i-1,n} \quad \text{. . (33a)}$$

$$\frac{V_{pi}}{\Delta t}(\varepsilon 2_{i,n+1} - \varepsilon 2_{i,n}) = - q'_{gwi,i+1}\,\varepsilon 1_{i+1,n} - (q'_{gwi,i} - q'_{gwi,i+1})\,\varepsilon 1_{i,n} + q'_{gwi,i}\,\varepsilon 1_{i-1,n} - q'_{ggi,i+1}\,\varepsilon 2_{i+1,n} - (q'_{ggi,i} - q'_{ggi,i+1})\,\varepsilon 2_{i,n} + q'_{ggi,i}\,\varepsilon 2_{i-1,n} \quad \text{. . . (33b)}$$

The Appendix describes application of the stability analysis to these two simultaneous equations to obtain the Condition 2 and Eq. 3.

Derivation of *F* for Effects of Explicit Compositions

This section ignores the effects of explicit treatment of saturation-dependent terms on stable step size. For the assumptions mentioned in the Introduction, the mass conservation equations in differential form for 1D flow are

$$\frac{\partial}{\partial t}(S_o\rho_o x_I + S_g\rho_g y_I) = -\frac{\partial}{\partial x}(q_o\rho_o x_I + q_g\rho_g y_I), \quad I = 1,2,..,N_c \quad \text{. . . . (34a)}$$

$$\frac{\partial}{\partial t}(S_o\rho_o) = -\frac{\partial}{\partial x}(q_o\rho_o) \quad \text{. . . . . (34b)}$$

$$\frac{\partial}{\partial t}(S_g\rho_g) = -\frac{\partial}{\partial x}(q_g\rho_g) \quad \text{. . . . . (34c)}$$

If $q_o = 0$, these equations give

$$\frac{\partial y_I}{\partial t} + \frac{q_g\rho_g y_I}{S_o\rho_o x_I + S_g\rho_g y_I}\frac{\partial y_I}{\partial x} = 0 \quad \text{. . . . . (35a)}$$

and if $q_g = 0$,

$$\frac{\partial x_I}{\partial t} + \frac{q_o\rho_o x_I}{S_o\rho_o x_I + S_g\rho_g y_I}\frac{\partial x_I}{\partial x} = 0 \quad \text{. . . . . (35b)}$$

Eqs. 35 are obtained from Eqs. 34 using $y_I = K_I x_I$ and neglecting the dependence of the K-values on pressure and composition. Eqs. 35 are easily expressed in difference form, with explicit dating of the spatial derivatives and upstream weighting. Application of the von Neumann stability analysis, as in the Appendix, gives stability conditions

$$\frac{q_g\rho_g y_I}{S_o\rho_o x_I + S_g\rho_g y_I}\frac{\Delta t}{V_{pi}} < 1 \qquad (q_o = 0) \quad \text{. . . . . (36)}$$

$$\frac{q_o\rho_o x_I}{S_o\rho_o x_I + S_g\rho_g y_I}\frac{\Delta t}{V_{pi}} < 1 \qquad (q_g = 0) \quad \text{. . . . . (37)}$$

If S_o is 0 or S_g is 0, these further reduce to the well known stability conditions for multicomponent miscible flow,

$$\frac{q_g\Delta t}{S_g V_{pi}} < 1 \qquad (S_o = 0) \quad \text{. . . . . (38)}$$

$$\frac{q_o\Delta t}{S_o V_{pi}} < 1 \qquad (S_g = 0) \quad \text{. . . . . (39)}$$

The four stability conditions, just given for various choices of rates and saturations, are all included in the general stability condition, written for the general case of multidimensional flow with non-neighbors,

$$Max(I)\ \frac{Q_o\rho_o x_I + Q_g\rho_g y_I}{S_o\rho_o x_I + S_g\rho_g y_I}\frac{\Delta t}{V_{pi}} < 1 \quad \text{. . . . . (40)}$$

Q_o is the sum of all oil outflow rates from block i to its neighbors, rb/day. The rate Q_g is similarly defined. Condition 40 simply states that, for each component, the mols flowing out of a block in a time step cannot exceed the mols in place in the block. All saturations, densities, and compositions in Condition 40 are dated explicitly at time n, while all rates q and Q are dated at time $n+1$.

The writer knows of no theoretical basis for Condition 40 in the case where both Q_o and Q_g are nonzero. Young and Russell[6] derived a different, approximate stability condition for those conditions. Numerous 1D simulations were performed generating steady-state (uniform saturation) distributions corresponding to injection of different fixed-proportion gas-oil-water feed streams. The proportions were such that the throughput Eq. 8 dominated (i.e. gave smaller Δt than) the displacement Eq. 3. In all cases, non-oscillatory stability occurred with Δt given by the Condition 40 (using = in place of <). In cases of only one mobile hydrocarbon phase, instability arose for Δt values 20 % larger. With mobile oil and gas phases,

instability occurred in most cases at Δt values 20 % larger, but in some cases did not occur until Δt was 60 % larger.

Testing and Discussion

A large number of 1D test problems and numerous 3D field study datasets were run to test stability as a function of the CFL limit. The term CFL is $Max(i)\,\mathrm{CFL}_i$ where CFL_i is $F_i \Delta t / V_{pi}$. The tests included black oil and compositional, two-phase and three-phase cases, with cocurrent and countercurrent flow. Several of these tests are described below.

With one exception, CFL=1 gave non-oscillatory stability in all cases. The CFL limit for stability ranged from 1 to 2, depending upon the problem-dependent uniformity of the CFL_i values. The stable CFL limit was close to 1.0 for near-uniform CFL_i and close to 2.0 for the Buckley Leverett, variable CFL_i problem.[4] As capillary pressure becomes more dominant in the F_i of Condition 2, the CFL_i distribution becomes more uniform and the CFL limit tends toward 1.0, as shown by Problem 4 below. Tests of the heuristic Eq. 8, which perturbed uniform 1D, compositional gas-oil flowing mixtures, showed CFL limits ranging from 1.1 to 1.6.

The linear solver step of an Impes Newton iteration gives a new pressure distribution which may result in new upstream blocks for each phase. If more than one Newton iteration is taken, the model recalculates transmissibilities based on those new upstream blocks. In addition, loops may be performed within a Newton when stable step logic is used. After the linear solver is called, flow rates and stable step size are calculated using the upstream blocks and interblock rates corresponding to the new pressure distribution. If that stable step is less than the time step being used, the model recalculates transmissibilities using the new pressure distribution, resets Δt to the stable step value and returns to the solver. For most problems, the average number of Newtons per time step is close to 1.0 and the average number of loops per Newton is close to 0.

An option used in the tests discussed below is the forcing of one loop with Δt reset to stable step size regardless of whether the stable step is smaller or larger than the time step being used. This results in the CFL value each time step exactly equalling the value cited in the discussion. Unless noted otherwise, the runs described here used no constraints other than the stable step condition. That is, no change constraints or maximum time step constraints were used.

Problems 1-6 use the water-oil mobility data of Young and Russell's first two problems.[6] Problem 1 is their 20-block 1D vertical water-oil displacement with an implicit producer in block 1. After water breakthrough, the CFL occurred in block 2. For a CFL of 2.0, the after-breakthrough behavior was stable. For CFL = 2.1 and larger, S_w oscillated in block 2 and significant oscillations in oil production rate occurred. An explanation for this stability for CFL values up to 2.0 is given elsewhere.[4] Identical results were obtained when the 20-block column was horizontal.

Problem 2 simulates counter-current flow in a 20-block, 100 ft, closed vertical column with $k(\gamma_o - \gamma_g) = 274$ md-psi/ft. Runs (a) were made for initial S_{wi}=.45, and runs (b) for initial S_{wi} = 1.0 in the upper half and .2 in the lower half of the column. The stable CFL limit was 1.6 for runs (a) and 1.2 for runs (b). In runs (a), CFL_i decreased significantly with distance from the CFL block. Runs (b) exhibited more uniform CFL_i distributions.

Problem 3 illustrates stability behavior when the CFL_i values are uniform throughout the grid. Water and oil were co-injected to establish uniform saturations through a 1D 20-block horizontal grid. The injection mix was then perturbed from 50 % (by volume) water to 45 % water. After the new steady-state, the injection mix was again perturbed, from 45 % to 55 % water. For CFL=1.2, the results were stable with no perturbations. However, the perturbations travelled unstably through the grid with the amplitude of S_w oscillations at the perturbation front increasing with distance travelled. Large water-cut oscillations occurred after perturbation breakthrough, followed by a return to stability. This behavior is not tolerable if significant oscillations in water-cut and/or gas-oil ratio (gor) are to be avoided. These results exhibit the stable flow which can occur with uniform CFL_i and CFL considerably above 1. In numerical tests of stability, perturbations should be introduced before concluding that a uniform CFL_i, CFL > 1 flow regime is "stable".

Problem 4 includes capillary pressure in a 1D 20-block horizontal grid with initial S_{wi} = .2. The diffusivity or capillary term is $k(P_{cwo})_{\max}/uL\mu_w$=13.5 with $P_{cwo}=(P_{cwo})_{max}((1-S_w)/(1-S_{wc}))^3$. Water breakthrough occurred at about .24 pore volumes water injected. Instability at CFL = 1.1 gave fractional water-cut oscillations with an amplitude of about .04 when the water-cut was .4.

Three-phase Problem 5 simulates gas injection in a 1D horizontal 20-block grid with initial S_{wi} = .4 and S_{oi} = .6. Gas viscosity μ_g=.05 cp, S_{org}=.2, S_{gc}=0, and $k_{rg}=(S_g/(1-S_{org}-S_{wc}))^3$. Gas, oil, and water are incompressible and there is no dissolved gas or oil content of the gas phase. After gas breakthrough at .177 pore volumes injected, Δt was controlled each time step with one loop to exactly satisfy the specified CFL value. Runs with CFL values up to 1.98 exhibited stability with no water-cut or gor oscillations. Using CFL = 2.2 gave large gor oscillations and significant water-cut oscillations, similar to the two-phase Problem 1 results.

Problem 6 is the same as Problem 5 except a water-oil-gas mixture containing equal parts by volume of each phase was injected at constant total rate to establish uniform saturations throughout the 20 blocks. Runs from that point were then made using the 1-loop option so that the Δt of each time step exactly satisfied the specified CFL value. With no perturbations of the injection mixture, a CFL of 1.5 gave large water-cut and gor oscillations which persisted indefinitely. A CFL of 1.2 gave stable behavior - no change in water-cut or gor. However, similar to the two-phase Problem 3, for CFL = 1.2 perturbations

in the injection mixture resulted in unstable travel of the perturbations and large gor oscillations after perturbation breakthrough, followed by a return to stability.

Problem 7 is Young and Russell's 1/4 five-spot waterflood problem. The block-centered 13x13x1 grid had uniform spacing with half edge and quarter corner blocks. That is, the volume of each interior block was four times that of each corner well block and twice that of each other edge block. Wells are located at grid points. The time step was controlled by changes $(\Delta S_{max}, \Delta x_{max})=(.1,.1)$ and the specified CFL limit. The time step was limited by the CFL limit alone after water breakthrough at .339 pore volumes injected. The 1-loop option was activated at that point and runs were made with various CFL limits. As the CFL limit was increased above 1.88, the water-cut oscillated, with an amplitude increasing with increasing CFL. The run to .4 pore volumes injected, using CFL=1.88 (without the 1-loop option) took 93 steps, 94 Newtons, and .44 seconds cpu on a 400 MHz Gateway PC with the Compaq Visual Fortran Compiler version 6.0.

The SPE Comparative Solution Project Problems 1,3, and 10 are denoted here as Problems SPE1, SPE3, and SPE10. Problem SPE1 was run using the above changes and a CFL limit of 1.5. All steps after gas breakthrough at about 1230 days were controlled by the CFL limit alone, with F_i from the displacement Eq. 3. The run took 145 steps, 154 Newtons, 1.42 seconds cpu and exhibited no gor oscillations. A gor reversal at 1300 days is real. Unstable gor behavior resulted for CFL > 1.5. The stable step of Condition 2 increased from 8 days at 1250 days to 32 days at 2200 days and varied by less than ± 10 % from an average 31 days value thereafter. This stable step vs time profile differs considerably from that reported by Young and Russell for this SPE1 problem. The controlling cell was generally a neighbor of the producing block.

The SPE3 9-component compositional problem was stable at a CFL = 2.0, requiring 105 steps, 106 Newtons, and 5.13 seconds cpu. All time steps were controlled by the CFL limit alone (Condition 2 with < 1 replaced by < 2), and with F_i given by the throughput Eq. 8. The displacement Eq. 3 F_i limited none of the steps. The stable step of Condition 2 declined from 32 to 24.5 days in the first ten years, and declined from 44 to 18.5 days in the last five years. Compared with Young and Russell, the last five year values are about the same, but the first ten year values here are about two times larger. Instabilities occurred for CFL=2.2.

Gas-oil Problem SPE10 simulates gas injection in a 100x1x20 grid with a geostatistical permeability distribution. The immiscible gas and oil are incompressible and there is no solution gas or interphase mass transfer. Implicit is faster than Impes for this problem, running in 718 steps, 816 Newtons, and 214 cpu seconds. The Impes run using CFL=2.0 was stable and required 7018 steps, 7059 Newtons, and 558 cpu seconds. Both runs gave nearly identical time plots of rates and gor, contradicting, in this case, the common belief that implicit results are less accurate than Impes results due to larger numerical dispersion error. An Impes run using CFL=2.2 gave rate and gor vs time plots which were smooth and identical to those for CFL=2.0. Nevertheless, the run was unstable, reflected in the 8494 steps, 8535 Newtons, and 849 cpu seconds. The instabilities were saturation oscillations at blocks not near the producer.

Conclusions

An Impes stability criterion is derived for the three-phase, cocurrent and countercurrrent, multidimensional flow occurring in black oil and compositional models. It accounts for viscous, gravity, and capillary forces in structured or unstructured grids and gives a stable step size for each grid block. These stable steps can be used as switching criteria in an adaptive implicit model or to set the time step size in an Impes formulation.

The criterion is $F_i \Delta t / V_{pi} < 1$. We define the terms $CFL_i = F_i \Delta t / V_{pi}$ and $CFL = Max(i)CFL_i$. A large number of 1D two- and three-phase numerical tests and numerous field study datasets were run using Impes to check the criterion. Non-oscillatory stability resulted for time steps obeying CFL = 1. Instability (oscillations) occurrred at CFL = 1.1 or 1.2 when the CFL_i values were nearly uniform through the grid. Non-oscillatory stability occurred for CFL values between 1.2 and 2.0 as the (problem-dependent) CFL_i distribution became more non-uniform. All runs with CFL > 2.0 exhibited oscillations.

Nomenclature

f_g = fractional flow of gas in a 1D gas-oil system, $\lambda_g(1 - gT\lambda_o/q)/\lambda_t$

f_g' = df_g/dS_g

F = stability function

g = $(\gamma_o - \gamma_g)(Z_{i+1} - Z_i)$

i = $\sqrt{-1}$

k = permeability

k_r = relative permeability, fraction

K_I = component I K-value, y_I/x_I

L = length of 1D column

$M_{I,n}$ = mass of component I in grid block at time level n

N = number of grid blocks

N_c = number of non-aqueous components

p = oil phase pressure

P_{cgo} = gas-oil capillary pressure, $p_g - p_o$, dependent upon s_g

P_{cgo}' = dP_{cgo}/dS_g

P_{cwo} = water-oil capillary pressure, $p_o - p_w$, dependent upon s_w

P_{cwo}' = dP_{cwo}/dS_w

q = interblock total flow rate, rb/day

Q_g = sum of gas outflow rates from a grid block, rb/day

Q_o = sum of oil outflow rates from a grid block, rb/day

q_w, q_o, q_g = phase interblock flow rates

q_{gi} = gas phase flow rate from block i to block i+1

$q'_{ggi,i} = \partial q_{gi}/\partial S_{gi}$

$q'_{ggi,i+1} = \partial q_{gi}/\partial S_{gi+1}$

$q'_{gwi,i} = \partial q_{gi}/\partial S_{wi}$

$q'_{gwi,i+1} = \partial q_{gi}/\partial S_{wi+1}$

q_{wi} = water phase flow rate from block i to block i+1

$q'_{wgi,i} = \partial q_{wi}/\partial S_{gi}$

$q'_{wgi,i+1} = \partial q_{wi}/\partial S_{gi+1}$

$q'_{wwi,i} = \partial q_{wi}/\partial S_{wi}$

$q'_{wwi,i+1} = \partial q_{wi}/\partial S_{wi+1}$

S = saturation, fraction

s_{gc} = critical gas saturation

s_{org} = residual oil saturation to gas

S_{wc} = connate water saturation

t = time

T = transmissibility, kA/l, rb-cp/day-psi

$T_{i+1/2}$ = transmissibility for flow between grid blocks i and i+1

T_{ij} = transmissibility connecting block i and neighbor block j

u = superficial velocity

V_p = grid block pore volume

x, y, z = Cartesian coordinates

x_l = mol fraction of component I in the oil phase

y_l = mol fraction of component I in the gas phase

Z = depth, measured vertically downward

Greek

β, β_1, β_2 = eigenvalues of von Neumann stability analysiss, Eqs. A-2, A-12

Δt = time step, $t_{n+1} - t_n$

$\Delta x, \Delta y, \Delta z$ = grid block dimensions, length

ΔS_{max} = change over time step of saturation, maximum over grid

Δx_{max} = change over time step of mol fraction, maximum over grid

ε = error in saturation

γ = phase density or gradient, psi/ft

λ = error amplification factor

λ_1 = error amplification factor for water saturation error

λ_2 = error amplification factor for gas saturation error

λ_g = gas phase mobility, k_{rg}/μ_g, dependent upon s_g

$\lambda'_g = d\lambda_g/dS_g$

λ_o = oil phase mobility, k_{ro}/μ_o, dependent upon s_w and s_g

$\lambda'_{ow} = \partial\lambda_o/\partial S_w$

$\lambda'_{og} = \partial\lambda_o/\partial S_g$

λ_t = total mobility, e.g. $\lambda_w+\lambda_o+\lambda_g$ in a three-phase system

λ_w = water phase mobility, k_{rw}/μ_w, dependent upon s_w and s_g

$\lambda'_{wg} = \partial\lambda_w/\partial S_g$

$\lambda'_{ww} = \partial\lambda_w/\partial S_w$

μ = viscosity

Φ = phase flow potential

$\Psi = \lambda_o\lambda_g/(\lambda_o+\lambda_g)$, dependent upon S_g

$\Psi' = d\Psi/dS_g$

ρ_g = molar density of gas phase, mols/rb

ρ_o = molar density of oil phase, mols/rb

Subscripts

g = gas phase

i , j = grid block indices

i, j, k = grid block indices in the x, y, and z directions, respectively

I = non-aqueous component number, I =1, 2, ..., N_c

n = time level

o = oil phase

w = water phase

x, y, z = Cartesian directions x, y, and z

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Appendix -- Stability Analysis

The method of stability analysis used here was developed by J. von Neumann and is described by many authors, including Richtmyer.[7]

The Two-Phase Case. We consider the difference equation

$$\varepsilon_{i,n+1} - \varepsilon_{i,n} = a\varepsilon_{i+1,n} - b\varepsilon_{i,n} + c\varepsilon_{i-1,n}, \quad \ldots\ldots\ldots\ldots \text{(A-1)}$$

similar to Eq. 22, where all coefficients a,b,c are ≥ 0. The von Neumann stability analysis method replaces $\varepsilon_{i,n}$ by the Fourier type term

$$\varepsilon_{i,n} = \lambda^n e^{\hat{i}i\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A-2)}$$

The ratio $\varepsilon_{i,n+1}/\varepsilon_{i,n}$ is λ so that the stability condition is $|\lambda| < 1$. Substituting $\varepsilon_{i,n}$ from Eq. A-2 into Eq. A-1 and using the identity $e^{\hat{i}\beta} = \cos\beta + \hat{i}\sin\beta$ gives

$$\lambda = 1 - b + (a+c)\cos\beta + \hat{i}(a-c)\sin\beta \quad \ldots\ldots\ldots\ldots \text{(A-3)}$$

and

$$|\lambda|^2 = (1 - b + (a+c)\cos\beta)^2 + ((a-c)\sin\beta)^2 \quad \ldots\ldots \text{(A-4)}$$

We seek the maximum value of $|\lambda|^2$ over the range of eigenvalues, or equivalently, the range of $\cos\beta$ values, $-1 < \cos\beta < 1$. The maximum value of $|\lambda|^2$ occurs at a β value which satisfies $d|\lambda|^2/d\beta = 0$. Taking this derivative from Eq. A-4, setting it to zero, and solving for β gives two roots. Using the first root, $\sin\beta = 0$, in Eq. A-3, we find that stability ($|\lambda| < 1$) requires the two conditions (for $\cos\beta = -1$ and $+1$, respectively)

$$a + b + c < 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A-5a)}$$

$$a + c < b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A-5b)}$$

Adding these two conditions gives $a+c<1$. The second root leads to nothing of consequence. For $N>1$, all difference forms of type Eq. A-1 in this paper satisfy $a+c=b$ so that the Condition A-5b is always satisfied. Therefore the stability condition for Eq. A-1 is A-5a or, using $a+c=b$,

$$a + c = b < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A-6)}$$

Discussion of the result $b < 2$ for the case $N = 1$ is dropped for brevity.

Many authors point out that the stability condition for two or three dimensions is simply the sum of two or three terms, each of which is identical in form to that derived for the 1D case. For example, consider Eq. A-1 written for three dimensions

$$\begin{aligned}\varepsilon_{ijk,n+1} - \varepsilon_{ijk,n} &= a_x\varepsilon_{i+1,jkn} - b_x\varepsilon_{ijkn} + c_x\varepsilon_{i-1,jkn} \\ &+ a_y\varepsilon_{i,j+1,kn} - b_y\varepsilon_{ijkn} + c_y\varepsilon_{i,j-1,kn} \\ &+ a_z\varepsilon_{ij,k+1,n} - b_z\varepsilon_{ijkn} + c_z\varepsilon_{ij,k-1,n} \quad \ldots \text{(A-7)}\end{aligned}$$

Substituting $\varepsilon_{ijkn} = \lambda^n e^{\hat{i}i\beta_x + \hat{i}j\beta_y + \hat{i}k\beta_z}$ leads to the amplificaction factor

$$\begin{aligned}|\lambda|^2 = [1 &+ (a_x+c_x)\cos\beta_x - b_x + (a_y+c_y)\cos\beta_y - b_y \\ &+ (a_z+c_z)\cos\beta_z - b_z]^2 \\ &+ (a_x-c_x)^2\sin\beta_x^2 + (a_y-c_y)^2\sin\beta_y^2 \\ &+ (a_z-c_z)^2\sin\beta_z^2 \quad \ldots\ldots\ldots\ldots\ldots \text{(A-8)}\end{aligned}$$

The maximum value of $|\lambda|$ occurs for $\cos\beta_x = \cos\beta_y = \cos\beta_z = -1$ and the condition that $|\lambda|$ be < 1 gives (provided $a+c<b$ each direction)

$$a_x+b_x+c_x + a_y+b_y+c_y + a_z+b_z+c_z < 2 \quad \ldots\ldots\ldots \text{(A-9)}$$

which is simply the 1D result Condition A-5a with an additional term of identical form for each additional direction.

The Three-Phase Case. The three-phase case involves the two error Eqs. 33a and 33b which are of the form

$$\begin{aligned}\varepsilon 1_{i,n+1} - \varepsilon 1_{i,n} &= a11\,\varepsilon 1_{i+1,n} - b11\,\varepsilon 1_{i,n} + c11\,\varepsilon 1_{i-1,n} \\ &+ a12\,\varepsilon 2_{i+1,n} - b12\,\varepsilon 2_{i,n} + c12\,\varepsilon 2_{i-1,n} \text{(A-10a)} \\ \varepsilon 2_{i+1,n} - \varepsilon 2_{i,n} &= a21\,\varepsilon 1_{i+1,n} - b21\,\varepsilon 1_{i,n} + c21\,\varepsilon 1_{i-1,n} \\ &+ a22\,\varepsilon 2_{i+1,n} - b22\,\varepsilon 2_{i,n} + c22\,\varepsilon 2_{i-1,n} \text{(A-10b)}\end{aligned}$$

Comparing the coefficients in Eqs. 33 with those in Eqs. A-10 shows that the latter satisfy

$$a_{rs} + c_{rs} = b_{rs} \qquad rs = 11, 12, 21, 22 \quad \ldots\ldots \text{(A-11)}$$

Substituting $\varepsilon 1_{i,n} = \lambda_1^n e^{\hat{i}i\beta_1}$ and $\varepsilon 2_{i,n} = \lambda_2^n e^{\hat{i}i\beta_2}$ in these equations gives

$$\boldsymbol{\varepsilon}_{n+1} = E\boldsymbol{\varepsilon}_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A-12)}$$

where $\boldsymbol{\varepsilon}_n$ is the error vector $(\varepsilon 1_{i,n}, \varepsilon 2_{i,n})^T$ and the elements εrs of the amplification matrix E are

$$\varepsilon 11 = 1 - d11 \quad \text{(A-13a)}$$

$$\varepsilon 12 = -d12 \quad \text{(A-13b)}$$

$$\varepsilon 21 = -d21 \quad \text{(A-13c)}$$

$$\varepsilon 22 = 1 - d22 \quad \text{(A-13d)}$$

and, for rs=11,12,21,22,

$$drs = -(ars+crs)\cos\beta_s + brs - \hat{i}(ars-crs)\sin\beta_s \quad \text{(A-14)}$$

The spectral radius ρ_E of E is the eigenvalue of E of maximum absolute value. We impose the stability requirement $\rho_E < 1$. The value of ρ_E obviously depends upon the eigenvalues β_1, β_2, which assume all values corresponding to the ranges $-1 < \cos\beta_1 < 1$ and $-1 < \cos\beta_2 < 1$. In the two-phase case above, the most restrictive stability condition $|\lambda| < 1$ resulted for the eigenvalue $\cos\beta = -1$, $\sin\beta = 0$. Assuming that is also true for this three-phase case gives the eigenvalues $\cos\beta_1 = \cos\beta_2 = -1$, $\sin\beta_1 = \sin\beta_2 = 0$. Using these eigenvalues in Eq. A-14 gives real drs values and Eqs. A-13 give the (real) εrs elements of the amplification matrix E. The two eigenvalues of the E matrix are then easily calculated as the roots of E's quadratic characteristic equation. The stability condition that $\rho_E < 1$ is obtained as

$$d11 + d22 + \sqrt{(d11+d22)^2 - 4\det(D)} < 4 \quad \text{(A-15)}$$

where $\det(D) = d11\,d22 - d12\,d21$. The drs satisfy

$$drs = 2brs \qquad rs = 11, 12, 21, 22 \quad \text{(A-16a)}$$

$$d11 > 0 \quad \text{(A-16b)}$$

$$d22 > 0 \quad \text{(A-16c)}$$

$$\det(D) > 0 \quad \text{(A-16d)}$$

These drs are related to the coefficients in the original Eqs. 33 by

$$d11 = (q'_{wwi,i} - q'_{wwi,i+1}) * 2\Delta t / V_{pi} \quad \text{(A-17a)}$$

$$d12 = (q'_{wgi,i} - q'_{wgi,i+1}) * 2\Delta t / V_{pi} \quad \text{(A-17b)}$$

$$d21 = (q'_{gwi,i} - q'_{gwi,i+1}) * 2\Delta t / V_{pi} \quad \text{(A-17c)}$$

$$d22 = (q'_{ggi,i} - q'_{ggi,i+1}) * 2\Delta t / V_{pi} \quad \text{(A-17d)}$$

As in the two-phase case discussed above, it might appear that the form of each of these drs terms would depend upon which of the 12 possible three-phase cocurrent/countercurrent flow configurations is assumed. Again, however, that is not the case. Each drs is independent of the flow configuration. Eqs. A-17 together with differentiation of Eqs. 30 give the drs values. That tedious process together with the Condition A-15 gives the final stability Condition 2 with Eqs. 3-5.

For each of the three two-phase cases, the stability Condition 2 is both necessary and sufficient. For the three-phase case, the condition is a necessary one, valid for the above described choice of eigenvalues β_1 and β_2. However, there is no proof here that the condition is also sufficient for the three-phase case. Numerous numerical tests indicate the condition is both sufficient and necessary in the three-phase case.

SI Metric Conversion Factors

bbl	x 1.589 873	E-01 = m^3
ft^3	x 2.831 685	E-02 = m^3
lbm	x 4.535 924	E-01 = kg
psi	x 6.894 757	E+00 = kPa