Comparison of Alternating-Direction and Successive Overrelaxation Techniques in Simulation of Reservoir Fluid Flow

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ABSTRACT

This study was performed to compare the capability and computing efficiency of successive overrelaxation (SOR) and alternating-direction (ADI) techniques in simulating pressure maintenance by water and gas injection. The calculations simulated two-phase flow and accounted for effects of capillary pressure, relative permeability, gravity and reservoir heterogeneity.

The two techniques investigated were applied to the iterative, simultaneous solution of the two flow equations. Several variations of the SOR method were used: point (PSOR), point symmetric (PSSOR), line (LSOR) and line symmetric (LSSOR). The SOR methods were applied in simultaneous solution of the two partial difference equations describing the two-phase flow.

Results showed that, for the oil-water simulation problems investigated here, the ADI iterative technique is superior to all variations of the SOR technique employing single relaxation factors. For all three oil-water problems the best single-value relaxation factor in the SOR technique was found to be unity. The total computing time required for simultaneous solution with ADI ranged from approximately 45 to 75 percent of that required using the best SOR technique, namely, LSOR, when the unity relaxation factor was employed in the latter technique. A significant improvement in the SOR computational requirements was obtained in the PSOR and LSOR simulation of one of the three oil-water problems—a 100 grid point two-dimensional simulation. The improved program, using combinations of relaxation factors, resulted in the reduction of LSOR computing requirements to approximately 94 percent of that required using ADI. Due to the relative complexity of the procedures involved in producing the improved SOR simulation programs, it was not considered feasible to apply these methods to the simulation of the other oil-water problems. Comparative results indicate that similar improvements in the LSOR simulation of the 300 and 625 grid point oil-water problems would still leave LSOR inferior to ADI on a computing time basis.

In the simulation of a 100 grid point gas-oil cross-section, an optimized LSOR simulation using a number of relaxation factors required approximately 76 percent of the computing time that was used in the ADI simulation. The best LSOR run employing a single relaxation factor ($\omega = 1.65$) required approximately 83 percent of the ADI computing time. A satisfactory PSOR simulation of this problem could not be obtained.

INTRODUCTION

A variety of mathematical techniques are available for numerical solution of the partial differential equations governing multidimensional multiphase fluid flow in reservoirs. This work was performed to compare the capability and computing efficiency of two such techniques.

The model (set of equations) employed simulates the three-dimensional, unsteady-state flow of two immiscible, incompressible phases and is applicable to pressure maintenance-type problems involving flank or pattern water injection or gas injection. The equations account for effects of gravity, capillarity, relative permeability and arbitrary reservoir geometry and heterogeneity. The model consists of two partial differential equations expressing conservation of mass of each flowing phase.

The model equations were expressed in implicit finite difference form and solved simultaneously for the wetting and nonwetting phase flow potentials. This simultaneous solution at each time step was performed using (1) an alternating-direction technique[1,2] (ADI), and (2) several

Original manuscript received in Society of Petroleum Engineers office Sept. 11, 1967. Revised manuscript received June 14, 1968. Paper (SPE 1880) was presented at SPE 42nd Annual Fall Meeting held in Houston, Tex., Oct. 1-4, 1967.

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[1]References given at end of paper.

variants of the successive overrelaxation method[3-7] (SOR). The following variations of the latter technique were employed: point (PSOR), point symmetric (PSSOR), line (LSOR) and line symmetric (LSSOR). The use of the symbols ADI, PSOR, etc., in the remainder of this paper will refer specifically to the simultaneous solution using that particular iterative technique.

The application of SOR in simultaneous solution of the two flow equations simply means that we actually employed block SOR. That is, the difference equation solved is itself a matrix equation with 2×2 matrix coefficients. The scope of this work did not include a theoretical analysis of convergence rates for this block SOR method. This paper simply reports numerical experiments comparing ADI and the particular block SOR scheme defined clearly in the Appendix.

The reservoir fluid flow problems considered involve the simulation of two types of pressure maintenance — water injection and gas injection. The water injection systems investigated involved both two-dimensional areal and three-dimensional grids, while in the gas injection case a vertical slice (cross-section) was simulated.

The effect of grid refinement (number of blocks or grid points) on the relative efficiency of ADI and SOR iterative techniques was examined in a two-dimensional areal problem by using grids of 100 and 625 blocks.

DESCRIPTION OF THE MODEL

The following partial differential equations describe unsteady-state, incompressible, two-phase flow.

$$\nabla \cdot \left[\frac{k k_{rw}}{\mu_w} \nabla \Phi_w \right] + \frac{B_w Q_w}{V_b} = \phi \frac{\partial S}{\partial t}, \quad \ldots (1a)$$

$$\nabla \cdot \left[\frac{k\, k_{rn}}{\mu_n} \nabla \Phi_n \right] + \frac{B_n Q_n}{V_b} = - \phi \frac{\partial S}{\partial t} \quad \ldots (1b)$$

The difference approximations to Eqs. 1a and 1b used are

$$\Delta A_w \Delta \Phi_w + B_w Q_w = \frac{V_p}{\Delta t} \Delta_t S, \quad \ldots (2a)$$

$$\Delta A_n \Delta \Phi_n + B_n Q_n = - \frac{V_p}{\Delta t} \Delta_t S \quad \ldots (2b)$$

where:

$$\Delta_t S \equiv S_{i,j,k,m+1} - S_{i,j,k,m}, \quad \ldots (3a)$$

$$\Delta A \Delta \Phi = \Delta_x A_x \Delta_x \Phi + \Delta_y A_y \Delta_y \Phi + \Delta_z A_z \Delta_z \Phi, \quad \ldots (3b)$$

$$\Delta_x A_x \Delta_x = A_{xi+\frac{1}{2},j,k} \left(\Phi_{i+1,j,k} - \Phi_{i,j,k} \right)_{m+1} - A_{xi-\frac{1}{2},j,k} \left(\Phi_{i,j,k} - \Phi_{i-1,j,k} \right)_{m+1} \quad \ldots (4)$$

The interblock transmissibilities A_{xw}, A_{xn} are defined as indicated by

$$A_{xwi-\frac{1}{2},j,k} = \left(\frac{k\, \Delta y\, \Delta z}{\Delta x} \frac{k_{rw}}{\mu_w} \right)_{i-\frac{1}{2},j,k}, \quad \ldots (5)$$

and A_y, A_z are similarly defined. Subscripts denote position and time as

$$x = i\Delta x, \quad y = j\Delta y, \quad z = k\Delta z, \quad t = m\Delta t.$$

The difference Eqs. 2a and 2b are "implicit" because of the "backward" difference relative to the index $m + 1$ associated with the space differences, Eq. 4.

The interblock transmissibility defined in Eq. 5 involves a relative permeability value, k_{rw}. The value used is the relative permeability corresponding to the saturation of the upstream block.[8] For example, the relative permeability for flow between adjacent Blocks 1 and 2 is k_{r_1} if potentials obey $\Phi_1 > \Phi_2$, while k_{r_2} is used if $\Phi_2 > \Phi_1$. This handling of interblock relative permeability is intuitively correct if we consider the extreme case of oil draining from a block (1) of low oil saturation to an adjacent block (2) of high oil saturation. If k_{ro_1} were 0.05 and k_{ro_2} were 0.8, and simple arithmetic averaging were used to obtain $k_{ro_{1\frac{1}{2}}}$, a relative permeability of 0.425 would be used and oil would drain out of Block 1 at a rapid, highly erroneous rate. However, an upstream weighting would give an interblock transmissibility corresponding to a relative permeability of 0.05, which would result in a more realistic, retarded rate of gravity drainage. More pertinent to this weighting problem, however, is a comparison between a series of Buckley-Leverett profiles and one-dimensional calculations using the model described here with various weight factors on upstream and downstream relative permeabilities. In all cases best agreement between the correct (Buckley-Leverett) profiles and one-dimensional calculations was obtained for a 1.0 (or very nearly so) upstream weighting factor.

The term $\Delta_t S$ in Eqs. 2a and 2b is replaced in terms of potentials by use of the capillary pressure definition

$$P_c = p_n - p_w = \Phi_n - \Phi_w - \frac{\Delta \rho\, g}{144\, g_c} Z. \quad \ldots (6)$$

Use of this relation gives

$$\Delta_t S = S' \Delta_t P_c = S' (\Delta_t \Phi_n - \Delta_t \Phi_w) \quad (7)$$

where S' is a chord of the capillary pressure curve

$$S' \equiv \frac{S_{m+1} - S_m}{P_{cm+1} - P_{cm}} \quad \ldots\ldots\ldots \quad (8)$$

and where m is the time index, $t = m\Delta t$.

In the systems analyzed, capillary pressure has been assumed to be a single-valued function of saturation over the whole range of potentials encountered.

Inserting $\Delta_t S$ from Eq. 7 into Eqs. 2a and 2b gives

$$\Delta A_w \Delta \Phi_w + B_w Q_w = G \Delta_t \Phi_w - G \Delta_t \Phi_n \quad (9a)$$

$$\Delta A_n \Delta \Phi_n + B_n Q_n = - G \Delta_t \Phi_w + G \Delta_t \Phi_n \quad (9b)$$

where

$$\Delta_t \Phi \text{ is } \Phi_{i,j,k,m+1} - \Phi_{i,j,k,m}$$

and

$$G = -V_p S'/\Delta t.$$

This development of the three-dimensional Eqs. 9a and 9b is included here only for the sake of completeness since the two-dimensional forms of these equations were given by Douglas, Peaceman and Rachford.[2]

For all the example problems investigated, Neumann boundary conditions (zero flow across the boundaries) were used.

METHOD OF SOLUTION

In this study Eqs. 9a and 9b were solved simultaneously by iterative techniques (ADI and block SOR) for the wetting and nonwetting phase potentials. The formulation of these iterative techniques is given in the Appendix.

In connection with this use of iterative methods, it is of interest to note that Eqs. 9a and 9b may be represented in matrix form as

$$\Delta A \Delta \underline{\Phi} + \underline{Q} = C \Delta_t \underline{\Phi} \quad \ldots\ldots\ldots \quad (10)$$

where

$$A = \begin{Bmatrix} A_w & 0 \\ 0 & A_n \end{Bmatrix} \qquad C = \begin{Bmatrix} G & -G \\ -G & G \end{Bmatrix},$$

$$\underline{\Phi} = \begin{Bmatrix} \Phi_w \\ \Phi_n \end{Bmatrix} \quad \text{and} \quad \underline{Q} = \begin{Bmatrix} B_w Q_w \\ B_n Q_n \end{Bmatrix}.$$

Since the determinant of matrix C is zero, iterative techniques are usually required for efficient solution. The alternative is direct solution by Gaussian elimination, which, in general, requires considerably more computation than iterative ADI or SOR. Exceptions to this statement occur for grid systems that have a relatively small number of blocks (points) in one of the dimensions.

If fluid compressibility effects were included here, then the determinant of the matrix C would be non-zero and the system of Eqs. 9a and 9b would be parabolic rather than elliptic.* The significance of this fact is that a parabolic system can be solved by an alternating-direction calculation without iteration. However, as discussed in the literature,[9] whether the equations are elliptic or parabolic (whether compressibility effects are absent or present), the use of an iterative as opposed to a noniterative technique in general allows use of a larger time step and results in lower computer expense for solution of the problem.

Although the *system* of equations solved simultaneously in this work is elliptic, the *individual* equations are of a parabolic form. Because of this, the range of values of the relaxation factor that will produce convergence in the block SOR solution of this simultaneous system may differ markedly from the range that yields convergence in the scalar (single-unknown) elliptic problem. (This latter range is for relaxation factors between 0 and 2, with the optimum range between 1 and 2, that is, with overrelaxation.) A theoretical treatment of the convergence analysis in the case of point and line overrelaxation as applied to the scalar parabolic equation is included in Ref. 10. This analysis shows that in cases where the parabolic term dominates, convergence with both point and line SOR methods is most rapid with relaxation factors close to unity. A theoretical analysis of the simultaneous system actually investigated in this work is considerably more complex and was not undertaken. However, if an analogy with regard to "parabolic" dominance holds between the scalar single equation and the simultaneous system treated here, then a theoretical analysis of convergence might indicate an optimum ω value near unity.

When the iterated solutions for the wetting and nonwetting phase potentials meet the applicable closure criteria for a given time step, the new wetting phase saturation at each grid point is determined by applying Eq. 7 in the following form.

$$S_{m+1} = S_m + S' (\Phi_{nm+1} - \Phi_{nm} - \Phi_{wm+1} + \Phi_{wm}) \quad \ldots\ldots \quad (11)$$

*The terms "elliptic" and "parabolic" are used loosely here. If Eq. 10 were a scalar equation (e.g., single-phase flow), then it would be elliptic if the scalar C were 0, parabolic if scalar C were positive. For the case here where C is a matrix, we call Eq. 10 "elliptic" then, by analogy, if the determinant $|C| = 0$ and parabolic if $|C| > 0$.

During the iteration procedure each new pair of phase potentials provides the basis for updating S'. The following damping procedure was found to be beneficial in calculating the new value of S'.

$$S'^{(\ell+1)} = w_1 \left(\frac{S^{(\ell)} - S_m}{P_c^{(\ell)} - P_{cm}} \right) + (1 - w_1) S'^{(\ell)} \quad \ldots (12)$$

where w_1 = damping factor, a function of the current iteration number ℓ ($0 < w_1 \leq 1$).

As discussed in the Appendix, the variables actually solved for are changes in potentials over the iteration, rather than the potentials themselves. This results in greater accuracy in relation to round-off error.

PROCEDURE FOR COMPARISON

The evaluation of the various SOR iterative schemes proceeded along the following lines.

1. By trial and error determine the single average optimum value of ω for each problem that applied over several, if not all, of the time steps required.

2. Investigate the possibility of improving convergence by varying relaxation factors iteration-by-iteration during a given time step, and by applying different relaxation factors to the wetting and nonwetting phases.

Success in applying the ADI technique in other problems of the type investigated here indicated that uniformly good convergence rates can be obtained with a single set of iteration parameters applied over the duration of the simulation. For this reason the computer program used did not include any procedure where the iteration parameters depend upon functions defining the state of convergence (e.g., absolute residual sums, incremental material balance error). Accordingly, the same approach was taken at the outset in applying the various SOR techniques. It was found during the course of the work that considerable improvement in the rate of convergence of the SOR techniques could be effected by varying the relaxation factor, depending upon the relative magnitude of the incremental material balance error and the absolute residual sums. For some of the problems analyzed special techniques were programmed into the SOR models to vary the relaxation factors as functions of these convergence criteria.

An experimental approach was taken throughout for the determination of the optimum relaxation factors in the various SOR techniques. The variation of the nonlinear coefficient S' iteration-by-iteration prevents a rigid convergence or stability analysis.

Experience with ADI simulation of similar problems indicated that the application of single sets of iteration parameters provided a more efficient solution routine than attempted stepwise optimization.

CRITERIA FOR CONVERGENCE

The model for simultaneous solution permits closure or convergence to be determined by any combination of the following:

1. The sum of each of the absolute residuals over the grid (Σ $|B1X|$ or Σ $|B2X|$, see Appendix) must be less than a specified tolerance.

2. The incremental balance error (over the time step) must be less than a specified tolerance. The incremental material balance error in the wetting and nonwetting phases is equivalent to the arithmetic sum over the grid of the residuals $B1X$ and $B2X$, respectively, for the closed boundary systems considered here.

3. The maximum saturation change during the last iteration in a given time step must be less than a specified tolerance.

It should be noted that each of the above closure criteria are related to the flow equations themselves (Eqs. 9a and 9b) and hence are applied in exactly the same manner regardless of which iterative scheme is being used.

OPERATIONAL FEATURES OF THE MODEL

In order to ensure that a valid comparison could be made between the various iterative techniques applied, the same computer program was employed in each case, except for the differences in the iteration procedure. This program has the following features:

1. The grid employed is three-dimensional, orthogonal cartesian (rectangular). Areally, any number of blocks may be excluded from the calculation procedure in simulating the actual reservoir configuration. Roughly 1,000 blocks may be handled in-core on a 32K machine, with up to 2,500 blocks on a 64K machine.

2. Since closed boundary conditions were employed, fluid movement across reservoir boundaries is represented by "wells" in the edge blocks.

3. Production and injection is distributed among the various layers on the basis of fluid mobilities and the layer's permeability-thickness products. This procedure assumes uniform potential in all producing layers at the well, which was nearly the case in the problems treated.

4. The reservoir to be simulated may be treated as: a) completely homogeneous, b) consisting of several layers of different properties, or c) heterogeneous. In the latter case, individual block properties (e.g., pore volume, permeability) are assigned. Capillary pressure curves are assigned to each layer using appropriate Leverett-type functions.

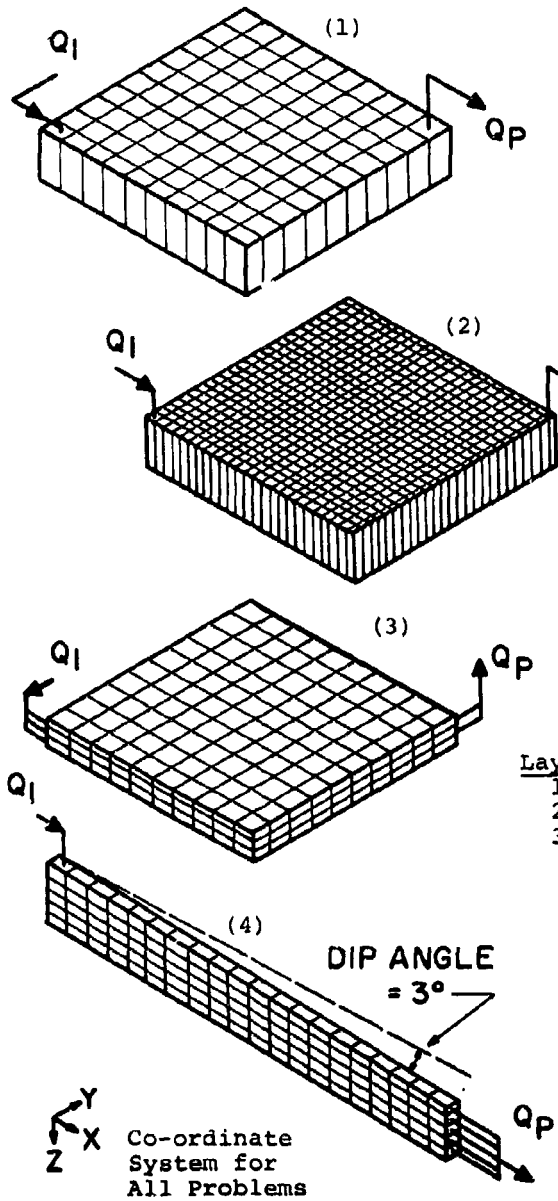
5. The interblock transmissibilities (A_x's, etc.) to each phase are computed at the first of each time step and are held constant throughout that step. An alternative would have been their variation

(updating) within the time step by iteration.

Experience has shown that use of explicit interblock transmissibilities (evaluated at the beginning of the time step) yields accurate results for problems of the type treated here. For these problems, iteration on transmissibility would increase required computing time without significantly changing the computed saturations and pressures. There is no evidence to indicate that the operational features used are inherently prejudicial to either of the basic iterative schemes investigated.

RESERVOIR PROBLEMS ANALYZED

A graphical representation of the four problems considered in this analysis, together with the basic data, are given in Fig. 1. The first three problems are of the oil-water, pressure maintenance type. Each of these three problems represents the simulation of one-quarter of a 20-acre five-spot waterflood. Two-dimensional Problems 1 and 2 are identical except for the grid size employed. Three-dimensional Problem 3 represents a three-layered homogeneous reservoir with injection into



OIL-WATER SYSTEMS

ρ_w=60.0 lb/ft^3 ρ_n=48.0 lb/ft^3

μ_w=0.5 cp. μ_n=2.0 cp.

Problem (1)

NX=10 NY=10 NZ=1

Δx=46.7' Δy=46.7' Δz=100'

k_x=100 md k_y=100 md

$Q_I=Q_p$=100 BPD ϕ=0.20

Initial Conditions:

S_w=0.22 p_w=0.0 psi p_n=6.5 psi

Problem (2)

NX=25 NY=25 NZ=1

Δx=18.68' Δy=18.68' Δz=100'

k_x=100 md k_y=100 md

$Q_I=Q_p$=100 BPD ϕ=0.20

Initial Conditions:

S_w=0.22 p_w=0.0 psi p_n=6.5 psi

Problem (3)

NX=10 NY=1 NZ=3

Δx=46.7' Δy=46.7' Δz=20'

Data by Layer

Layer	ϕ	k_x md	k_y md	k_z md	Capillary Pressure Multiplier	Initial S_w
1	0.16	40	40	1.0	2.2	0.252
2	0.20	200	200	2.0	1.0	0.234
3	0.12	10	10	2.0	4.5	0.516

$Q_I=Q_p$=60 BPD

Initial Conditions

p_w=0.0 psi p_n=6.5 psi, at datum

GAS-OIL SYSTEM

ρ_w=47.52 lb/ft^3 ρ_n=10.08 lb/ft^3

μ_w=1.0 cp. μ_n=0.02 cp.

Problem (4)

NX=20 NY=1 NZ=5

Δx=40' Δy=1' Δz=5'

k_x=500 md k_z=500 md

$Q_I=Q_p$=0.23 BPD ϕ=0.20

Initial Conditions: S_w=1.00

FIG. 1 — PROBLEMS ANALYZED.

all three layers and production from the top two layers.

For the purposes of this study, Problems 1 and 3 were analyzed through 1,080 days of injection, which represents an advance of the flood to approximately one-half of the distance between wells. Problem 2 was run on a comparative basis to only 260 days due to the smaller grid (block) size and corresponding reduced time steps. The relative permeability and capillary pressure curves for these problems are given in Fig. 2.

The following closure criteria were used in the oil-water problems.

	Problem 1	2	3
1. Max. $\Sigma \lvert B1X \rvert$ or Max. $\Sigma \lvert B2X \rvert$	5.0	5.0	5.0
As percent of ΣQ_w	5.0	5.0	12.0
2. Max. incremental material balance error (during any given time step) as percent of ΣQ_w	0.1	0.1	0.1

As shown in Fig. 1, Problem 4 represents the simulation of gas injection in the vertical slice of a slightly tilting, homogeneous reservoir. A uniform permeability of 500 md was taken and injection occurred in the top block at the crest of the reservoir while production is distributed uniformly along the downdip edge. This gas injection problem was analyzed through 360 days. At the end of this time the gas front had advanced along approximately 90 percent of the top layer. The relative permeability and capillary pressure curves for this problem are given in Fig. 3. Closure criteria used for this problem were

1. Max. $\Sigma \lvert B1X \rvert$ or Max. $\Sigma \lvert B2X \rvert$	0.1
As percent of ΣQ_w	4.35
2. Max. incremental material error (during any given time step) as percent of ΣQ_w	0.1
3. Max. saturation change in last iteration	0.001

RESULTS

OIL-WATER PRESSURE MAINTENANCE PROBLEMS

ADI Method

In each of Problems 1, 2 and 3 a maximum of three cycles of five or six iteration parameters each were used. Except for the first time step in Problems 1 and 2, during which the residual sum closure tolerance was not met, convergence in less than the maximum allotted iterations was obtained in every time step. Closure was controlled in all but a few time steps by the residual sums tolerance (or a minimum number of iterations), incremental material balance error being negligible.

For each of the problems, S' was weighted cyclically, such that w_1 = iteration number/iterations/cycle.

No attempts were made to optimize the iteration parameters used, the actual values applied coming virtually directly from the two-dimensional formulas (see Appendix) with the number of iterations per

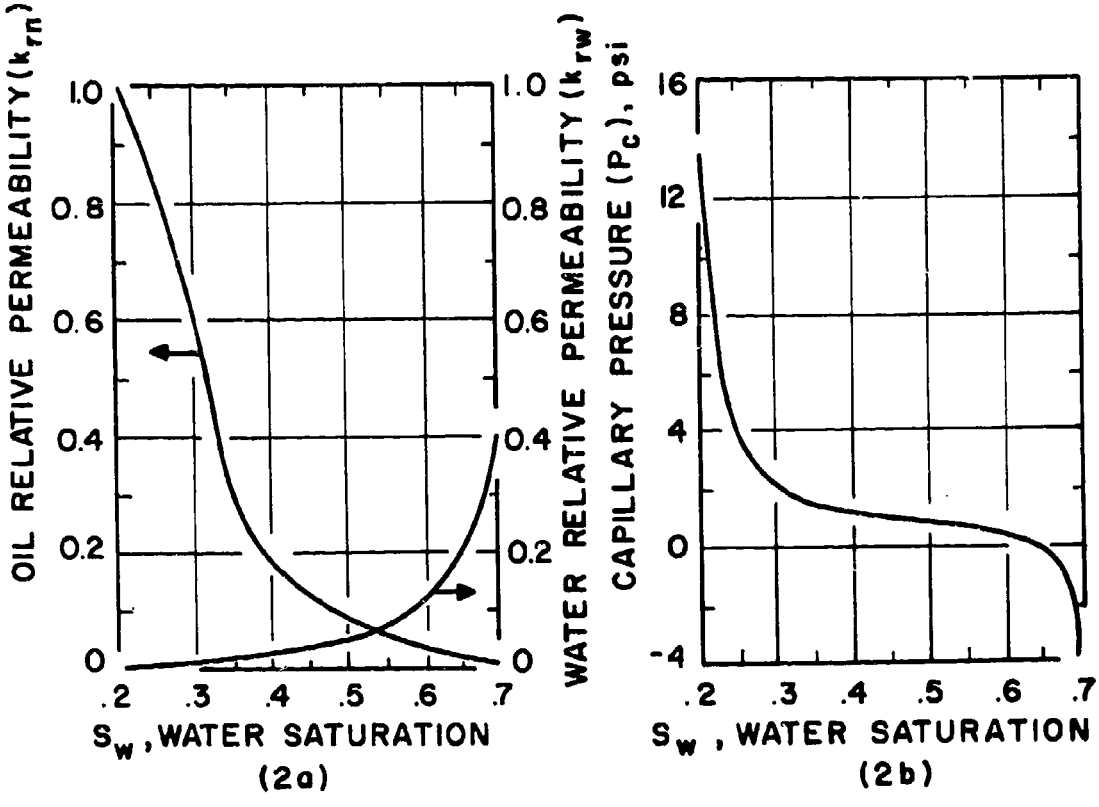


FIG. 2—RELATIVE PERMEABILITY AND CAPILLARY PRESSURE CURVES FOR OIL-WATER SYSTEM.

cycle selected as an average figure for this type of problem.

The summary of the computer run characteristics for ADI applied to the oil-water problems is given in Table 1.

SOR Method

With SOR the maximum number of iterations allowed was 40. It was found that this number was normally adequate to permit closure in the oil-water problems provided a satisfactory relaxation factor was used.

Single-Valued Relaxation Factors. For each of the three water-oil problems, a number of constant relaxation factors lying in the usual optimum range of 1.0 to 2.0 were applied for each of the SOR techniques. In all cases a factor other than 1.0 resulted in increasingly poorer incremental material balance error as the simulation proceeded. This error became worse as the value of the relaxation factor was increased above unity. For a factor of 1.8, an incremental material balance error of about 200 percent resulted during the second time step when applying PSOR to Problem 1. Results with the other SOR techniques on this and other problems were comparable.

In contrast to the incremental material balance behavior, higher values of the relaxation factor would yield initially faster rates of reduction of

TABLE 1 — SUMMARY OF COMPARATIVE RESULTS

Problem No.	No. of Time Steps	Iteration Method	Net Run Time, sec[1] (CDC 6600)	Total Iterations Required	Average Calculated Time per Iteration (sec.)[2]
		Original Simultaneous Model			
1	60	ADI	53.6	492	0.1089
		LSOR (ω = 1.0)	69.9	806	0.0866
		PSOR (ω = 1.0)	101.5	1,352	0.0752
2	46	ADI	308.8	530	0.5827
		LSOR (ω = 1.0)	500.4	1,067	0.4689
		PSOR (ω = 1.0)	630.1	1,523	0.4137
3	60	ADI	215.2	592	0.3635
		LSOR (ω = 1.0)	349.9	1,322	0.2646
		PSOR (ω = 1.0)	405.5	1,783	0.2274
		Modified Simultaneous Model[3]			
1	60	ADI	49.5	492	0.1001
		Improved LSOR	46.6	539	0.0865
		LSOR (ω = 1.0)	65.2	806	0.0809
		Improved PSOR	61.1	827	0.0739
		PSOR (ω = 1.0)	94.6	1,352	0.0700
4	49	ADI (4 cycle)	72.6	831	0.0874
		LSOR (ω = 1.65)	60.5	829	0.0730
		Improved LSOR	55.0	741	0.0742
		PSOR	unsuccessful		

[1]*Net run time* = total computing time, iterations plus all other routines, excluding compile time.

[2]This figure represents *total net run time* averaged for each iteration. The actual time devoted explicitly to the iteration routine differs for each technique, varies from problem to problem, and was not determinable.

[3]The modified simultaneous model differs from the original only in that some procedures were made more efficient to reduce computation time; no changes in the iteration routines themselves were made.

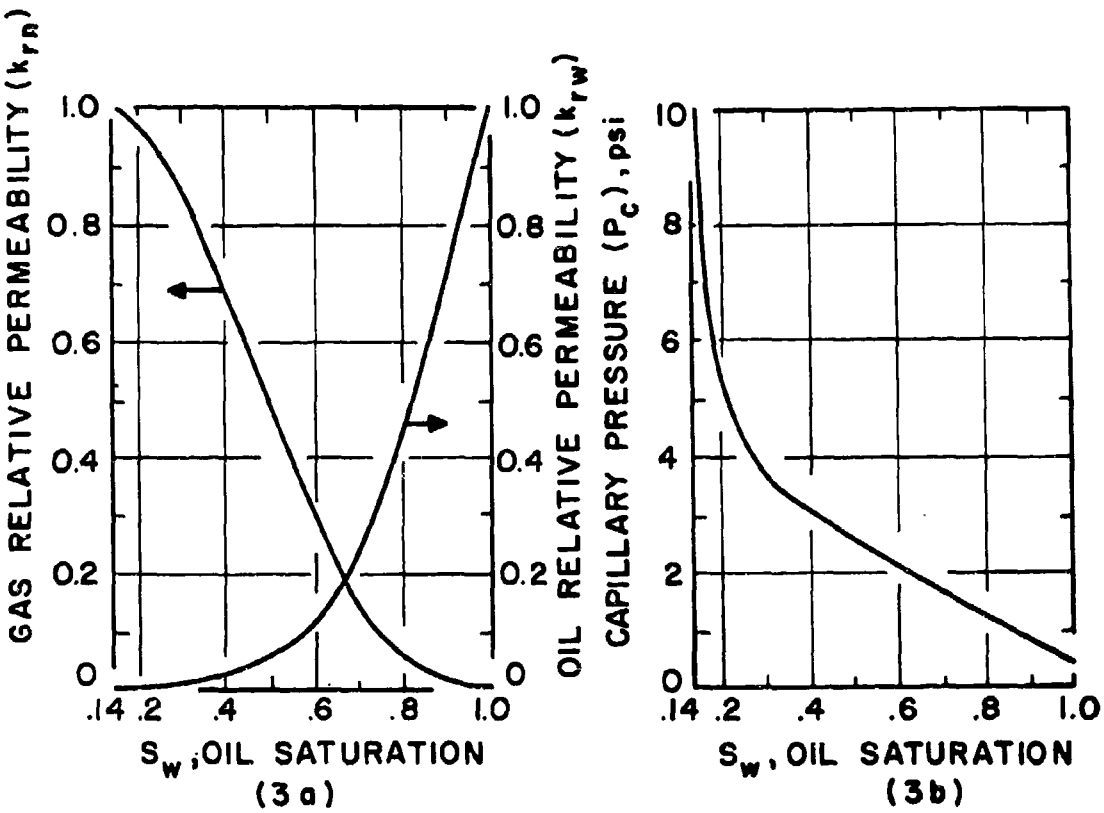


FIG. 3 — RELATIVE PERMEABILITY AND CAPILLARY PRESSURE CURVES FOR GAS-OIL SYSTEM.

the residuals ($B1X$ and $B2X$) at least until the material balance error rendered all results meaningless.

Comparisons run over the first several time steps showed conclusively that for all the SOR variants and for all three problems, the only value of the relaxation factor that would permit a continuation of the simulation to any reasonable number of time steps was unity. Even for a relaxation factor of 1.0 incremental material balance error became the controlling factor in satisfying the closure criteria.

For each of the three problems a comparison between the symmetric and nonsymmetric forms of both point and line overrelaxation was made for a relaxation factor of 1.0. In each case better convergence rates were obtained with the nonsymmetric technique. As a typical result, in the case of PSOR vs PSSOR for $\omega = 1.0$ applied over the first 10 time steps in Problem 3; total number of iterations required equalled 132 for PSOR, 158 for PSSOR. Comparisons of symmetric vs nonsymmetric were made at relaxation factors other than 1.0 and results similar to the above were obtained. However, material balance problems prevented a comparison over more than a few time steps.

As a result of the inability of relaxation factors other than unity being able to maintain an adequate material balance, and of the proven superiority of the nonsymmetric techniques, all final simulation runs for the oil-water problems using constant single-valued relaxation factors were made with the PSOR and LSOR techniques with $\omega = 1.0$.

It should be noted that in the oil-water problems analyzed the parabolic terms in the individual wetting- and nonwetting-phase equations (Eqs. 2a and 2b) were predominant. As noted earlier, the convergence analysis for the single-unknown parabolic equation showed that for cases of parabolic dominance, convergence with the SOR methods should be obtained most rapidly with relaxation factors close to unity. The experimental results described above indicate that this behavior may also be true in general for the two-equation case treated here.

Variable Relaxation Factors. The experimental results with single relaxation factors showed that higher rates of residual reduction could be achieved as the relaxation factors were increased above unity. However, since these higher relaxation factors produced greater incremental material balance errors than with $\omega = 1.0$, more iterations were required to satisfy the convergence criteria. A complete breakdown of the solution would then result with the higher relaxation factors when the incremental material balance error became excessive. It thus appeared that a combination of relaxation factors — those greater than unity to achieve favorable rates of reduction of absolute residual sums, and unity to control the incremental material balance error — would result in improvements in the iteration and computing time requirements for the SOR methods.

Several experiments were conducted using combinations of relaxation factors in the PSOR and LSOR simulation of Problem 1. Of all the combinations tested the best results were obtained using basic relaxation factors of 1.4 and 1.6, with a factor of unity used to control the incremental material balance error. Details on the special programming procedures used in the combination factor runs will be found in Ref. 10. Due to the considerable amount of detailed analysis and experimentation required in developing the improved program using combinations of relaxation factors, it was not found feasible to simulate Problems 2 and 3 with this procedure.

A summary of the results of the SOR simulations is given in Table 1.

Comparison Between ADI and SOR

With reference to the comparative data included in Table 1, it is seen that the computing requirements for ADI simulation of the three oil-water problems is consistently better than that of the best SOR method *using single-valued relaxation factors,* namely, LSOR with $\omega = 1.0$. The comparison to the final times actually used in these problems is

Problem No.	ADI Computing Time as Percent of LSOR Time
1	76.7
2	61.7
3	61.5

Since the simulation of Problem 2 was carried out to only 260 days as compared to 1,080 days for Problem 1, the effect of grid spacing upon the computing requirements of ADI relative to LSOR cannot be made directly from the Table 1 data. However, from the available comparative data through 260 days and the estimated increase in LSOR iteration requirements from 260 through 1,080 days, it has been estimated that the ADI computing requirements for Problem 2 simulation through 1,080 days would be only from 45 to 50 percent of that required for LSOR with $\omega = 1.0$.

As a result of the improvements introduced into the LSOR simulation of Problem 1 through the use of a combination of relaxation factors, the computing requirements for this LSOR simulation were reduced to only 94 percent of that of the ADI simulation.

For all the calculations on the three problems (both ADI and SOR), excellent agreement in the computed pressures and saturations were obtained, except, of course, in those SOR simulations that broke down as a result of excessive material balance errors. In the absence of any noniterative solution against which the results could be checked, the consistency in the results, regardless of which iterative method or iteration parameters were used, is interpreted as implicit proof of the correctness of the solutions.

GAS-OIL PRESSURE MAINTENANCE PROBLEM

ADI Method

This problem presented somewhat greater

difficulty for simulation by the ADI technique than was the case with the simulation of the oil-water problems. Although the closure criteria on absolute residual sums and incremental material balance error were roughly equivalent on a percentage basis to those used for the oil-water problems, a considerably greater number of average iterations per time step were required to effect closure. Closure was controlled throughout the ADI simulation by the incremental material balance error. The simulation was performed with a maximum of four cycles of seven iteration parameters each during every time step.

Computer run characteristics for this ADI simulation are given in Table 1.

SOR Methods

As in the oil-water problems, the initial attempts at the simulation of Problem 4 with the PSOR technique were carried out using constant relaxation factors. For this problem, however, not a single set of such factors would give even an indication that convergence would be possible with PSOR. In each case the sum of absolute residuals could not be reduced below the initial level during the first time step in any reasonable number of iterations. Although a maximum number of 40 iterations/time step was employed as in the oil-water problems, the detailed results from each iteration showed conclusively that, based on the changes during the last several iterations, further improvement would be negligible with increased iterations using PSOR. In attempting to obtain a convergent PSOR simulation of this problem, a number of other special procedures were investigated, including cyclically applied relaxation factors and various grid point ordering systems. None of these special methods resulted in a successful PSOR simulation.

In the case of LSOR it was found that the original x-directional line iteration applied successfully to the oil-water problems failed to give satisfactory convergence when applied to this gas-oil simulation. The use of z-directional lines suggested by Sheffield,[13] however, yielded very successful convergence rates for the LSOR simulation of this problem. A number of final simulations were obtained using both constant-value relaxation factors and combinations of factors. The results obtained with optimum single relaxation factor (ω = 1.65) and with the optimum combination of relaxation factors and combinations of factors (ω = 1.0, 1.6 and 1.65) are given in Table 1. Successful LSOR simulation of this problem with single relaxation factors greater than unity is believed to follow from the relatively small parabolic term in the flow equations for this vertical cross-section.

Comparison Between ADI and SOR Results

As indicated by the data in Table 1, the LSOR simulation of Problem 4 using z-directional lines required less computing time than was the case with ADI simulation. For the optimum single relaxation factor cases the LSOR computing time was 83 percent of the ADI time, while in the combination factor simulation the LSOR computing time was 76 percent of that for ADI simulation.

CONCLUSIONS

The following conclusions are based on the solution of four reservoir problems involving two- and three-dimensional two-phase flow. The extent to which these conclusions may be generalized must take into account this specific nature of the problems.

1. Higher convergence rates, and thus lower computing times, were obtained with nonsymmetric (PSOR and LSOR) as opposed to symmetric (PSSOR and LSSOR) successive overrelaxation techniques.

2. Higher convergence rates were obtained for line (LSOR) as opposed to point (PSOR) SOR techniques.

3. In the simulation of three oil-water problems the alternating-direction (ADI) technique was superior to all variations of the SOR technique when the latter employed single relaxation factors. The ADI computing times were approximately 45 to 75 percent of those required using the best SOR technique with single relaxation factors (LSOR).

4. In order to prevent the growth of material balance errors, with resulting increases in iterations and in solution stability, all final SOR simulations of the oil-water problems using single relaxation factors were conducted with ω = 1.0 (without overrelaxation). A significant improvement in the convergence rates of PSOR and LSOR was obtained in the simulation of the 100-block oil-water problem, the improved LSOR simulation requiring only 94 percent of the ADI computing time. While similar improved SOR simulations could be effected for the other oil-water problems, comparable reductions in computing times would not be sufficient to yield computing times lower than those required with ADI simulation.

5. Convergence in each problem was obtained using ADI with an easily determined set of iteration parameters held constant throughout the run.

6. The LSOR technique was superior to ADI in the simulation of a 100-grid block gas-oil cross-section. The computing time requirements for LSOR using a single optimum relaxation factor and using a combination of factors were 83 and 76 percent, respectively, of the ADI computing time. PSOR simulation of this problem was unsuccessful.

NOMENCLATURE

B = formation volume factor, reservoir volume/unit std vol
k = absolute permeability, md × 0.00633
k_r = relative permeability
p = pressure, psi
P_c = capillary pressure, psi
Q = injection rate, STB/D

V_p = pore volume of the block, bbl

V_b = bulk volume of block, bbl, $(\Delta x)(\Delta y)(\Delta z)/5.6146$

$S = S_w$ = wetting phase saturation

t = time, days

S' = chord of capillary pressure curve, psi^{-1} (see Eq. 8)

Z = vertical position measured positively downward, ft

ϕ = porosity

$\Phi = p - \rho g Z/144 g_c$ = potential, psi

ρ = density, lb/cu ft

$\Delta\rho = \rho_w - \rho_n$

μ = viscosity, cp

Special symbols for difference equation formulation are defined in the text.

SUBSCRIPTS

w = wetting phase

n = nonwetting phase

ACKNOWLEDGMENTS

The authors gratefully acknowledge coverage of the computer cost and the second author's released time through a research project at The U. of Texas sponsored by Continental Oil Co. (Ponca City), Shell Development Co. (Houston) and the Sinclair Oil Co. (Tulsa). The first author was provided financial assistance by INCE, an agency of the Government of Venezuela, while on educational leave from Texas Petroleum Co., Venezuelan Div.

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APPENDIX

FORMULATION OF ITERATIVE TECHNIQUES

THREE-DIMENSIONAL DOUGLAS-RACHFORD ALTERNATING DIRECTION IMPLICIT PROCEDURE

Under this iteration scheme the difference forms of Eqs. 9a and 9b, together with appropriately selected iteration parameters, are solved implicitly during successive sweeps of the grid in the x, y and z directions.[1,2] During the sweep of a particular direction only those potentials entering the gradient terms for that direction are considered as unknowns. Specifically, for the x-direction sweep during the $\ell + 1$ iteration, the iterative forms of Eqs. 9a and 9b are

$$\Delta_x A_{xw} \Delta_x \Phi^*_w + \Delta_y A_{yw} \Delta_y \Phi^{(\ell)}_w + \Delta_z A_{zw} \Delta_z \Phi^{(\ell)}_w - G\Phi^*_w + G\Phi^*_n = H_{\ell+1}(\Sigma A_w)(\Phi^*_w - \Phi^{(\ell)}_w) - B_w Q_w - G\Phi_{wm} + G\Phi_{nm} \quad \text{(A-1)}$$

$$\Delta_x A_{xn} \Delta_x \Phi^*_n + \Delta_y A_{yn} \Delta_y \Phi^{(\ell)}_n + \Delta_z A_{zn} \Delta_z \Phi^{(\ell)}_n + G\Phi^*_w - G\Phi^*_n = H_{\ell+1}(\Sigma A_n)(\Phi^*_n - \Phi^{(\ell)}_n) - B_n Q_n + G\Phi_{wm} - G\Phi_{nm} \quad \text{(A-2)}$$

where $H_{\ell+1}$ = iteration parameter for iteration $\ell + 1$

$\Sigma A_w, \Sigma A_n$ = sum of transmissibilities to the wetting and nonwetting phases, respectively, in all directions about the reference grid point

Φ^* = new potential during the $\ell + 1$ iteration following the x-direction sweep.

For the y- and z-direction sweeps, respectively, for the wetting phase

$$\Delta_x A_{xw} \Delta_x \phi^*_w + \Delta_y A_{yw} \Delta_y \phi^{**}_w + \Delta_z A_{zw} \Delta_z \phi^{(\ell)}_w - G\phi^{**}_w + G\phi^{**}_n = H_{\ell+1}(\Sigma A_w)(\phi^{**}_w - \phi^{(\ell)}_w) - B_w Q_w - G\phi_{wm} + G\phi_{nm} \quad (A\text{-}3)$$

$$\Delta_x A_{xw} \Delta_x \phi^*_w + \Delta_y A_{yw} \Delta_y \phi^{**}_w + \Delta_z A_{zw} \Delta_z \phi^{(\ell+1)}_w - G\phi^{(\ell+1)}_w + G\phi^{(\ell+1)}_n = H_{\ell+1}(\Sigma A_w)(\phi^{(\ell+1)}_w - \phi^{(\ell)}_w) - B_w Q_w - G\phi_{wm} + G\phi_{nm} \quad (A\text{-}4)$$

Analogous expressions will result for the nonwetting phase sweeps in the y- and z-directions.

To enhance accuracy in the results computed at the new iteration level, it is convenient to replace the new potentials after each directional sweep by the difference in potentials with respect to the previous iteration. This is, defining

$$PX = \phi^*_w - \phi^{(\ell)}_w \qquad RX = \phi^*_n - \phi^{(\ell)}_n$$

$$PY = \phi^{**}_w - \phi^{(\ell)}_w \qquad RY = \phi^{**}_n - \phi^{(\ell)}_n$$

$$PZ = \phi^{(\ell+1)}_w - \phi^{(\ell)}_w \qquad RZ = \phi^{(\ell+1)}_n - \phi^{(\ell)}_n \quad (A\text{-}5)$$

Eqs. A-1 and A-2 become

$$\Delta_x AX_w \Delta_x PX - (G + H_{\ell+1}\Sigma A_w)PX + G(RX) = -\left[\Delta A_w \Delta\phi^{(\ell)}_w + B_w Q_w - G(\phi^{(\ell)}_w - \phi_{wm}) + G(\phi^{(\ell)}_n - \phi_{nm})\right] = -B1X^{(\ell)} \quad (A\text{-}6)$$

$$\Delta_x AX_n \Delta_x RX - (G + H_{\ell+1}\Sigma A_n)RX + G(PX) = -\left[\Delta A_n \Delta\phi^{(\ell)}_n + B_n Q_n + G(\phi^{(\ell)}_w - \phi_{wm}) - G(\phi^{(\ell)}_n - \phi_{nm})\right] = -B2X^{(\ell)} \quad (A\text{-}7)$$

B1X and *B2X* are the *residuals* of Eqs. 9a and 9b; that is, the residuals at any iteration level ℓ, which indicate the amount by which the current approximations to the solution are in error, are used directly in the $\ell + 1$ iteration for the determination of new *PX* and *RX* values.

Analogous expressions for the y- and z-direction sweeps are derived in a similar manner, but will nor be included here for the sake of brevity.

Eqs. A-6 and A-7, and the analogous equations for the y- and z-directions, each representing a bitridiagonal system, are solved simultaneously by Gaussian elimination, using the recurrence relationships formulated by Richtmyer.[12] Iteration parameters (H_k's) used in the above equations are determined from relationships developed for two-dimensional Alternating Direction Implicit Procedure, since no theoretical determinations have yet been made for three dimensions. These two dimensional formulas are

1. For two-dimensional system:

$$(H)_{\text{minimum}} = \text{Minimum Of:} \left[\frac{\pi^2}{(2N_x^2)\left(1 + \frac{(\Delta x)^2 k_y}{(\Delta y)^2 k_x}\right)}, \frac{\pi^2}{(2N_y^2)\left(1 + \frac{(\Delta y)^2 k_x}{(\Delta x)^2 k_y}\right)}\right]$$

$$(H)_{\text{maximum}} = 1$$

2. For two-dimensional cross-section (also used for three-dimensional system):

$$(H)_{\text{minimum}} = \frac{\pi^2}{(2N_x^2)\left(1 + \frac{(\Delta x)^2 k_z}{(\Delta z)^2 k_x}\right)}$$

$$(H)_{\text{maximum}} = 2$$

A set of iteration parameters is determined by spacing geometrically between the minimum and maximum values, for an assigned numbers of iteration parameters per cycle. The parameters are then applied cyclically until the necessary closure tolerances have been met.

SUCCESSIVE OVERRELAXATION TECHNIQUES

For sake of brevity, the SOR equations will be illustrated schematically only for a two-dimensional system similar to that of Eq. 10. The detailed formulation of the SOR equations actually used in

this work, derived in a manner analogous to that for ADI presented above, are included in Ref. 10.

A system of equations similar to Eq. 10 may be represented as

$$A\Delta^2 \underline{u} + \underline{Q} = C\Delta_t \underline{u} \quad \ldots\ldots\ldots \text{(A-8)}$$

The form of this equation in two spatial dimensions is

$$A\underline{u}_{i+1,j} + A\underline{u}_{i-1,j} + A\underline{u}_{i,j+1} + A\underline{u}_{i,j-1} - (4A+C)\,\underline{u}_{i,j} + \underline{b}_{i,j} = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(A-9)}$$

where $\underline{b}_{i,j} = \underline{Q}_{i,j} + C\underline{u}_{i,j,m}$.

The *Point SOR* method for Eq. A-9 is defined as

$$\underline{u}_{i,j}^{(\ell+1)} = \underline{u}_{i,j}^{(\ell)} + \omega\,(4A+C)^{-1} \left[\begin{array}{l} A\underline{u}_{i+1,j}^{(\ell)} + A\underline{u}_{i-1,j}^{(\ell+1)} \\ \\ + A\underline{u}_{i,j+1}^{(\ell)} + A\underline{u}_{i,j-1}^{(\ell+1)} \\ \\ - (4A+C)\,\underline{u}_{i,j}^{(\ell)} + \underline{b}_{i,j} \end{array} \right] \quad \ldots\ldots\ldots\ldots\ldots \text{(A-10)}$$

The term $(4A + C)^{-1}$ denotes the inverse of the matrix $(4A + C)$. Eq. A-10 represents *block* SOR. [See Eq. (3.61), page 80, of Varga.[14]) The block consists of the pair of values $(u_1, u_2)_{i,j}$ at each grid point, where the $\underline{u}$ vector is $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

The Line SOR used herein for solution of Eq. A-8 is

$$\underline{u}^{*(\ell+1)}_{i,j} = (4A+C)^{-1} \left[\begin{array}{l} A\underline{u}^{*(\ell+1)}_{i+1,j} + A\underline{u}^{*(\ell+1)}_{i-1,j} \\ \\ + A\underline{u}^{(\ell)}_{i,j+1} + A\underline{u}^{(\ell+1)}_{i,j-1} \\ \\ + \underline{b}_{i,j} \end{array} \right], \quad \ldots\ldots\ldots\ldots\ldots \text{(A-11a)}$$

and

$$\underline{u}_{i,j}^{(\ell+1)} = \omega\,\underline{u}^{*(\ell+1)}_{i,j} + (1-\omega)\,\underline{u}_{i,j}^{(\ell)}. \quad \ldots\ldots\ldots\ldots\ldots \text{(A-11b)}$$

Several different variations of the solution using Point SOR (Eq. A-10) and Line SOR (Eqs. A-11) are possible. The chief variations include:

1. If Eqs. A-10 and A-11 are solved at each iteration ℓ starting from the same point, or along the same line, the solution scheme will be the regular or nonsymmetric forms of point and line SOR (PSOR and LSOR). If the direction of solution is reversed every second iteration, by commencing with points or lines of the highest i and j indexing, the solution scheme becomes *symmetric* point and line SOR (PSSOR and LSSOR).

2. Different directions of orientation of the point and line SOR solutions may be used. That is, the point SOR may be progressed pointwise in the x, y or z directions (in three dimensions). Similarly, the implicit line SOR solutions may be carried out along x-, y- or z-directional lines. In this study, for example, both x- and z-directional line solutions were employed.

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