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Simulation of Three-Dimensional, Two-Phase Flow In Oil and Gas Reservoirs

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ABSTRACT

Two computer-oriented techniques for simulating the three-dimensional flow behavior of two fluid phases in petroleum reservoirs were developed. Under the first technique the flow equations are solved to model three-dimensional flow in a reservoir. The second technique was developed for modeling flow in three-dimensional media that have sufficiently high permeability in the vertical direction so that vertical flow is not seriously restricted. Since this latter technique is a modified two-dimensional areal analysis, suitably structured three-dimensional reservoirs can be simulated at considerably lower computational expenses than is required using the three-dimensional analysis. A quantitative criterion is provided for determining when vertical communication is good enough to permit use of the modified two-dimensional areal analysis.

The equations solved by both techniques treat both fluids as compressible, and, for gas-oil applications, provide for the evolution of dissolved gas. Accounted for are the effects of relative permeability, capillary pressure and gravity in addition to reservoir geometry and rock heterogeneity. Calculations are compared with laboratory waterflood data to indicate the validity of the analyses. Other results were calculated with both techniques which show the equivalence of the two solutions for reservoirs satisfying the vertical communication criterion.

INTRODUCTION

Obtaining the maximum profits from oil and gas reservoirs during all stages of depletion is the fundamental charge to the reservoir engineering profession. In recent years much quantitative assistance in evaluating field development programs has been provided by computerized techniques for predicting reservoir flow behavior. Because of the spatially distributed and dynamic nature of producing operations, automatic optimization procedures, such as those now in use for planning refining operations, are not now available for planning reservoir development. However, present mathematical simulation techniques do furnish powerful means for making case studies to help in planning primary recovery operations and in selecting and timing supplemental recovery operations.

A number of methods have been reported which simulate the flow of one, two or three fluid phases within porous media of one or two effective spatial dimensions.[1-4] However, applying computer analyses to actual reservoirs have been limited mostly to two-dimensional areal or cross-sectional flow studies for two immiscible reservoir fluids. To obtain a three-dimensional picture of reservoir performance using such two-dimensional techniques, it has been necessary to interpret the calculations by combining somehow the results from essentially independent areal and cross-sectional studies. To the authors' knowledge, the only other three-dimensional computational procedure, in addition to those presented here, was developed by Peaceman and Rachford[8] to simulate the behavior of a laboratory waterflood.

Two computational techniques which may be used to simulate three-dimensional flow of two fluid phases are described in this paper. The first method, called the "three-dimensional analysis", employs a fully three-dimensional mathematical model that treats simultaneously both the areal and cross-sectional aspects of reservoir flow. The second method, called the vertical equilibrium (VE) analysis", is applicable

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[1]References given at end of paper.

only to reservoirs having good vertical communication. It adapts a two-dimensional areal calculation by providing special techniques to account for flow and saturation variation in the direction normal to reservoir bedding planes. A quantitative criterion based on reservoir parameters is provided for determining when reliable results can be expected from the second method. The general character of the analyses and applications are discussed. Mathematical descriptions and developments are included as appendices.

PROPERTIES OF ANALYSES

The three-dimensional analysis and the VE analysis are difference equation analogs of the dynamic two-phase flow systems occurring in oil and gas reservoirs. The difference equations employed in the analyses are finite approximations to the standard differential equations which govern the simultaneous flow of two fluid phases in porous media. The following characteristics are built into each of the analyses: (1) broad flexibility with respect to reservoir structure, thickness variation and well locations, (2) capacity for modeling three-dimensional distributions of porosity and directional permeability, (3) consideration of rock compressibility and pressure-dependent fluid densities and viscosities, (4) the ability to account for the evolution of dissolved gas for gas-oil systems which accompanies locally declining pressure, (5) treatment of the effects of relative permeability and the interaction of gravitational, capillary and viscous forces, (6) use of reflection conditions normal to all boundaries of the calculation grid with influx or efflux accounted for by assigning source or sink terms to boundary blocks and (7) simultaneous solution of the linked difference equations employing alternating direction implicit (ADI) procedures in an iterative manner.

The fundamental dissimilarity between the two analyses is that the three-dimensional nature of calculations in the former is reduced to two-dimensional in the latter through the VE concept. This concept is developed mathematically in Appendix B.

Owing to the three-dimensional vs two-dimensional nature of the calculations performed, the two analyses employ somewhat different ADI solution procedures. In the three-dimensional analysis, for which difference equations and method of solution are detailed in Appendix A, the Douglas-Rachford technique[5] is used. However, the two-dimensional, modified areal model embodied in the VE analysis employs the Peaceman-Rachford technique which is similar to that described for incompressible fluids.[1]

VE CONCEPT

Reservoir flow systems are termed in VE whenever the pressure in each fluid phase varies hydrostatically along any line traversing the sand thickness normal to its underlying and overlying structural confinements and/or bedding planes. Experience gained from numerous two-dimensional cross-sectional and three-dimensional simulations of reservoir and laboratory model behavior have shown that VE frequently occurs. Even in cases where significant underrunning by water or overriding by gas were calculated, the observance of VE has been quite common.

Factors which favor VE are (1) low resistance to flow normal to the bedding planes, (2) sands thin in the direction normal to the bedding planes and (3) low areal rates of fluid movement. A quantitative criterion, which may be used to assess when the VE assumption is valid, is presented in terms of the above variables in the next section.

Physically, the occurrence of VE implies that the rate of redistribution toward a capillary-gravitational equilibrium configuration within a dip-normal column of fluids is high relative to the rate at which saturation fronts advance areally. Alternatively, VE implies that the dip-normal components of gravity and capillary force are balanced and that, for all practical purposes, the dip-normal component of viscous force is zero.

Provided that VE obtains, knowledge of the value of capillary pressure at any reference point within the sand layer establishes the variation of capillary pressure along the dip-normal line through that point and, hence, the phase saturations along that line. Therefore, to simulate the flow behavior for any qualifying three-dimensional porous medium, multiphase calculations must be carried out only over a two-dimensional reference surface within the medium.

Eqs. 1 and 2 express the VE condition mathematically.

$$\frac{\partial \Phi_n}{\partial z} = \frac{\partial \Phi_w}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$P_c\ (x, y, z) = P_{cp}\ (x,y) - \cos a_d \int_o^z (\rho_{sw} - \rho_{sn})\, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The z coordinate is oriented in the dip-normal direction (vertical if the dip angle is zero) and measures distance below the reference surface. For convenience, the reference surface was taken to connect the mid-points of all dip-normal line segments passing through the sand. Eq. 2 follows from Eq. 1 and the definitions of flow potential and capillary pressure. In Eq. 2, P_{cp} is the value of capillary pressure at any areal point (x, y) on the reference surface. Eqs. 1 and 2 apply at all points within the three-dimensional medium.

Implementation of the VE concept involves integration over the third dimension (z, or reservoir thickness) of each term remaining after substitution of Eq. 1 into the three-dimensional partial differential equations governing two-phase flow. The

integration process (Appendix B) effectively modifies the functional relationships between relative permeability, capillary pressure and saturation. The resulting relationships allow the two-dimensional calculations to account for the dip-normal variation of saturation, and thereby simulate both the vertical and areal character of reservoir flow behavior if VE prevails. Eqs. 3 and 4 illustrate how the integration process is applied to obtain a volumetrically averaged saturation and an effective relative permeability for wetting phase flow parallel to the x-z plane at any areal point on the reference surface.

$$\overline{S}_w = \int_{-h/2}^{h/2} \phi(z)\, S_w(z)\, dz \Big/ \int_{-h/2}^{h/2} \phi(z)\, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$k_{prw} = \int_{-h/2}^{h/2} k_{xy}(z)\, k_{rw}(z)\, dz \Big/ \int_{-h/2}^{h/2} k_{xy}(z)\, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

In Eqs. 3 and 4 h is the dip-normal sand thickness. Porosity ϕ and absolute permeability for flow parallel to the x-y plane k_{xy} vary with z to account for stratification within the sand thickness. If anisotropy exists in the x-y planes, absolute permeabilities in the x and y directions may be handled separately. Saturation S_w and relative permeability k_{rw} are indirect functions of z since each vary functionally with capillary pressure, which in turn varies directly with z according to Eq. 2. Relative permeability for nonwetting phase flow is modified in a manner similar to that shown in Eq. 4.

Successive applications of Eqs. 3 and 4 over a range of values for $\overline{P}_c$ allow construction of pseudo or equilibrium relative permeability and capillary pressure curves from corresponding rock or laboratory curves. Fig. 1 presents typical rock capillary pressure and relative permeability curves and the pseudocurves which result for two particular sand thicknesses. The figures show that the integration process causes the slope of the pseudo relationships to become more nearly linear than the corresponding rock relationships, and that the tendency toward linearity increases with increasing sand thickness.

As presented quantitatively in Appendix B, the shape of the pseudo relative permeability and capillary pressure curves is affected by (1) density difference $\rho_{sw} - \rho_{sn}$, (2) dip angle α_d, (3) sand thickness h and (4) stratification $k(z)$ and $\phi(z)$. Several important properties and programming considerations associated with the VE analysis are discussed in the section Computational Properties.

A further application of the VE concept is in association with the Buckley-Leverett technique for modeling two-phase flow in one dimension. Use of pseudo relative permeability curves in place of rock curves in constructing the fractional flow function permits the technique to partially account for the effects of vertical saturation variations on flow behavior.

The VE concept can be extended readily to provide a concept of "block" relative permeability and capillary pressure relationships. Such an extension can be of value in cases where the VE assumption is not valid and rigorous attention must be given to vertical flow. In these cases, the procedure allows coarser grid spacing in the dip-normal direction to be used in two-dimensional cross-sectional or full three-dimensional calculations than otherwise would be required to model flow behavior satisfactorily. This extension is based on assuming equilibrium only within an individual grid block rather than through the entire sand thickness.

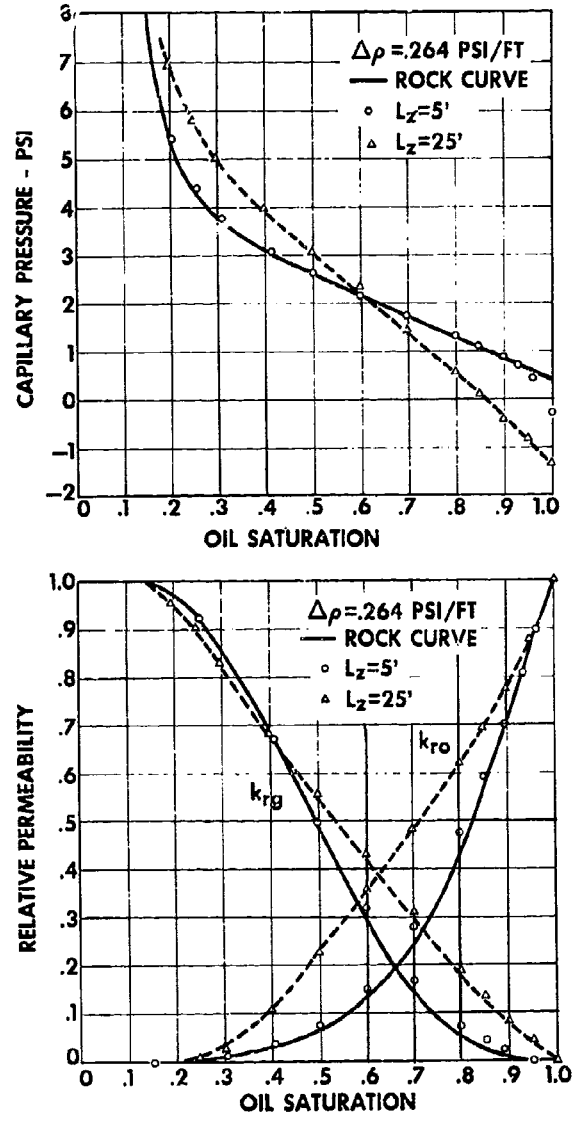

FIG. 1 — SENSITIVITY OF EQUILIBRIUM RELATIVE PERMEABILITY AND CAPILLARY PRESSURE CURVES TO DEPTH OF INTEGRATION.

VE CRITERION

The VE criterion was developed to provide a measure for predicting the quality of VE results for particular reservoir situations. The functional form of the criterion was derived analytically from the partial differential equations which describe two-phase flow due to gravitational segregation in a closed, one-dimensional sand column (Appendix C). The dimensionless criterion group τ defined in Eq. 5 corresponds physically to the ratio of the time constant associated with the decay of vertical transients to the time required for the fluids to advance areally through a distance equal to the reservoir thickness.

$$\tau = h\,u/0.00633\,k_z\,\psi \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$\psi = -\frac{dP_c}{dS_w}\;\frac{k_{rn}\,k_{rw}}{k_{rn}\,\mu_w + k_{rw}\,\mu_n} \quad \ldots\ldots \quad (6)$$

The absolute permeability that influences τ is that which applies for flow in the dip-normal or z direction. Since relative permeability and capillary pressure relationships (rock basis) enter the definition of ψ, this parameter varies with saturation. The recommended procedure for obtaining an appropriate value of ψ is to average the values that result over the mid-range of saturation. It is important to note that the influence of relative permeability and viscosity associated with the wetting and nonwetting phases is symmetric, i.e., subscripts w and n applied to these parameters in Eqs. 5 and 6 may be reversed. This characteristic reveals that high viscosity for either phase affects τ adversely.

A number of three-dimensional and companion VE calculations provided the correspondence between the value of τ and the quality of VE results shown in Table 1.

COMPUTATIONAL PROPERTIES

Computational procedures for each of the analyses were developed and programmed in FORTRAN IV for running on an IBM 7044 computer with 32,000 words of core storage. The extended grid size version of the three-dimensional program employs an IBM 1301 disk file to provide additional storage.

MAXIMUM GRID SIZE

The basic version of the three-dimensional program operates in-core and handles up to an 800-block calculation grid. Through utilization of disk storage, the alternate version of this program extends the maximum grid size to 69,750 blocks for problems requiring further definition. The program developed for the VE analysis operates in-core and can accomodate a 600-block calculation grid.

TABLE 1 — SENSITIVITY OF VE QUALITY TO τ

Value of τ	Quality of VE Results
Under 50	Very good
50 to 200	Good to fair
Over 200	Unreliable

SPECIAL CONSIDERATIONS: VE ANALYSIS

The procedures for constructing the pseudo capillary pressure and relative permeability relationships which follow from the VE condition were illustrated by Eqs. 3 and 4. In the most general case, such integration operations must be carried out for each block of the areal grid at each time step in the numerical solution because of spatial and dynamic variations of fluid densities and areal heterogeneity of rock properties. However, in many cases density variations are slight enough and rock property variations are regular enough to permit use of an externally generated set of pseudo relative permeability and capillary pressure relationships. To take advantage of cases of this type, the VE program was equipped to handle as input data up to nine sets of externally generated pseudo relationships. Where applicable, this approach may reduce VE computational time requirements by as much as 70 percent.

Several advantages which translate into savings of computational expense are associated with using the VE analysis instead of the three-dimensional analysis for qualifying reservoirs. The most obvious is due to the reduced size of the VE grid as compared with the three-dimensional grid for equivalent areal definition because of layer consolidation. Two further advantages are due to the improved convergence of the VE analysis during the iterative ADI solution because the capillary pressure-saturation relationship is more nearly linear than in the three-dimensional analysis, and the length-to-width dimension ratios of the areal grid blocks normally are closer to unity than the height-to-width and height-to-length dimension ratios of three-dimensional grid blocks.

APPLICATIONS

Results calculated under the three-dimensional analysis and the VE analysis are presented for a laboratory waterflood and a hypothetical water-drive reservoir. For these cases the values for the VE criterion parameter τ were 15 and 210. Validity of the calculations is indicated by comparing results with data from the laboratory waterflood. Calculated results for each of the two systems are compared to indicate the quality of the VE results relative to the three-dimensional results. Where saturation distributions are compared, the three-dimensional results were reduced to an areal basis by depth averaging.

LABORATORY WATERFLOOD: FIVE-SPOT PATTERN (τ = 15)

Oil recovery curves from several laboratory waterfloods were presented by Gaucher and

Lindley.[6] Calculations were made using the three-dimensional and VE programs to simulate the flood which was scaled to represent a 42 B/D water injection rate into a 16-md, 20-acre 5-spot 20 ft thick. Properties reported by Gaucher and Lindley for the fluids and unconsolidated sand were employed in the calculations.

The experimental data, the results calculated in three-dimensional and under VE and those calculated in three-dimensional by Peaceman and Rachford several years ago are compared in Fig. 2. The agreement between the experimental data and the three calculations is generally good although the calculations predict slightly higher recoveries than were observed experimentally beyond 0.9 PV of water injection.

The close agreement between the Peaceman-Rachford and the present three-dimensional results provide a favorable cross-check between the techniques that is especially interesting since different methods of numerical solution were used in the two programs — leap-frog solution of P and R· equations in the former, and simultaneous solution of Φ_w and Φ_n equations in the latter. Results calculated by the present three-dimensional and VE programs were virtually identical (Fig. 2). Under the conditions of this flood, the value of the VE criterion parameter r was 15. The three-dimensional and VE calculations were performed over $10 \times 10 \times 5$ and 10×10 grids, and computational costs to simulate 1.2 PV injection (50 years of field production) were $250 and $37.

NATURAL WATER DRIVE: HETEROGENEOUS RESERVOIR (r = 210)

This example illustrates the application of three-dimensional methods to a typically hetrogeneous reservoir. Production is taken from 19 wells distributed over the crest of the structure, and water influx due to a natural water drive occurs along the southern and eastern boundaries of the hypothetical reservoir (Fig. 3). Within each of the six areal regions, rock properties are layered uniquely (Table 2). Rate schedules for the wells and influx points are not tabulated, but during the 10-year study period production and influx were essentially in balance at a rate of 1 PV in 65 years. The areal sweep velocity at this influx rate is about 1 ft/day.

The fluid distributions after 10 years of production as calculated using the three-dimensional and VE programs and the initial saturations are presented in Fig. 3. The somewhat irregular pattern of fluid movement predicted by the three-dimensional analysis can be seen most clearly in

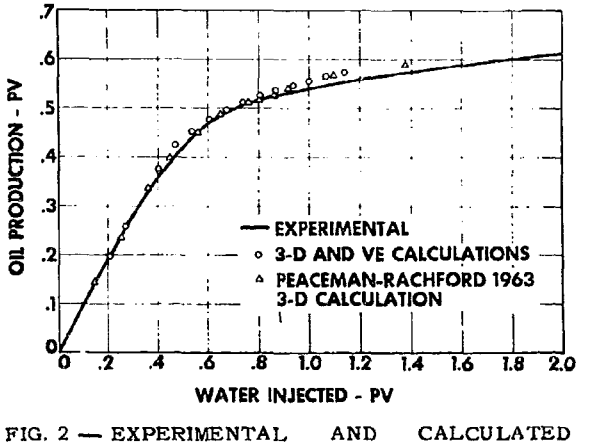

FIG. 2 — EXPERIMENTAL AND CALCULATED BEHAVIOR OF LABORATORY FIVE-SPOT WATERFLOOD.

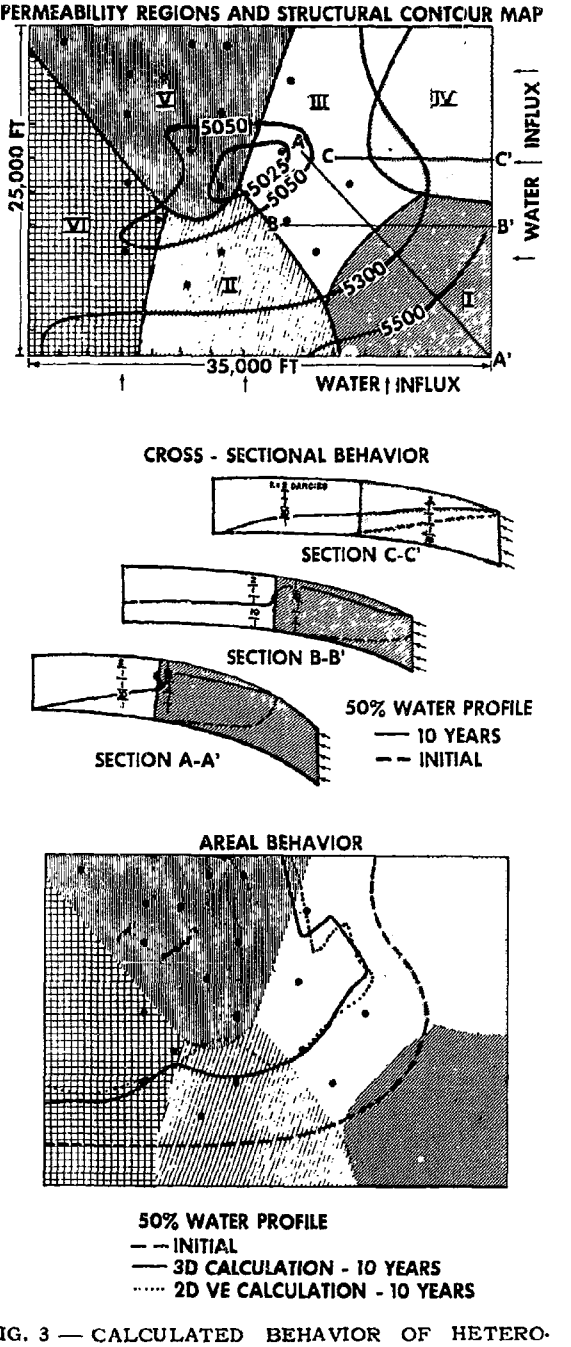

FIG. 3 — CALCULATED BEHAVIOR OF HETEROGENEOUS RESERVOIR UNDER NATURAL WATER DRIVE.

the cross-sectional projections taken normal to the structural surfaces. As expected, the advancing fluids flowed preferentially within the high-permeability streaks. Since the permeability governing drainage out of the loose sand layer was markedly lower than that for areal advance (lower by a factor of 500 between a 10- and 0.1-darcy layer because of the harmonic averaging) vertical redistribution proceeded slowly. Thus, as shown by Sections A-A′ and B-B′ in Fig. 3, where the loose layer was near the sand top, water override developed in some regions of the reservoir. However, in other regions where the loose layer was near the sand bottom (Section C-C′), pronounced underrunning by water occurred.

Considering the degree of heterogeneity in this reservoir and the definitely non-VE character of the three-dimensional results, it is surprising that the VE and three-dimensional results agreed so well. Of course, where water override occurred, its effects were obscured by depth-averaging saturations. The r value of 210 for this problem indicates that the VE simplification is invalid and that VE results would not be reliable. Computational costs for the three-dimensional and VE calculations were approximately $400 and $60.

CONCLUSIONS

1. The validity of the three-dimensional analysis was indicated by experimental data. Good agreement between calculated and experimental oil recoveries was obtained for a laboratory waterflood.

2. Applicability of the VE analysis for simulating three-dimensional flow behavior in reservoirs having good vertical communication was demonstrated. Results calculated using this modified two-dimensional areal analysis compared favorably with those from the more expensive but more rigorous three-dimensional analysis in several example applications.

3. The VE criterion parameter provides a quantitative test for determining whether a particular reservoir can be modeled reliably using the VE analysis. Experience from numerous case studies indicates that reservoirs having parameter values greater than 200 may require three-dimensional modeling.

4. Both the three-dimensional and VE analyses provide the reservoir engineer with reliable, comprehensive techniques for predicting three-dimensional reservoir flow behavior. From a computational cost standpoint, the VE analysis, if applicable, should be used in preference to the three-dimensional analysis. However, even for reservoirs which require the three-dimensional analysis, computational costs normally need not be prohibitive.

TABLE 2 — DATA FOR HETEROGENEOUS RESERVOIR UNDER NATURAL WATER DRIVE

Water viscosity	.40 cp
Oil viscosity	.55 cp
Rock compressibility	3×10^{-6} psi^{-1}
L_x	35,400 ft
L_y	25,600 ft
h	300 ft
Three-dimensional grid	15 × 10 × 5
VE grid	15 × 10

Pressure (psi)	B_w (res. bbl/STB)	ρ_{sw} (psi/ft)	B_o (res. bbl/STB)	ρ_{so} (psi/ft)
600	1.005	.4324	1.316	.2923
4,000	.9956	.4371	1.244	.3093

	Areal Permeability (k_{xy}) Layers					Porosity (ϕ) Layers				
Region	1	2	3	4	5	1	2	3	4	5
I	1	10	.1	1	1	.1	.25	.2	.12	.12
II	1	2	10	.1	1	.2	.12	.1	.25	.12
III	2	1	1	10	.1	.12	.12	.25	.2	.1
IV	2	1	1	1	10	.2	.12	.1	.12	.25
V	1	1	10	.2	1	.12	.1	.25	.2	.12
VI	1	8	.1	10	.2	.12	.25	.1	.25	.2

Vertical permeability: $k_z = 0.1\ k_{xy}$.
Interblock permeability: harmonic mean.

NOMENCLATURE

$(A_{xw})_{i+\frac{1}{2},j,k} = \dfrac{0.00633}{5.614}\dfrac{\Delta y \Delta z}{\Delta x}\left(k_x \dfrac{k_{rw}}{\mu_w}\rho_{sw} b_w\right)_{i+\frac{1}{2},j,k}$ transmissibility of wetting phase in x direction, bbl/ft-day

b = formation volume factor, STB/reservoir bbl or Mcf/reservoir bbl

B = formation volume factor, reservoir bbl/STB or reservoir bbl/Mcf

$b' = db/dp$

c_f = rock compressibility, psi^{-1}

D = depth, or vertical position, ft, measured positively downward

g = acceleration of gravity, ft/sq second

g_c = gravitational conversion constant, 32.2 lb$_m$ ft/lb$_f$ sq second

$G_1 = V_p \rho_{sw} S_w b_w \left(c_f + \dfrac{b_w'}{b_w} - \dfrac{S_w'}{S_w}\right) \Big/ \Delta t$, STB/ft-day

$G_2 = V_p \rho_{sw} S_w b_n \left[\dfrac{S_w'}{S_w} + c_f \dfrac{S_n}{S_w} + R_s \dfrac{b_w}{b_n}\left(c_f - \dfrac{S_w'}{S_w} + \dfrac{b_w'}{b_w} + \dfrac{R_s'}{R_s}\right)\right] \Big/ \Delta t$, STB/ft-day

$H_1 = V_p \rho_{sn} S_w' b_w / \Delta t$, STB/ft-day

$H_2 = V_p \rho_{sn} S_n b_n \left[\dfrac{S_w'}{S_n} - \dfrac{b_n'}{b_n} - R_s \dfrac{S_w'}{S_n}\dfrac{b_w}{b_n}\right] \Big/ \Delta t$, STB/ft-day

H_k = iteration parameter for k^{th} iteration

h = reservoir thickness, ft

i_n = nonwetting phase injection rate, STB/D of Mcf/D

i_w = wetting phase injection rate, STB/D

i_v = injection rate, STB/D/bbl of bulk reservoir volume

k = absolute permeability, md

k_r = relative permeability

k_{rp} = pseudo relative permeability

L_x = reservoir length, ft

L_y = reservoir width, ft

P_c = capillary pressure, $p_n - p_w$, psi

P_{cp} = pseudo capillary pressure, i.e., P_c at reference plane of reservoir, psi

$P_c' = dP_c/dS_w$

p = pressure, psi

R_s = solution gas ratio, Mcf/STB

$R_s' = dR_s/dp$

S or S_w = wetting phase saturation

$\bar{S}$ = average wetting phase saturation through reservoir thickness

$S' = dS_w/dP_c$

Δt = time step, days

t = time, days

u = volumetric or Darcy velocity, volume/sq ft-day

V_p = block pore volume, $\phi\ \Delta x \Delta y \Delta z/5.614$, bbl

Δx, Δy, Δz = block dimensions, ft

α_d = dip angle

ϕ = porosity

Φ = flow potential, $\int_0^p \frac{dp}{\rho_s(p)} - D$, ft

ρ_s = specific weight, psi/ft

$\Delta\rho = \rho_{sw} - \rho_{sn}$

μ = viscosity, cp

$$\psi = - \frac{k_{rn}\, k_{rw}}{k_{rn}\, \mu_w + k_{rw}\, \mu_n} \frac{dP_c}{dS_w}$$

SUBSCRIPTS

w = wetting phase

n = nonwetting phase

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APPENDIX A

THREE-DIMENSIONAL FLOW EQUATIONS

DERIVATION OF EQUATIONS

The basic equations governing two-phase flo in porous media are (1) the continuity or materi balance equation for each phase

$$-\nabla \cdot (b_w \vec{u}_w) + (i_v)_w = \frac{\partial}{\partial t} (\phi\, b_w\, S_v$$

. . . . . (A-1

$$-\nabla \cdot (b_w R_s \vec{u}_w + b_n \vec{u}_n) + (i_v)_n =$$

$$\frac{\partial}{\partial t} (\phi\, b_n S_n + \phi\, b_w R_s S_w) \quad . .(A-1$$

(2) Darcy's law relating superficial velocities flow potential

$$\vec{u}_w = - k \frac{k_{rw}}{\mu_w} \rho_{sw} \nabla \Phi_w \quad . . .(A-2$$

$$\vec{u}_n = - k \frac{k_{rn}}{\mu_n} \rho_{sn} \nabla \Phi_n \quad . . . .(A-2$$

and (3) the capillary pressure definition

$$P_c = p_n - p_w \quad . . . . . (A-$$

Eq. A-3, along with the definition of Φ,

$$\Phi_w = \int_o^{p_w} \frac{dp}{\rho_{sw}} - D$$

$$\Phi_n = \int_o^{p_n} \frac{dp}{\rho_{sn}} - D \quad \ldots . \quad (A\text{-}4)$$

allows expression of the saturation derivative $\partial S_w/\partial t$ in terms of the potentials

$$\frac{\partial S_N}{\partial t} = S' \left[\rho_{sn} \frac{\partial \Phi_n}{\partial t} - \rho_{sw} \frac{\partial \Phi_w}{\partial t}\right] \quad \ldots . \quad (A\text{-}5)$$

Substituting Eqs. A-2 and A-5 into Eq. A-1 yields two equations in the two dependent variables Φ_w and Φ_n,

$$\nabla \cdot \left(k \frac{k_{rw}}{\mu_w} b_w \rho_{sw} \nabla \Phi_w\right) + (i_v)_w =$$

$$\phi S \rho_{sw} b_w \left(c_f - \frac{S'}{S} + \frac{b'_w}{b_w}\right) \frac{\partial \Phi_w}{\partial t} + \phi S' \rho_{sn} b_w \frac{\partial \Phi_n}{\partial t} \quad \ldots . \quad (A\text{-}6a)$$

$$\nabla \cdot \left(k \frac{k_{rw}}{\mu_w} b_w R_s \rho_{sw} \nabla \Phi_w\right) + \nabla \cdot \left(k \frac{k_{rn}}{\mu_n} b_n \rho_{sn} \nabla \Phi_n\right) + (i_v)_n =$$

$$\phi S \rho_{sw} b_w \left[\frac{S'}{S} \frac{b_n}{b_w} + \frac{1 - S}{S} c_f \frac{b_n}{b_w} + R_s \left(c_f - \frac{S'}{S} + \frac{R'_s}{R_s} + \frac{b'_w}{b_w}\right)\right] \frac{\partial \Phi_w}{\partial t} - \phi \rho_{sn} (1 - S) b_n \left(\frac{S'}{1-S} - \frac{b'_n}{b_n} - R_s \frac{S'}{1-S} \frac{b_w}{b_n}\right) \frac{\partial \Phi_n}{\partial t} \quad \ldots . \quad (A\text{-}6b)$$

where $d\phi/dp_w = c_f \phi$, and $S \equiv S_w$.

Multiplying Eqs. A-6 by the block volume $\Delta x \Delta y \Delta z$ and writing derivatives in difference form yields

$$\Delta A_w \Delta \Phi_w + i_w = G_1 \cdot \Delta_t \Phi_w + H_1 \cdot \Delta_t \Phi_n \quad \ldots . \quad (A\text{-}7a)$$

$$\Delta A_w R_s \Delta \Phi_w + \Delta A_n \Delta \Phi_n + i_n = G_2 \cdot \Delta_t \Phi_w + H_2 \cdot \Delta_t \Phi_n \quad \ldots . \quad (A\text{-}7b)$$

where difference notation is defined by $\Delta A \Delta \Phi = \Delta_x A_x \Delta_x \Phi + \Delta_y A_y \Delta_y \Phi + \Delta_z A_z \Delta_z \Phi$; $\Delta_x A_x \Delta_x \Phi = A_{x\, i+\frac{1}{2},\, j,\, k} (\Phi_{i+1,\, j,\, k} - \Phi_{i,\, j,\, k}) - A_{x\, i-\frac{1}{2},\, j,\, k} (\Phi_{i,\, j,\, k} - \Phi_{i-1,\, j,\, k})$; $\Delta_t \Phi = \Phi_{i,\, j,\, k,\, m+1} - \Phi_{i,\, j,\, k,\, m}$, and A_x, A_y, A_z, G_1, G_2, H_1 and H_2 are defined in the Nomenclature. The source terms i are in units of STB/D or Mcf/D injected for the block. All unsubscripted terms such as i, G_1, etc., in Eq. A-7 are understood to apply at spatial position i, j, k. The potential Φ on the left-hand side is understood to apply at the new time $m + 1$; Eqs. A-7 are, in this sense, implicit. Transmissibilities A_w, $A_w R_s$ and A_n are all evaluated at old time m.

Application of the Douglas-Rachford[5] alternating-direction procedure to Eqs. A-7 will now be described. Eqs. A-7 can be written more simply in matrix form as

$$\Delta A \Delta \underline{\Phi} + \underline{i} = G \Delta_t \underline{\Phi} \quad \ldots . \quad (A\text{-}8)$$

where

$$A = \begin{bmatrix} A_w & 0 \\ A_w R_s & A_n \end{bmatrix}$$

$$\underline{i} = \begin{bmatrix} i_w \\ i_n \end{bmatrix}$$

$$G = \begin{bmatrix} G_1 & H_1 \\ G_2 & H_2 \end{bmatrix}$$

$$\underline{\Phi} = \begin{bmatrix} \Phi_w \\ \Phi_n \end{bmatrix} \quad \ldots . \quad (A\text{-}9)$$

The x-direction calculation is

$$\Delta_x A_x \Delta_x \underline{\Phi}^* + \Delta y\, Ay\, \Delta y\, \underline{\Phi}^k + \Delta_z A_z \Delta_z \underline{\Phi}^k + \underline{i} = G(\underline{\Phi}^* - \underline{\Phi}_m) + H(\underline{\Phi}^* - \underline{\Phi}^k) \quad \text{(A-10a)}$$

The y- and z-direction calculations are

$$\Delta_x A_x \Delta_x \underline{\Phi}^* + \Delta_y A_y \Delta_y \underline{\Phi}^{**} + \Delta_z A_z \Delta_z \underline{\Phi}^k + \underline{i} = G(\underline{\Phi}^{**} - \underline{\Phi}_m) + H(\underline{\Phi}^{**} - \underline{\Phi}^k) \quad \text{(A-10b)}$$

and

$$\Delta_x A_x \Delta_x \underline{\Phi}^* + \Delta_y A_y \Delta_y \underline{\Phi}^{**} + \Delta_z A_z \Delta_z \underline{\Phi}^{k+1} + \underline{i} = G(\underline{\Phi}^{k+1} - \underline{\Phi}_m) + H(\underline{\Phi}^{k+1} - \underline{\Phi}^k) \quad \text{(A-10c)}$$

The matrix H is given by

$$H = \begin{bmatrix} H_k \Sigma A_w & o \\ H_k \Sigma A_w R_s & H_k \Sigma A_n \end{bmatrix} \quad \text{(A-11)}$$

where the Σ term denotes summation of the six transmissibilities on the six faces of the block. The term H_k is an iteration parameter discussed below. Index k is iteration number; $\underline{\Phi}^k$ is the k^{th} iterate of $\underline{\Phi}$. The solution of equations identical in form to Eqs. A-10a, A-10b and A-10c for $\underline{\Phi}^*$, $\underline{\Phi}^{**}$ and Φ^{k+1}, respectively, is described in detail in the appendix to Ref. 1.

Iteration parameters are obtained from a relation that results from an analysis of convergence of the two-dimensional (Peaceman-Rachford) iterative ADI.[1] The convergence analysis is beyond the scope of this paper.[7] The relation employed here for minimum iteration parameter is

$$H_{min} = \frac{\pi^2}{2N_x^2} \cdot \frac{1}{1 + \frac{k_z}{k_x}\frac{\Delta x^2}{\Delta z^2}} \quad \text{(A-11)}$$

where N_x is number of blocks in the x direction, k_x and k_z are rock permeabilities in the x and z directions, respectively, and Δx and Δz are block dimensions. Maximum H_k is unity and intermediate values are spaced geometrically. One *cycle* is defined as, say, six iterations where each iteration involves solving Eqs. A-10 with a single H_k value. Cycles are then repeated using the same set of six iteration parameters until convergence is attained.

APPENDIX B

VE EQUATIONS AND SATURATION FUNCTIONS

ADJUSTMENT OF RELATIVE PERMEABILITY AND CAPILLARY PRESSURE CURVES TO REFLECT EQUILIBRATED SATURATION DISTRIBUTION

Mathematical expression of the VE assumption simply involves integration over z (reservoir thickness direction) of each term in the three-dimensional partial differential equations governing two-phase flow. This integration process yields integrals of the type $\int_{z=-h/2}^{h/2} k\, k_{rw} dz$ and $\int_{-h/2}^{h/2} \phi S\, dz$, where h is reservoir thickness. The first integral relates to *effective* permeability for areal wetting phase flow, taking into account the distribution of absolute and relative permeability through reservoir thickness. The second integral relates to the *average* saturation of the areal block corresponding to a capillary pressure value of P_{cp} at block center. Eq. 2 gives $dz = -dP_c / \Delta\rho \cos \alpha_d$ where $\Delta\rho$ is $\rho_{sw} - \rho_{sn}$ so that these integrals can be written as

$$\frac{1}{\Delta\rho \cos \alpha_d} \int_{P_{cp} - \Delta\rho \cos \alpha_d \frac{h}{2}}^{P_{cp} + \Delta\rho \cos \alpha_d \frac{h}{2}} k(z)\, k_{rw}\, d P_c$$

$$\frac{1}{\Delta\rho \cos \alpha_d} \int_{P_{cp} - \Delta\rho \cos \alpha_d \frac{h}{2}}^{P_{cp} + \Delta\rho \cos \alpha_d \frac{h}{2}} \phi(z)\, S\, d P_c$$

Equilibrium or pseudo relative permeability and capillary pressure curves are defined as

$$k_{prw} = \frac{1}{\Delta\rho \cos \alpha_d} \int_{P_{cp} - \frac{\Delta P_c}{2}}^{P_{cp} + \frac{\Delta P_c}{2}} k(z)\, k_{rw}\, d P_c \Big/ \int_{-\frac{h}{2}}^{\frac{h}{2}} k(z)\, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}1)$$

$$\bar{S} = \frac{1}{\Delta\rho \cos \alpha_d} \int_{P_{cp} - \frac{\Delta P_c}{2}}^{P_{cp} + \frac{\Delta P_c}{2}} \phi(z)\, S\, d P_c \Big/ \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi(z)\, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}2)$$

where $\Delta P_c \equiv \Delta\rho \cos \alpha_d\, h/2$.

These equations define $k_{prw} - \bar{S}$ and $P_{cp} - \bar{S}$ curves which reflect reservoir stratification and vertical saturation distribution. The integrations are easily performed since k_{rw} and S are single-valued functions of P_c (from rock curves), and z is a function of P_c from Eq. 2.

These $k_{prw} - \bar{S}$ and $P_{cp} - \bar{S}$ relationships are dependent upon (1) density difference $\Delta\rho$, (2) dip angle α_d, (3) reservoir thickness h and (4) stratification $k(z)$, $\phi(z)$. Thus, the general case of compressible fluids and areal variation of stratification requires separate pseudo relative permeability and capillary pressure curves for each areal grid point.

If fluids are incompressible, or nearly so, and the reservoir is homogeneous with constant thickness and dip angle, then the pseudo relationships are unique over the entire reservoir. In this case, letting $\alpha \equiv h\, \Delta\rho \cos \alpha_d$,

$$k_{prw} = \frac{1}{\alpha} \int_{P_{cp} - \frac{\alpha}{2}}^{P_{cp} + \frac{\alpha}{2}} k_{rw}\, d P_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}3)$$

$$\bar{S} = \frac{1}{\alpha} \int_{P_{cp} - \frac{\alpha}{2}}^{P_{cp} + \frac{\alpha}{2}} S\, d P_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}4)$$

and these $k_{prw} - \bar{S}$, $P_{cp} - \bar{S}$ relationships may be fed into the VE program and used directly, i.e., the internal integration can be bypassed.

The pseudo or equilibrium relative permeability and capillary pressure curves (Eqs. B-2 through B-4) are compared with rock curves in Fig. 1. The pseudo relationships are always more nearly linear than the rock curves.

APPENDIX C

DERIVATION OF A CRITERION FOR VALIDITY OF VE ASSUMPTION

Consider a sand column, closed at both ends, nearly vertical and saturated with a mixture of two incompressible immiscible fluids. Such a column may be approximately identified with a column of fluid moving areally through a reservoir. As this column moves, the fluid distribution tends toward one of capillary-gravitational equilibrium since the ends are represented by the top and bottom of the reservoir and are therefore closed. The existence of equilibrated fluid distributions through reservoir thickness will depend on the time necessary for transients in such a closed column to decay.

If z denotes distance down the column, then Darcy's law for two-phase flow gives phase velocities as

$$u_w = -\frac{k_{rw}}{\mu_w} k \left(\frac{\partial p_w}{\partial z} - \rho_{sw} \frac{\partial D}{\partial z} \right) \quad \ldots\ldots\ldots\ldots (C\text{-}1)$$

$$u_n = -\frac{k_{rn}}{\mu_n} k \left(\frac{\partial p_n}{\partial z} - \rho_{sn} \frac{\partial D}{\partial z} \right) \quad \ldots\ldots\ldots\ldots (C\text{-}2)$$

where k is permeability in the direction of the column.

Since fluids are incompressible and the column is closed at both ends, the sum of these velocities must be zero,

$$u_w + u_n = 0 \quad \ldots\ldots\ldots\ldots (C\text{-}3)$$

Substituting Eqs. C-1 and C-2 into Eq. C-3 and using the definition $P_c = p_n - p_w$ yields

$$\frac{\partial p_w}{\partial z} = \left[\left(\frac{k_{rn}}{\mu_n}\rho_{sn} + \frac{k_{rw}}{\mu_w}\rho_{sw}\right)\frac{\partial D}{\partial z} - \frac{k_{rn}}{\mu_n}\frac{\partial P_c}{\partial z}\right] \div \left[\frac{k_{rw}}{\mu_w} + \frac{k_{rn}}{\mu_n}\right] \quad \ldots (C\text{-}4)$$

The continuity or material balance equation for water flow is

$$\frac{\partial u_w}{\partial z} + \phi \frac{\partial S}{\partial t} = 0 \quad \ldots (C\text{-}5)$$

Combining Eqs. C-1, C-4 and C-5 yields

$$\frac{\partial}{\partial z}\psi_1 \frac{\partial P}{\partial z} = \frac{\phi}{k} S' \frac{\partial P}{\partial t} \quad \ldots (C\text{-}6)$$

where

$$\psi_1 = -\frac{\frac{k_{rw}}{\mu_w}\frac{k_{rn}}{\mu_n}}{\frac{k_{rw}}{\mu_w} + \frac{k_{rn}}{\mu_n}}$$

$$S' = dS/dP_c$$

$$P = P_c + \Delta\rho \quad D \quad \ldots (C\text{-}7)$$

$$\Delta\rho = \rho_{sw} - \rho_{sn}$$

Taking mean values of ψ_1 and S' for the purpose of analysis, Eq. C-6 can be written

$$\frac{\partial^2 P}{\partial z_D^2} = \frac{\partial P}{\partial t_D} \quad \ldots (C\text{-}8)$$

where $z_D = z/h$, $t_D = k\psi_1 t/\phi h^2 S'$, and ψ_1 and S' now denote mean values. h is column length, i.e., reservoir thickness. The boundary conditions for Eq. C-8 are expressions of the no-flow conditions at the closed column ends $z_D = 0, 1$. These conditions are

$$\frac{\partial P}{\partial z_D} = 0 \quad \text{at } z_D = 0, 1 \quad \ldots (C\text{-}9)$$

The solution of Eq. C-8 subject to boundary conditions in Eq. C-9 and a nonequilibrium initial condition

$$P(z_D, 0) = f(z_D) \quad \ldots (C\text{-}10)$$

is

$$P(z_D, t_D) = A_o + \sum_{n=1}^{\infty} A_n \cos n\pi z_D e^{-n^2\pi^2 t_D} \quad \ldots (C\text{-}11)$$

where

$$A_o = \int_0^1 f(z_D)\, dz_D$$

$$A_n = \frac{\int_0^1 f(z_D) \cos n\pi z_D}{\int_0^1 \cos^2 n\pi z_D\, dz_D} \quad \ldots (C\text{-}12)$$

At large time (equilibrium), P is equal to A_0 at all z_D. The distance from equilibrium at any z_D, t_D is therefore

$$P(z_D, t_D) - A_o = \sum_{1}^{\infty} A_n \cos n\pi z_D e^{-n^2\pi^2 t_D} \quad \ldots (C\text{-}13)$$

The time necessary for $1/e$ decay toward equilibrium of the primary harmonic is then

$$\pi^2 t_D = 1 \quad \ldots (C\text{-}14)$$

or, using the definition of dimensionless time t_D,

$$t = \frac{\phi h^2}{k \psi \pi^2} \quad \ldots\ldots\ldots \text{(C-15)}$$

where

$$\psi = \frac{dP_c}{dS} \quad \psi_1 \quad \ldots\ldots\ldots \text{(C-16)}$$

Eq. C-15 gives an estimate of the time necessary for a $1/e$ decay of a nonequilibrium fluid distribution. If u is an average areal superficial velocity, then $\ell_x \phi/u$ is a measure of the time to flow ℓ_x feet areally. Thus,

$$\tau = \frac{h^2 u}{k \psi \ell_x} \quad \ldots\ldots\ldots \text{(C-17)}$$

is a measure of the ratio of time necessary for vertical direction transient decay to time necessary for a given areal advance. A convenient measure ℓ_x is simply h, meaning that we compare time necessary for vertical transient decay to time necessary to advance an areal distance equal to reservoir thickness. Thus, the validity of the vertical equilibrium assumption should be inversely proportional to the value of

$$\tau = \frac{h u}{k \psi} \quad \ldots\ldots\ldots \text{(C-18)}$$

This group scales compatibly with the scaling laws for reservoirs. That is, if two reservoirs are scaled, they will have the same value of τ.

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