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Comparison of Alternating Direction Explicit and Implicit Procedures in Two-Dimensional Flow Calculations

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ABSTRACT

Analysis and example applications have been performed to compare the accuracy and computing speed of alternating-direction explicit and implicit procedures (ADEP and ADIP) in numerical solution of reservoir fluid flow problems. ADIP yields significantly greater accuracy and requires about 60 per cent more computing time than ADEP, not 300 or 500 per cent more as reported elsewhere.[1,2]

INTRODUCTION

Several recent papers[1-3] discuss an alternating-direction explicit difference approximation (ADEP) to the diffusion equation. Example applications of ADEP and ADIP[4] were reported to support conclusions that ADEP is comparable in accuracy to ADIP and requires one-fifth to one-third the computing time of ADIP. Applications of ADEP in calculation of two-phase flow in reservoirs was also proposed.[3]

This study was performed to compare further the relative merits of ADEP and ADIP in simulation of two-dimensional flow of one and two fluid phases in reservoirs. Since two-phase flow equations are often essentially elliptic rather than parabolic, the efficiency of ADEP in solving the elliptic equation was also examined.

ADIP AND ADEP DIFFERENCE EQUATIONS

The diffusion equation:

$$u_{xx} + u_{yy} + q = u_t \quad \ldots \ldots \ldots (1)$$

governs heat conduction, molecular diffusion and slightly compressible fluid flow through porous media for the case of homogeneous, isotropic media. The ADEP procedure[1-3] involves replacement of Eq. 1 at odd time steps by:

$$L_{2n+1}(u) \equiv \frac{\Delta_x u_{ij}^{2n} - \Delta_x u_{i-1,j}^{2n+1}}{\Delta x^2} + \frac{\Delta_y u_{ij}^{2n} - \Delta_y u_{i,j-1}^{2n+1}}{\Delta y^2} - \frac{u_{ij}^{2n+1} - u_{ij}^{2n}}{\Delta t} + q = 0, \quad \ldots \ldots \ldots (2)$$

and at even time steps by:

$$L_{2n+2}(u) \equiv \frac{\Delta_x u_{ij}^{2n+2} - \Delta_x u_{i-1,j}^{2n+1}}{\Delta x^2} + \frac{\Delta_y u_{ij}^{2n+2} - \Delta_y u_{i,j-1}^{2n+1}}{\Delta y^2} - \frac{u_{ij}^{2n+2} - u_{ij}^{2n+1}}{\Delta t} + q = 0, \quad \ldots \ldots \ldots (3)$$

where

$$u_{ij}^{n} \cong u(i\Delta x,\ j\Delta y,\ n\Delta t)$$

$$\Delta_x u_{ij}^{n} \equiv u_{i+1,j}^{n} - u_{ij}^{n}$$

$$\Delta_y u_{ij}^{n} \equiv u_{i,j+1}^{n} - u_{ij}^{n}$$

Sweeping a two-dimensional grid from southwest to northeast using Eq. 2 and from northeast to southwest using Eq. 3 allows direct (explicit) calculation of *u* at the new time step at each grid point.

ADIP[4] replaces Eq. 1 by:

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[1]References given at end of paper.

$$R_{2n+1}(u) = \frac{\Delta_x^2 u_{ij}^{2n+1}}{\Delta x^2} + \frac{\Delta_y^2 u_{ij}^{2n}}{\Delta y^2} - \frac{u_{ij}^{2n+1} - u_{ij}^{2n}}{\Delta t} + q = 0 \quad . . . (4)$$

at odd time steps, and by:

$$R_{2n+2}(u) = \frac{\Delta_x^2 u_{ij}^{2n+1}}{\Delta x^2} + \frac{\Delta_y^2 u_{ij}^{2n+2}}{\Delta y^2} - \frac{u_{ij}^{2n+2} - u_{ij}^{2n+1}}{\Delta t} + q = 0 \quad . (5)$$

at even time steps, where

$$\Delta_x^2 u_{ij}^n = u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n$$

$$\Delta_y^2 u_{ij}^n = u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n$$

Each of Eqs. 4 or 5 implicitly relates three unknown (new time step) u values at each grid point and leads to a set of simultaneous equations easily solved by Gaussian elimination.[6]

TRUNCATION ERRORS

The truncation error of ADEP, defined as

$$[L_{2n+1}(u) + L_{2n+2}(u) - 2(u_{xx} + u_{yy} - u_t)_{t = 2n\Delta t}],$$

is:

$$\frac{\Delta t^2}{\Delta x} u_{xtt} + \frac{\Delta t^2}{\Delta y} u_{ytt} + \mathcal{O}\left(\frac{\Delta t^3}{\Delta x}\right) + \mathcal{O}\left(\frac{\Delta t^3}{\Delta y}\right) \quad . . . . (6)$$

The ADIP truncation error, defined as

$$[R_{2n+1}(u) + R_{2n+2}(u) - 2(u_{xx} + u_{yy} - u_t)_{t = 2n\Delta t}],$$

is:

$$\Delta t^2 u_{yytt} - \frac{\Delta t^2}{3} u_{ttt} + \mathcal{O}(\Delta t^3) \quad . (7)$$

Truncation error contribution of order Δx^2 and Δy^2 are identical for both techniques and are not included in Eqs. 6 and 7. The presence of Δx in the denominator of the leading term in the ADEP truncation error indicates inferior accuracy to ADIP because $\Delta t^2/\Delta x$ is an order of magnitude larger than Δt^2, the leading term in ADIP truncation error.

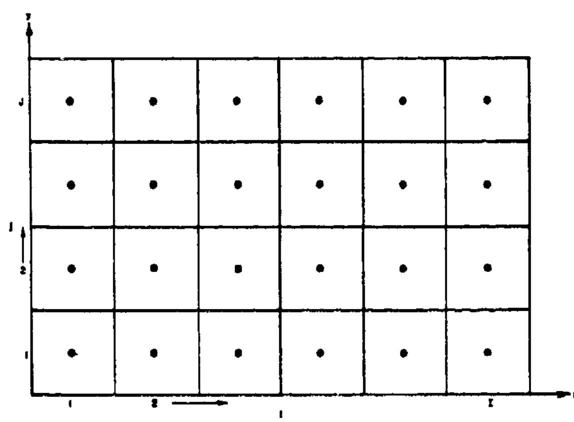
REPRESENTATION OF INSULATED BOUNDARIES WITH ADEP

Fig. 1 shows two types of spatial grids for numerical simulation of flow in reservoirs. For problems involving closed exterior boundaries, the difference representation $\Delta^2 u_{ij}$ of $u_{xx} + u_{yy}$ in Eq. 1 must satisfy

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \Delta^2 u_{ij} = 0 \quad . . . . (8)$$

to preserve the no-flow condition at the boundaries. If a grid of type shown in Fig. 1b is employed, the side points must be weighted in Eq. 8 by a factor of one half and corner points by one fourth.

As shown in Appendix A, ADIP satisfies Eq. 8 exactly for either type of grid. ADEP, however, yields an error term of order $(\Delta t)^2$ on the right side of Eq. 8 for grids of type shown in Fig. 1a. This error will cause the material balance in ADEP



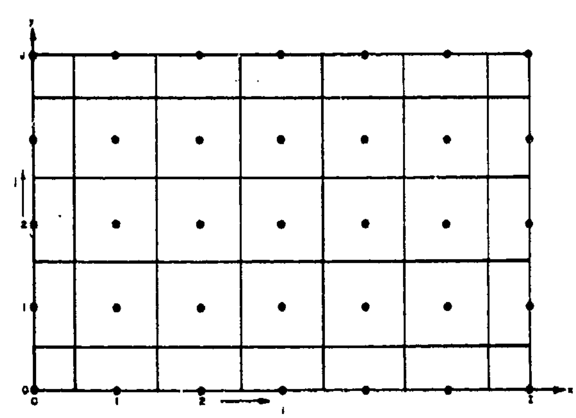


FIG. 1 — TWO TYPES OF GRIDS.

calculations to differ from 1.0 and may give rise to extremely serious errors if wells (sources or sinks) are placed close to the insulated boundaries. This statement is illustrated in the following example calculations.

Representation of insulated boundaries by ADEP with grids of type shown in Fig. 1b may be achieved in several ways. Four cases are described in detail in Appendix A and are briefly summarized here, since consideration of the first example problem discussed below requires a choice of one of these alternatives. This problem involves a unit square with two insulated and two constant potential boundaries (Fig. 2). As shown in Eq. 2, ADEP represents $\Delta_x^2 u_{oj}$ at odd time steps by (with the j index suppressed):

$$u_1^n - u_o^n - (u_o^{n+1} - u_{-1}^{n+1}) \quad . \; . \; . \; . \quad (9)$$

Case 1 insulates the boundary $x = i = 0$ of Fig. 1b setting u_{-1}^{n+1} equal to u_1^n leaving Eq. 9 otherwise unchanged. Case 2 is more consistent in preserving the time level of the first difference in x by replacing Eq. 9 by:

$$u_1^n - u_o^n - (u_o^n - u_1^n) \quad . \; . \; . \; . \; . \; . \quad (10)$$

Case 3 avoids the insulated boundary difficulty (with ADEP) by treating the square of side 2 with zero potential imposed on all sides. The unit square with two insulated boundaries is simply the upper right-hand quadrant of this larger square.

Case 4 preserves Eq. 9, reflecting u_{-1}^{n+1} to u_1^{n+1}, but results in $2I + 2J - 2$ equations requiring simultaneous solution, where I and J are the total numbers of grid points in the x and y directions, respectively. None of Cases 1, 2 or 4 satisfies Eq. 8 (Appendix A).

COMPARISON OF ADIP AND ADEP USING LARKIN'S EXAMPLE

Larkin[1] applied ADIP and ADEP to the diffusion equation in the unit square for conditions noted on Fig. 2. Mathematical statement of the problem is:

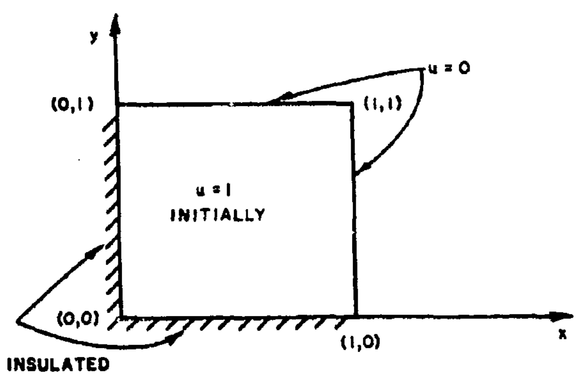


FIG. 2 — LARKIN'S EXAMPLE PROBLEM.

$$u_{xx} + u_{yy} = u_t \quad 0 \le x \le 1, \quad 0 \le y \le 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$u(x, y, 0) = 1 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad x = 0, \; 0 \le y \le 1$$
$$\text{and } y = 0, \; 0 \le x \le 1, \text{ all } t \quad . \; . \quad (12)$$

$$u(1,y,t) = u(x,1,t) = 0 \quad . \; . \; . \; . \quad (13)$$

The analytical solution to this problem is:

$$u(x,y,t) = \sum_{m,n=1}^{\infty} \frac{16}{mn\,\pi^2} \sin\frac{m\pi x}{2} \sin\frac{n\pi y}{2} \, e^{-(n^2+m^2)\frac{\pi^2}{4} t} \quad . \; . \quad (14)$$

which gives 0.62177 at $x = y = 0.5$ and $t = 0.08$. Larkin compared the ADIP and ADEP numerical solutions at this position and time in the form of Table 1 which gives the difference between the numerical solutions and 0.62177. Spatial increments of 0.1 in each direction were used. On the basis of this comparison at $t = 0.08$, Larkin concluded the methods were of roughly equivalent accuracy.

ADIP and ADEP were programmed in this study for Larkin's problem, and the differences between numerical results and the analytical solution (Eq. 14) are plotted vs time in Figs. 3 through 8 for time increments of 0.0025 and 0.02. The plotted per cent error is defined by $100 \times (u^* - u_{\text{Eq.}14}/1 - u_{\text{Eq.}14})$ which is actual error expressed as a per cent of the total change in u from the initial value of 1.0; u^* is the numerical ADIP or ADEP solution.

Figs. 3 and 4 compare ADIP and ADEP errors at the center point $x = y = 0.5$ for Case 1 treatment of ADEP differences at the insulated boundaries. Fig. 3 also shows the analytical solution. These figures show the pronounced superior accuracy of ADIP at small (0.0025) and more practical (0.02) time steps for Case 1 ADEP.

Figs. 5 and 6 show that the Case 2 treatment of the ADEP scheme at the insulated boundary is superior to that of Case 1. At the 0.0025 (critical) time step, ADEP is comparable or even slightly superior in accuracy to ADIP but this is of little

TABLE 1 — LARKIN'S ERROR COMPARISON

	Δt			
	0.02	0.01	0.005	0.0025
ADEP	−0.0255	0.0012	0.0000	−0.0013
ADIP	0.0032	−0.0039	−0.0021	−0.0019

practical interest. At the 0.02 time step, however, the results again show the superior accuracy of ADIP.

Figs. 7 and 8 correspond to Case 3 ADEP treatment where the question of difference form at insulated boundaries is avoided by solving the larger problem of a 2 × 2 square with all sides maintained at zero potential. These figures again show the superior ADIP accuracy at the 0.02 time step.

Figs. 3 through 8 show the erroneous conclusions that can be reached by simply comparing errors at the single time $t = 0.08$. The critical time increment is defined as the maximum Δt at which the normal explicit method $(u_{xx} + u_{yy})_n \Delta_t = (u^{n+1} - u^n)/\Delta t$ is stable. This increment is:

$$\Delta t_c = \frac{\Delta x^2}{2\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right]} \quad \ldots\ldots (15)$$

or $\Delta t_c = [(0.1)^2]/4 = 0.0025$ for this problem. Thus, the above comparisons of the two methods were made for time increments up to 0.02/0.0025 or eight times the critical.

Relative accuracies of methods such as ADIP and ADEP at the critical time step are of little practical interest since the sole advantage of these techniques is their provision of stability at considerably larger than critical Δt's and attendant reduced computing time requirements.

ANALYSIS OF QUON *ET AL.*[2] RESERVOIR EXAMPLE

Quon *et al.*[2] simulated a heterogeneous under-

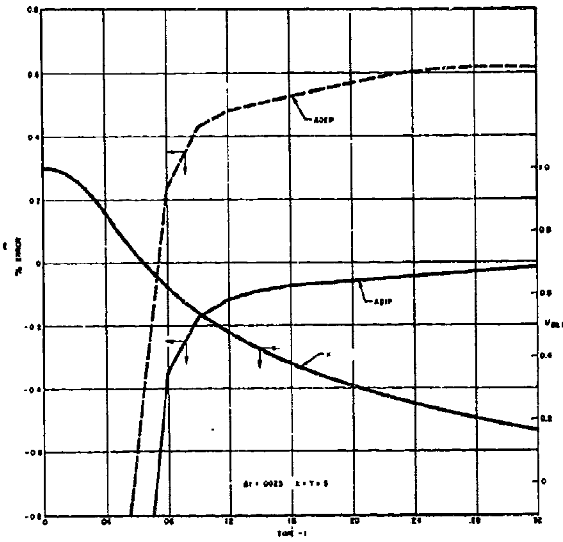


FIG. 3 — COMPARISON OF ADIP AND ADEP ERRORS (CASE 1).

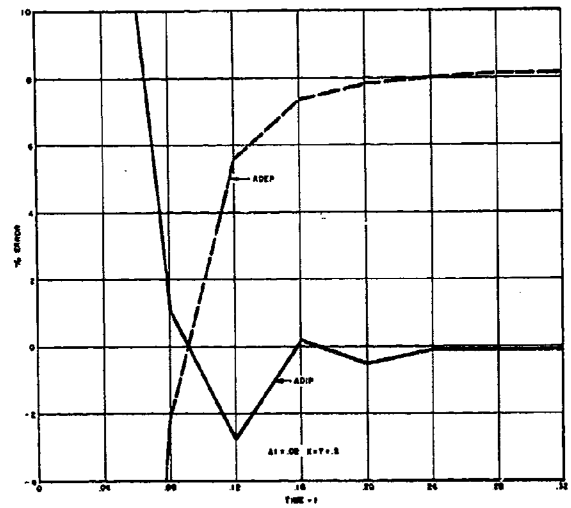


FIG. 4 — COMPARISON OF ADIP AND ADEP ERRORS (CASE 1).

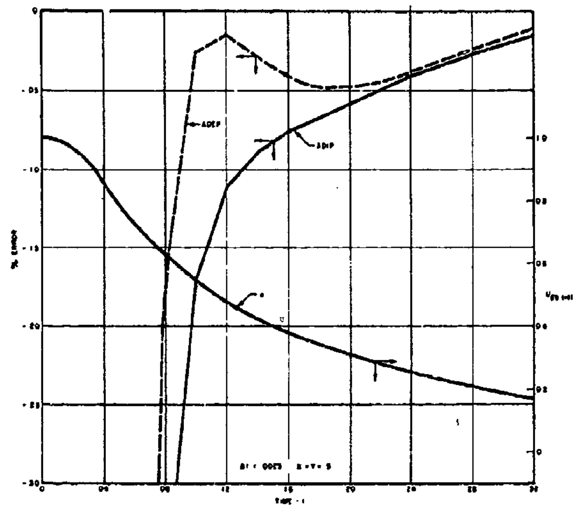


FIG. 5 — COMPARISON OF ADIP AND ADEP ERRORS (CASE 2).

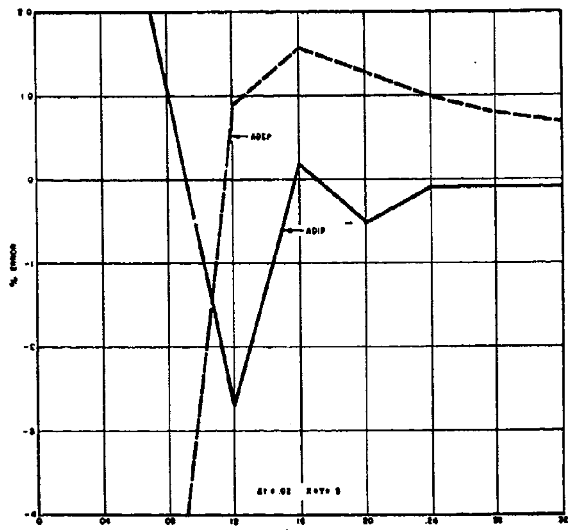


FIG. 6 — COMPARISON OF ADIP AND ADEP ERRORS (CASE 2).

saturated oil reservoir about three miles wide by five miles long. The reservoir was heterogeneous with the following properties:

$$\Delta t = 15 \text{ days}$$
$$\Delta x = \Delta y = 1{,}320 \text{ ft}$$
$$\mu_{avg} = 2.5 \text{ cp}$$
$$kh_{avg} = 70{,}000 \text{ md-ft}$$
$$\phi h_{avg} = 7 \text{ ft}$$
$$c = 7 \times 10^{-6} \text{ psi}^{-1}$$
$$p_{initial} = 1{,}065 \text{ psia,}$$

with six wells producing at 625 B/D and eight wells producing at 375 B/D.

A well-known relationship[7] giving the time necessary for a well producing from a closed reservoir to reach quasi-steady state is:

$$t^* = \frac{\mu \phi c r_e^2}{4(0.00633)k} \text{ days} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (16)$$

Insertion of the above data with the maximum possible distance of five miles used for r_e gives:

$$t = \frac{2.5\ (7)\ (7 \times 10^{-6})\ (5 \times 5280)^2}{4\ (.00633)\ (70{,}000)}$$
$$= 48 \text{ days} \quad . \; . \; . \; . \; . \; . \; . \; (17)$$

Thus, one time step of 15 days represents about 30 per cent of the time necessary for a quasi-steady-state regime to occur. A comparison between ADIP and ADEP at a time corresponding to 96 of these increments (1,440 days) thus has questionable significance. A more meaningful comparison would be one at times less than 48 days, using time increments considerably less than 15 days.

The critical time increment for this problem is given by

$$\frac{0.00633\ k\ \Delta t_c}{\mu \phi c} = \frac{\Delta x^2}{2\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right]}$$

or $\Delta t_c = 0.12$ day. A rough guide for selection of Δt is derived in Appendix B; this guide indicates that time steps of about 30 Δt_c, or about four days in this case, are reasonable at times prior to the onset of quasi-steady-state.

Actually, a nearly exact solution to the difference equation for times greater than 50 days could be obtained by setting $\partial p/\partial t$ in the diffusion equation to a constant α which can be easily calculated from the total reservoir volume-compressibility product and the total production rate. An elliptic equation then results which need be solved only once (i.e., not repetitively at successive time steps) to obtain the pressure distribution which is then positioned for any given time about the average reservoir pressure at that time.

COMPARISON OF METHODS USING WELL PROBLEM

ADIP and ADEP were compared in this work for the problem of a well located in the center of a square reservoir containing undersaturated oil (Fig. 9). The governing equation is:

$$p_{xx} + p_{yy} = p_t \quad . \; . \; . \; . \; . \; . \; . \; (18)$$

with $p_x = 0$ at $x = 0$ and 1, and $p_y = 0$ at $y = 0$ and 1. Initially, p is zero and fluid injection occurred at $x = y = 0$. The injection rate used in the numerical solution was normalized so that the analytical solution:

$$p = \frac{1}{4\pi}\ E_1 \left(4\ \frac{r^2}{t}\right) \quad . \; . \; . \; . \; . \; . \; (19)$$

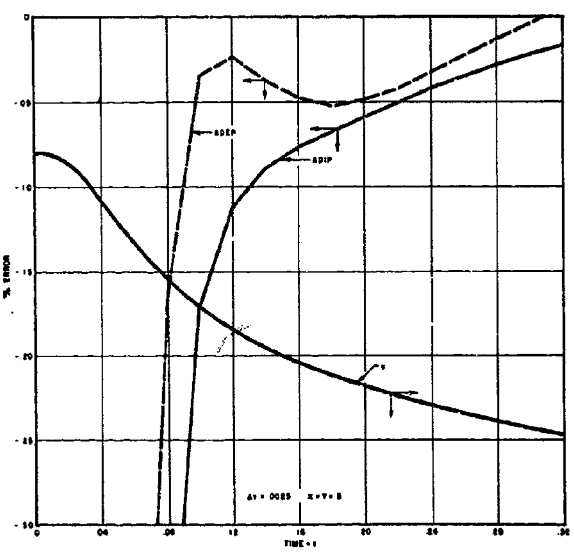


FIG. 7 — COMPARISON OF ADIP AND ADEP ERRORS (CASE 3).

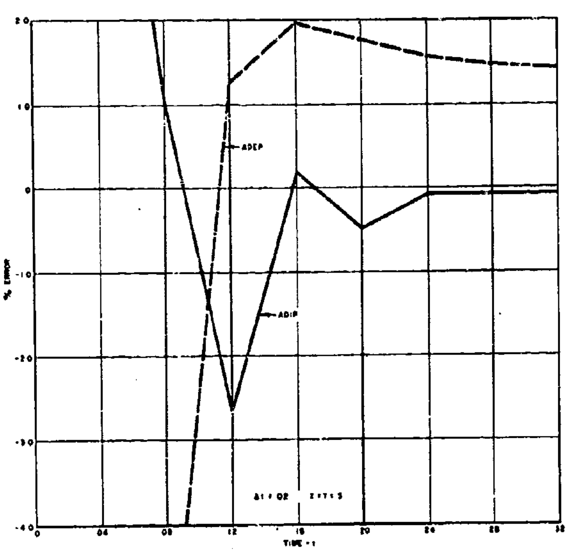


FIG. 8 — COMPARISON OF ADIP AND ADEP ERRORS (CASE 3).

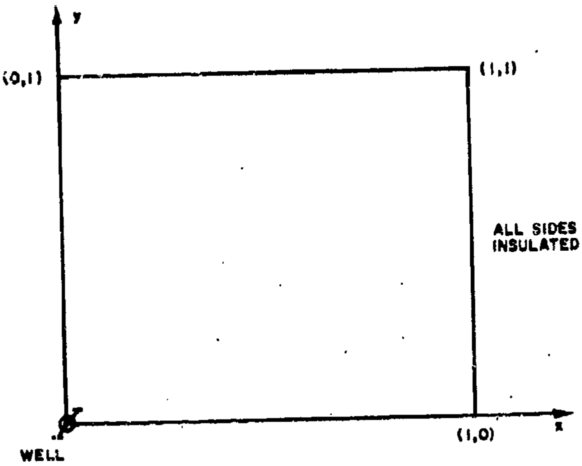


FIG. 9 — WELL PROBLEM CONFIGURATION.

applies at each point $r = \sqrt{x^2 + y^2}$ for times at which the effect of the exterior boundary is negligible. The effect of the exterior boundary is negligible at the well for times up to:

$$t = \frac{r_e^2}{4} \cong .25 \quad \ldots\ldots\ldots (20)$$

which is the time at which a quasi-steady-state regime begins.

Calculations were performed for increasing time steps as follows:

TABLE 2 — CASE 2 — ADEP RESULTS FOR FIG. 1b GRID WITH WELL AT $i = j = 0$ IN UNIT SQUARE-VARIABLE Δt CASE

t	$\frac{p_{ADEP} - p_{Eq.\,19}}{p_{Eq.\,19}} \times 100$	Material Balance for Unit Square
0.005	224.4	1.261
0.013	39.9	1.205
0.027	16.6	1.249
0.057	6.7	1.352
0.077	3.0	1.341
0.137	2.4	1.366
0.217	2.9	1.308
0.257	3.5	1.283

t	Δt
0 – 0.001	0.0005
0.001 – 0.005	0.001
0.005 – 0.017	0.002
0.017 – 0.037	0.005
0.037 – 0.097	0.01
0.097 – 0.657	0.02
0.657 – 0.857	0.05
0.857 – 1.457	0.1
1.457 – 2.957	0.15
2.957 – 5.457	0.25

ADIP calculations were performed with a grid of type shown in Fig. 1b and 20 increments along each side of the quarter square with side = 1. ADEP results were obtained by treating the square of side = 2 with the well at the center and with a grid of type shown in Fig. 1a. Preservation of the 0.05 Δx value used in the unit-square ADIP calculation

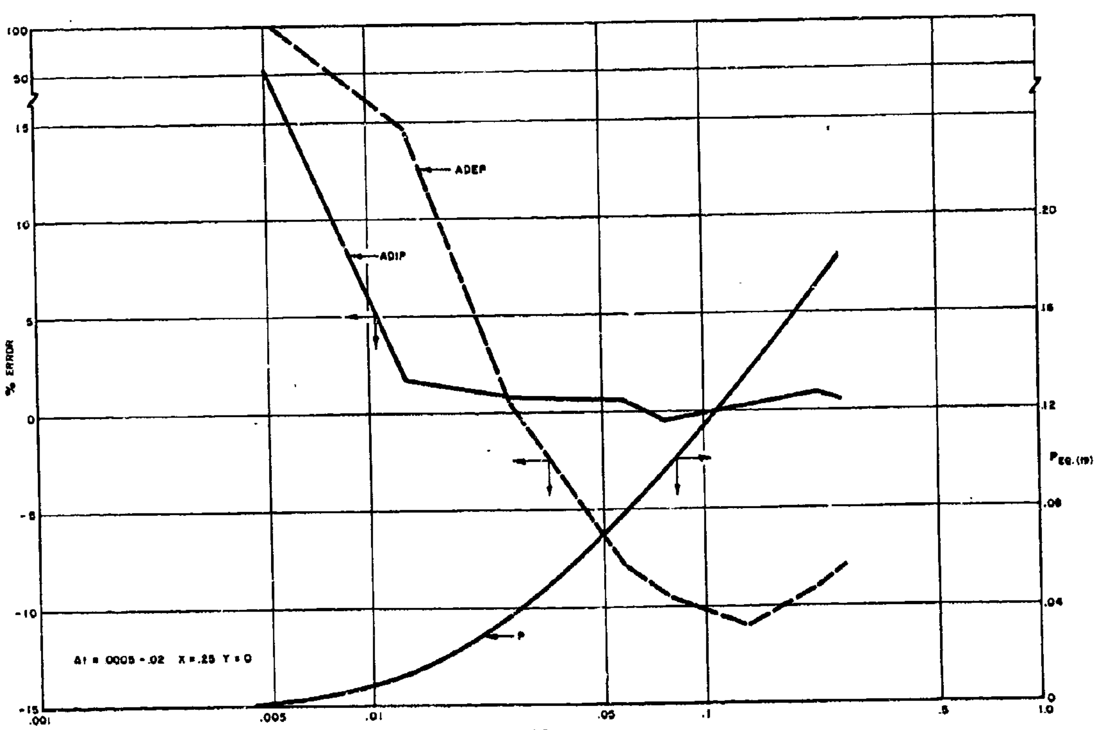


FIG. 10 — COMPARISON OF ADIP AND ADEP ERRORS (WELL PROBLEM).

required 41 spatial increments along each side of the square of side = 2.

Fig. 10 compares the percentage errors in ADIP and ADEP solutions at the point $x = 0.25$, $y = 0$ up to a time of 0.25. At this time, exterior boundary effects appeared at this position and the analytical solution (Eq. 19), no longer applied. The plotted errors are $100 \times \frac{p_{\text{numerical sol'n}} - p^*}{p^*}$ where p^* is the analytical solution (Eq. 19). The negligible effect on well pressure of the exterior boundary at times prior to 0.25 was checked by numerically solving Eq. 1 for a somewhat larger square and noticing the identity of the unit and larger square solutions (compared to the errors from Eq. 19) at $x = 0.25$, $y = 0$ for $t \leq 0.25$. The analytical solution for pressure is also shown on Fig. 10. The results for this case show a pronounced superior accuracy for ADIP. Attempts to use a grid of type shown in Fig. 1b with the Case 2 ADEP procedure gave excessively large errors (Table 2). The reason for this error of ADEP is discussed in Table 2 and in Appendix A: ADEP fails to preserve the no-flow condition at the insulated boundaries, and the error thus introduced is increased as the perturbing well is located closer to the boundary.

The more reasonable ADEP results shown in Fig. 10 correspond to a well in the center of a square of side 2; transients at the insulated boundaries were delayed in time and less severe than when the well was on the boundary. The attendant ADEP error induced by failure to preserve insulation was therefore reduced. Fig. 11 compares the 1.0000 ADIP material balance with the ADEP balance, which deviates from unity when transients reach the insulated boundaries.

Constant Δt simulations were also carried out on a unit square (well at corner) with Fig. 1b grid for ADIP and on a square of side 2 (well at center) with Fig. 1a grid for ADEP. The critical time increment for this problem is $\Delta t_c = \frac{\Delta x^2}{4} = \frac{0.0025}{4} = 0.000625$ while quasi-steady-state is reached at $t_{QSS} = \frac{r_e^2}{4} \cong 0.25$. Quon *et al.*[2] used a constant Δt about 30 per cent of their quasi-steady-state time. Here, a Δt of 0.06, about 25 per cent of t_{QSS}, was employed. This increment is about 100 times the critical time step.

Fig. 10 shows the variable time step ADIP results to be closing within 1 per cent of the true solution at time = 0.25. Error in the constant time step calculations was therefore defined as $100 \times \frac{p - p^*}{p^*}$ where p^* is Eq. 19 for time < 0.25 and is the ADIP solution using the variable time steps for $t > 0.25$.

Fig. 12 compares the ADIP and ADEP errors for the case of constant $\Delta t = 0.06$. The results again show ADIP to be considerably more accurate. Fig. 13 compares the 1.0000 ADIP material balance to the ADEP balance which immediately deviates over 40 per cent from unity since transients reach the insulated boundaries in one or two time steps of 0.06.

The Case 2 ADEP results for the case of the well at the corner of the unit square on a Fig. 1b grid with $\Delta t = 0.06$ were as follows:

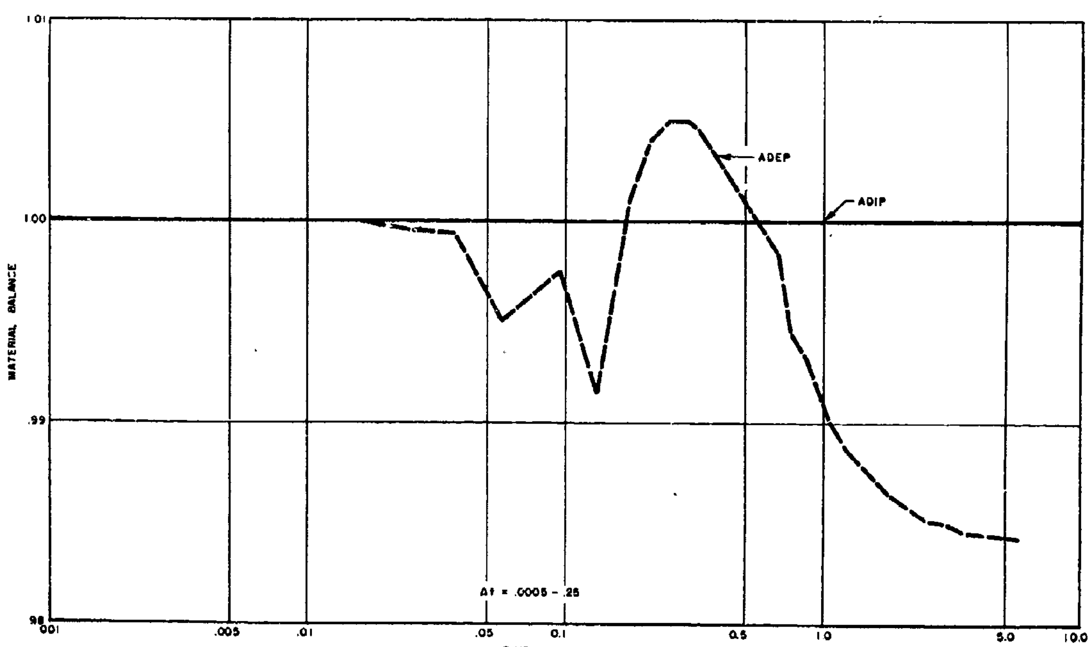


FIG. 11—COMPARISON OF ADIP AND ADEP MATERIAL BALANCES (WELL PROBLEM).

t	$\frac{p_{ADEP} - p^*}{p^*} \times 100$	Material Balance
0.12	19,307	180.664
0.36	11,469	61.084
0.60	7,413	30.983
1.2	3,380	11.381
2.4	1,482	4.587
3.6	1,038	3.145
4.8	860	2.577
5.52	773	2.372

As previously mentioned, the 0.06 time increment is about 100 times the critical Δt. Since increments of the order of 20 times the critical are more reasonable, a fairer comparison of ADEP and ADIP errors might be obtained by using a Δt value of 20 (0.0006) or 0.012. These errors are given in Table 3 for the point $x = 0.25$, $y = 0$. The ADEP results were again obtained from the square of side 2 with the well placed in the center. The ADEP material balance is also given in Table 3; ADIP material balance was 1.00000 over all time.

TABLE 3 — ADIP AND ADEP ERRORS FOR $\Delta t = 0.012$

time	$(p - p_{Eq.\ 19}) \times 100/p_{Eq.\ 19}$ ADIP	$(p - p_{Eq.\ 19}) \times 100/p_{Eq.\ 19}$ ADEP	ADEP Material Balance
0.024	48.96	34.28	0.9946
0.048	− 17.39	− 27.47	1.0020
0.072	7.27	− 23.35	1.0039
0.096	− 6.82	− 19.79	1.0048
0.12	2.64	− 16.07	1.0052
0.24	− 0.07	− 5.71	1.0033

UTILITY OF NONITERATIVE ADIP OR ADEP IN TWO-PHASE FLOW PROBLEMS

Quon *et al.*[3] proposed a noniterative application of ADEP in solution of the two partial differential equations governing two-phase flow in reservoirs. ADEP or ADIP can be employed in noniterative solution of these equations only if fluid compressibility is not zero. If compressibility is zero, then iteration is required at each time step. Even if compressibility is not zero, the noniterative approach will succeed only for a limited time step size; the limitation on time increment is far less severe if iteration is employed. As shown below, the limitation on time step in the noniterative approach is generally so severe that greater computing efficiency in two-phase flow problems is attained by iterating at each time step.

The question considered here is not whether ADEP is superior or inferior to ADIP in noniterative solution of two-phase flow problems. The question is whether the two-phase flow problem is essentially parabolic (i.e., subject to noniterative solution with reasonably large time steps) or elliptic (i.e., requiring iteration for use of a reasonable time step). In either case, the use of ADIP is indicated since, as shown above, ADIP is clearly superior to ADEP in the parabolic case and, as shown in Appendix C, in the elliptic case ADEP becomes identical to the extrapolated Liebmann method that has been proven inferior[4] to iterative ADIP for unit k, ϕ and c.

In earlier work the authors attempted to apply ADIP in noniterative solution of two- and three-

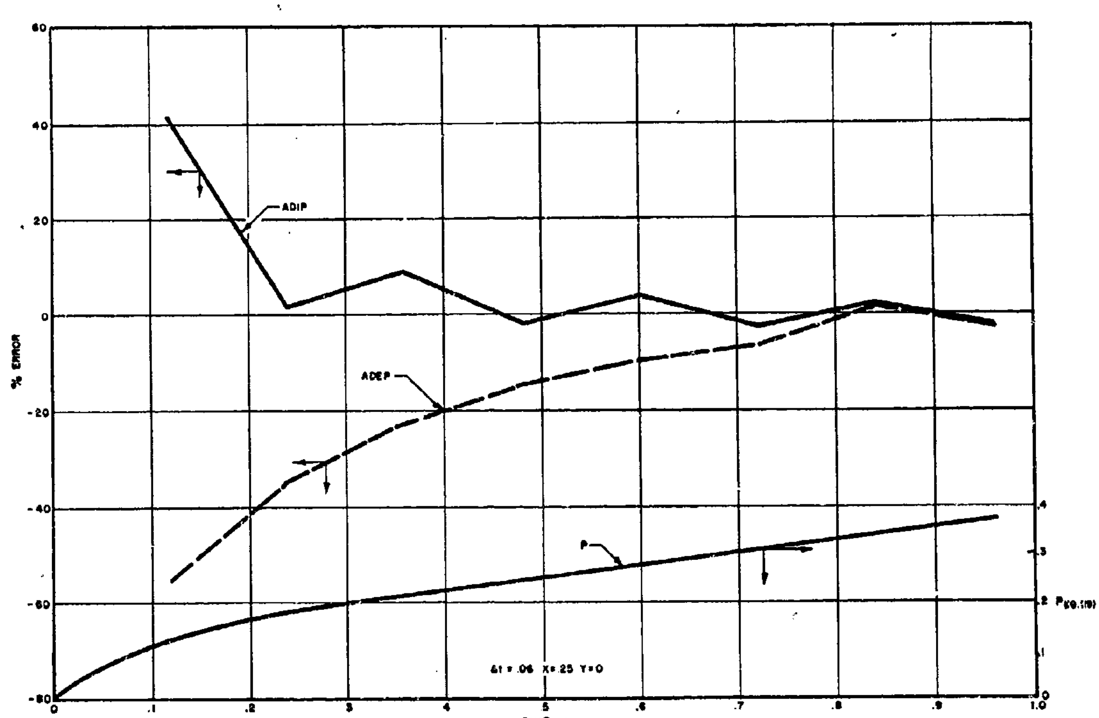


FIG. 12 — COMPARISON OF ADIP AND ADEP ERRORS (WELL PROBLEM).

dimensional two-phase flow problems. The approach succeeded only for the very limited class of large-block areal problems; iteration at each time step was found to be far more efficient in the majority of problems of interest. The reason for the advantage of iteration can be demonstrated by an analysis of the equation governing pressure distribution. Capillary pressure and saturation change terms may be dropped from the equation for simplification of the analysis without detraction from the results or conclusions. The resulting equation:

$$\frac{\partial}{\partial x} M \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} M \frac{\partial p}{\partial y} + \frac{\partial}{\partial z} M \frac{\partial p}{\partial z} = \phi \bar{c} \frac{\partial p}{\partial t} \quad . . . . . (21)$$

where $M = k(k_1/\mu_1 + k_2/\mu_2)$, subscripts refer to to fluid phases $\bar{c} = c_1 S_1 + c_2 S_2$ is parabolic if $\phi\bar{c} \neq 0$ and is elliptic otherwise. In the elliptic case, the difference form of Eq. 21 is augmented by a term $Hk\ (p^{k+1} - p^k)$ on the right-hand side and solved iteratively by the iterative ADIP method.[4]

The critical time increments for the explicit difference equivalent of Eq. 21, for one-, two- and three-dimensional flow, are:

$$\text{1D flow:} \quad \Delta t_c = \frac{\phi \bar{c}\, \Delta x^2}{2M} \quad . . . (22)$$

$$\text{2D flow:} \quad \Delta t_c = \frac{\phi \bar{c}\, \Delta x^2}{2M\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right]} \quad . (23)$$

$$\text{3D flow:} \quad \Delta t_c = \frac{\phi \bar{c}\, \Delta x^2}{2M\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + \left(\frac{\Delta x}{\Delta z}\right)^2\right]} \quad . . . . (24)$$

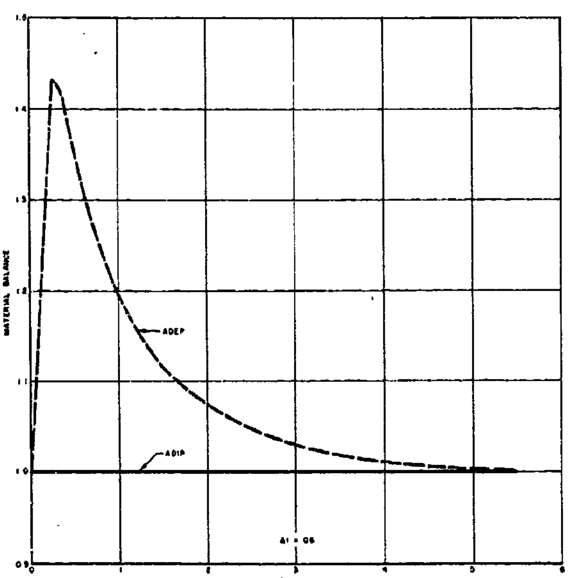


FIG. 13—COMPARISON OF ADIP AND ADEP MATERIAL BALANCES (WELL PROBLEM).

In Appendix B, a terminal time increment of the order of 50 Δt_c is shown to be reasonable in numerical solution of Eq. 21. In two-phase flow, however, perturbations are continually being produced by saturation and mobility changes; thus, a time increment more suitable to an early portion of the transient period should be employed. Therefore, let 50 Δt_c be selected as an upper bound on a suitable time increment for two-phase flow problems.

Eqs. 22 through 24 allow calculation of Δt_c for any given reservoir and fluid properties. Let $\bar{c} = 5 \times 10^{-5}$ 1/psi; $\phi = 0.20$; and $M = 200$ md/cp. Then, for a two-dimensional areal problem with $\Delta x = \Delta y$, Eq. 23 gives:

$$\Delta t_c = \frac{.2(5 \times 10^{-5})\, \Delta x^2}{2(200)(.0063)(2)} = .$$

$$2 \times 10^{-6}\, \Delta x^2 \text{ days} \quad . . . . . (25)$$

For a large reservoir with $\Delta x = 1{,}000$ ft, $\Delta t_c =$ two days and an upper limit of 50 Δt_c or 100 days per time step is estimated. Use of this or even smaller time steps in noniterative solution of the two-phase flow equations completes in efficiency with the iterative method where time steps of 360 days have typically been successfully employed in such large reservoirs.

Simulation of a quarter 20-acre five-spot using 10 increments on each of the 467-ft sides gives a Δx of about 50 ft which gives, from Eq. 25, $\Delta t_c =$ 0.005 days. In this case, 50 Δt_c is only 0.25 days, and a noniterative solution using this Δt is far more expensive than an actually performed iterative solution which employed a 30-day time step.

The above two cases indicate the applicability of the noniterative method in two-dimensional areal cases with sufficiently large Δx. The iterative method is far more efficient even for reservoirs of large areal extent, however, if a two-dimensional cross-section or three-dimensional simulation is performed. For the three-dimensional case with $\Delta z << \Delta x$, Eq. 24 gives $\Delta t_c \cong 4 \times 10^{-6}\, \Delta z^2$ and for a Δz of 10 ft, $\Delta t_c = 0.0004$ day. The tolerable increment of 50 Δt_c or 0.02 day is so small that the iterative method is two orders of magnitude cheaper than the noniterative.

To repeat, the above analysis simply indicates why in the writers' opinions, the noniterative method is generally inferior to the iterative solution. The analysis in no sense constitutes a proof of this conclusion. The validity of the conclusion rests on the writers' experience in actually solving two- and three-dimensional two-phase flow problems by both techniques.

COMPUTING TIME REQUIREMENTS

The numbers of arithmetic operations per grid point per time step for ADEP and ADIP are:

	Additions or Subtractions	Multi-plications	Divisions
ADEP $_{\Delta x = \Delta y}$	4	2	0
ADIP	5	3	1
ADEP $_{\Delta x \neq \Delta y}$	4	3	0
ADIP	5	4	1

Relative computing times for these floating-point, single-precision operations for the Burroughs 205 computer used by Larkin, the IBM 7040 used by Quon *et al.* and the IBM 7044 used here, are as follows:*

	Addition or Subtraction	Multi-plication	Division
Burroughs 205	2.5	10.1	13.5
IBM 7040	3	4.5	7
IBM 7044	5.5	10	18

Weighting the mix of operations for each method by these relative computing times gives the following ADIP:ADEP computing time ratios:

	Burroughs 205	IBM 7040	IBM 7044
$\Delta x = \Delta y$	1.87	1.69	1.8
$\Delta x \neq \Delta y$	1.65	1.57	1.65

If, as Peaceman[5] suggested, certain intermediate data are stored rather than recalculated each time step in the ADIP procedure, then one less multiplication and division are required. The above ADIP:ADEP ratios then become:

	Burroughs 205	IBM 7040	IBM 7044
$\Delta x = \Delta y$	1.08	1.14	1.13
$\Delta x \neq \Delta y$	1.06	1.12	1.11

These ADIP:ADEP ratios indicate that the ratios of five (Larkin) and three (Quon *et al.*) are excessive.

CONCLUSIONS

1. ADEP is nonconservative in that it fails to preserve no-flow conditions at exterior boundaries. This causes errors in potential and in material balance which can become extremely severe if wells are near the insulated boundaries.

2. ADIP accuracy is considerably superior to ADEP for Larkin's example problem of fluid flow or diffusion in a unit square.

3. ADIP accuracy was found to be considerably superior to that of ADEP in a closed-reservoir type of problem, even when the well was located as far from the boundary as possible in the ADEP case.

4. Two-phase flow problems are more efficiently treated by iteration except for cases of very large two-dimensional areal cases. The iterative adaptation of ADEP is identical to the well-known extrapolated Liebmann technique which has been proven inferior to iterative ADIP.

5. ADIP requires about 60 per cent more computing time than ADEP.

*The figures for the different computers bear no relation to each other; i.e., a multiplication on the IBM 7040 does not require 4.5/10 the time for a multiplication on the IBM 7044.

NOMENCLATURE

ADIP = alternating-direction implicit procedure
ADEP = alternating-direction explicit procedure
c = compressibility of fluid and rock, psia^{-1}
h = reservoir thickness, ft
i, j = grid indices, $x = i\Delta x$, $y = j\Delta y$
I = total number of grid points in x direction
J = total number of grid points in y direction
k = permeability, md
n = time index, $t = t_n$ or $t = n\Delta t$ for constant Δt
p = pressure, psi
p_w = well pressure
p_e = pressure at exterior boundary
q = injection rate, volume fluid/volume of reservoir-unit time
q_w = well injection rate, B/D
r = radius
r_w = well radius, ft
r_e = radius of closed exterior boundary
S_i = saturation of fluid phase i
t = time days, where units are implied
Δt_c = critical time increment for normal explicit difference scheme
t_D = $0.00633\, kt/\mu \phi c r_w^2$
x, y, z = spatial coordinates
ϕ = porosity
μ = viscosity, cp

ACKNOWLEDGMENT

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APPENDIX A

REPRESENTATION OF INSULATED BOUNDARY WITH ADEP

Consider solution of Eq. 1 in a rectangle with all four sides insulated. Fig. 1a shows the rectangle with a grid placing points ½-grid spacing in from the boundaries. A difference representation of $\Delta^2 u_{ij}$ of $u_{xx} + u_{yy}$ in Eq. 1 must satisfy the equation:

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \Delta^2 u_{ij} = 0 \quad . \ . \ . \ . \ . \ (A\text{-}1)$$

if the difference form preserves the no-flow condition at the boundaries. Satisfaction of Eq. A-1 by ADEP and ADIP can be examined with increased clarity and no loss in validity by considering satisfaction of the relation:

$$\sum_{i=1}^{I} \Delta_x^2 u_{ij} = 0 \quad . \ . \ . \ . \ . \ . \ (A\text{-}2)$$

The insulated boundary is represented by the difference boundary conditions:

$$u_{o,j}^n = u_{1,j}^n \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (A\text{-}3)$$

for $j = 1, 2, \ldots, J$. The ADIP differencing given in Eqs. 4 and 5 satisfies Eq. A-2 (and, therefore, Eq. A-1) exactly, since:

$$\sum_{i=1}^{I} (u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n) = 0$$

provided conditions in Eq. A-3 are imposed.

Substitution of the ADEP differencing into Eq. A-2 gives

$$\sum_{i=1}^{I} u_{i+1,j}^{2n} - u_{i,j}^{2n} - (u_{ij}^{2n+1} - u_{i-1,j}^{2n+1})$$

$$= (u_1^{2n+1} - u_1^{2n}) - (u_I^{2n+1} - u_I^{2n})$$

at odd time steps and

$$\sum_{i=1}^{I} (u_{i+1,j}^{2n+2} - u_{i,j}^{2n+2}) - (u_{ij}^{2n+1} - u_{i-1,j}^{2n+1})$$

$$= u_I^{2n+2} - u_I^{2n+1} - (u_1^{2n+2} - u_1^{2n+1})$$

at even time steps. For a full cycle:

$$\sum_{i=1}^{I} \Delta_x^2 u_{ij} = u_I^{2n+2} - 2u_I^{2n+1} + u_I^{2n}$$

$$- (u_1^{2n+2} - 2u_1^{2n+1} + u_1^{2n})$$

$$\cong \Delta t^2 \left[\left(\frac{\partial^2 u}{\partial t^2}\right)_{x = (I-\frac{1}{2})\Delta x} - \left(\frac{\partial^2 u}{\partial t^2}\right)_{x = \frac{1}{2}\Delta x} \right] \neq 0$$

$$. \ (A\text{-}4)$$

ADEP therefore fails to satisfy Eq. A-1 by an error of order Δt^2. The u_{tt} multiplier in Eq. A-4 implies that this error introduced by the ADEP procedure will increase as wells are placed closer to the boundaries. The failure of ADEP to satisfy Eq. A-1 will be reflected in material balances differing from 1.0.

Problems in which boundary values of u are specified are more satisfactorily treated by a grid of type shown in Fig. 1b which places points on the boundaries. If a problem is of mixed type with some insulated and some specified boundaries, then several variations of ADEP may be employed when the grid of Fig. 1b is used. Larkin's example problem is of this mixed condition type involving a unit square with two adjacent insulated sides with the two opposite sides held at zero potential. Four cases will be defined here for applying ADEP to such a problem with the grid of Fig. 1b.

Simplicity is served, with no loss in validity, if only $\Delta_x^2 u_{ij}$ portion of the ADEP difference form is discussed in relation to treatment of the insulated boundary. At odd time steps at the insulated boundary $x = i = 0$, with j suppressed:

$$\Delta_x^2 u_{ij}^{2n+1} = u_1^{2n} - u_0^{2n} - (u_0^{2n+1} - u_{-1}^{2n+1})$$

$$. \ . \ (A\text{-}5)$$

No implicit u_{-1}^{2n+1} value is available; thus, one possibility is reflecting u_{-1}^{2n+1} to u_1^{2n}. At even time steps $x = i = 0$.

$$\Delta_x^2 u_{ij} = u_1^{2n+2} - u_0^{2n+2} - (u_0^{2n+1} - u_{-1}^{2n+1})$$

and an explicit u_{-1}^{2n+1} value is available as u_1^{2n+1}. This procedure is labeled as Case 1 in the treatment of Larkin's example problem. Case 2 arises from the observation that a somewhat more consistent treatment of the difference form, Eq. A-5 results if the time level of the first difference $u_0^{2n+1} - u_{-1}^{2n+1}$ is preserved in both terms of the difference. Thus, if u_{-1}^{2n+1} is set equal to u_1^{2n}, then the u_0^{2n+1} term should be replaced by u_0^{2n}. Thus, Case 2 treats $\Delta_x^2 u_{ij}$ in Eq. A-5 as:

$$\Delta_x^2 u_{0j} = u_1^{2n} - u_0^{2n} - (u_0^{2n} - u_1^{2n}). \ (A\text{-}6)$$

Case 3 avoids the problem of treating $\Delta^2 u_{ij}$ on the insulated boundaries by solving Eq. 1 in a square with sides 2 units long and with all sides held at zero potential. The upper right-hand quadrant is then identical to the original problem of a unit square with sides $x = 0$ and $y = 0$ insulated.

Case 4 treats the unit square with no alteration of the ADEP scheme but requires simultaneous solution of a set of equations at points along the insulated boundaries. Thus, if the time indexing of Eq. A-5 is preserved, then $u_{-1}^{2n+1} = u_1^{2n+1}$ and:

$$\Delta_x^2 u_{0j} = u_1^{2n} + u_1^{2n+1} - u_0^{2n} - u_0^{2n+1} \quad \ldots \text{(A-7)}$$

Eq. 2 then contains two unknowns at each point $(0,j)$, $j = 0, 1 \ldots, J$ and $(i,0)$, $i = 1, 2, \ldots, I$. Additional equations at $(1,j)$, $j = 0, 1, \ldots, J$ and $(i, 1)$, $i = 1, 2, \ldots, I$ introduce no additional unknowns and provide a set of $2I + 2J - 2$ equations in the same number of unknowns. Simultaneous solution of this set of equations increases the computing time requirements of ADEP while still failing to preserve the insulated condition at the boundaries. In this case

$$\sum_{i=0}^{I-1} \Delta_x^2 u_{ij}$$

should contain no contribution from values of u_{ij} for $i = 0, 1, \ldots, I-2$. However, this sum for a full cycle, contains the terms $2\,u_1^{2n+1} - u_0^{2n} - u_0^{2n+2}$ which is approximately $2(\Delta x)\,u_x - \Delta t^2\,u_{tt}$. Thus, an error of order $(\Delta x + \Delta t^2)$ is incurred in failing to preserve insulation at the boundary.

APPENDIX B

TIME STEP SIZE FOR SINGLE-PHASE FLOW CALCULATIONS

A useful criterion for time step size in two-dimensional, single-phase flow calculations might be the ratio between the critical time increment and a Δt sufficient to give a desired maximum change in pressure over the grid.

An estimate of a practical time increment for two-dimensional single-phase reservoir problems is developed here by analyzing the case of a single well producing from a bounded reservoir. Consider the case of a well producing from a well in the center of a square reservoir of side $2L$. If $2N$ spatial increments on a side are employed and the well is located at $x = y = 0$, then the critical dimensionless time increment is:

$$\Delta t_{D_{\text{crit.}}} = 1 \quad \ldots \text{(B-1)}$$

where $t_D = 0.00633\,kt/\mu\phi c r_w^2$. This two-dimensional rectangular grid approximates the case of a well of radius $r_w = \dfrac{\Delta x}{2} = \dfrac{L}{2N}$ situated in the center of a bounded reservoir of exterior radius $r_e = L$. Thus, $R \equiv r_e/r_w = 2N$.

The well pressure is given approximately by the infinite reservoir solution:

$$p_w = \frac{70.7\,q_w\mu}{kh}\left[\ln t_D + 0.809\right], \quad \ldots \text{(B-2)}$$

for $25 \leq t_D \leq R^2/4 = N^2$ and by the quasi-steady-state solution:

$$p_w = 141.4\,\frac{q_w\mu}{kh}\,\frac{2}{R^2-1}\left(t_D + \frac{1}{4}\right) - \frac{3R^4 - 4R^4\,\ln R - 2R^2 - 1}{4(R^2-1)^2}, \quad \ldots \text{(B-3)}$$

for $t_D \geq N^2$. For $R = 2N$ of the order of 20 or larger, Eq. B-3 can be approximated by:

$$p_w = \frac{141.4\,q_w\mu}{kh}\,\frac{2}{R^2}\left(t_D + \frac{1}{2}\right) + \ln R - \frac{3}{4}, \quad \ldots \text{(B-4)}$$

for $t_D \geq N^2$. After quasi-steady-state occurs, the difference between well and exterior radius pressures is constant at:

$$p_w - p_e = 141.4\,\frac{q_w\mu}{kh}\left[\ln R - \frac{1}{2}\right] \quad \ldots \text{(B-5)}$$

The maximum rate of pressure change occurs at the well and is given by:

$$\frac{\partial P_w}{\partial t_D} = \frac{70.7\,q_w\mu}{kh}\,\frac{1}{t_D} \qquad 25 \leq t_D \leq N^2 \quad \text{(B-6)}$$

Let a dimensionless time increment Δt_D for the numerical solution be chosen so that the maximum pressure change in one time step is a fraction f of the ultimate pressure difference $p_w - p_e$. Then for $25 \leq t_D \leq N^2$, Eq. B-6 gives

$$\frac{\Delta p_w}{\Delta t_D} = \frac{f\left(\dfrac{141.4\,q_w\mu}{kh}\right)\left(\ln R - \dfrac{1}{2}\right)}{} \cong \frac{70.7\,q_w\mu}{kh}\,\frac{1}{t_D}$$

or

$$\Delta t_D \cong 2f\,(\ln R - \tfrac{1}{2})\,t_D \quad \ldots \text{(B-7)}$$

and from Eq. B-1

$$\frac{\Delta t_D}{\Delta t_{D_{\text{crit.}}}} = 2f(\ln 2N - \tfrac{1}{2})\,t_D \quad \ldots \text{(B-8)}$$

Thus, for $f = 0.02$, $N = 20$, $\Delta t_D/\Delta t_{D_{\text{crit}}} = 0.128\,t_D$ and ranges from 3.2 at $t_D = 25$ to 51 at $t_D = N^2 = 400$ when quasi-steady-state is achieved.

APPENDIX C

EQUIVALENCE OF ADEP AND EXTRAPOLATED LIEBMANN TECHNIQUE IN SOLUTION OF ELLIPTIC EQUATIONS

The extrapolated Liebmann or successive over-relaxation technique[8] treats the elliptic equation:

$$u_{xx} + u_{yy} = 0 \quad \text{. . . . . . .(C-1)}$$

by the iterative sequence:

$$u_{ij}^{k+1} = u_{ij}^{k} + \alpha \left(u_{i+1,j}^{k} + u_{i,j+1}^{k} + u_{i-1,j}^{k+1} + u_{i,j-1}^{k+1} - 4u_{ij}^{k} \right) \quad \text{. .(C-2)}$$

where u_{ij}^k = the k^{th} iterate at grid point (i,j), α = relaxation factor. Application of the ADEP differencing scheme to Eq. C-1 yields:

$$u_{ij}^{2k+1} = u_{ij}^{2k} + B \left(u_{i+1,j}^{2k} + u_{i,j+1}^{2k} + u_{i-1,j}^{2k+1} + u_{i,j-1}^{2k+1} - 2u_{ij}^{2k} - 2u_{ij}^{2k+1} \right) \quad \text{. . . . .(C-3)}$$

at odd iterations and:

$$u_{ij}^{2k+2} = u_{ij}^{2k+1} + B \left(u_{i+1,j}^{2k+1} + u_{i,j+1}^{2k+1} + u_{i-1,j}^{2k+2} + u_{i,j-1}^{2k+2} - 2u_{ij}^{2k+2} - 2u_{ij}^{2k+1} \right) \quad \text{. . . . .(C-4)}$$

at even iterations. Identifying α in Eq. C-2 with $B/(1+2B)$ causes Eq. C-2 to assume a form identical with Eq. C-3. Thus, the convergence rates of the extrapolated Liebmann and ADEP techniques in iterative solution of the elliptic Eq. C-1 are identical. This convergence rate is independent of the direction of calculations so that the use of Eq. C-4 in place of Eq. C-3 or alternate use of the two equations is immaterial.

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Further Discussion of a Paper Published in *Society of Petroleum Engineers Journal*, December, 1966

Comparison of Alternating Direction Explicit and Implicit Procedures in Two-Dimensional Flow Calculations

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DISCUSSION

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The diffusion equation dealt with by K. H. Coats and M. H. Terhune is similar to the equation governing isothermal flow of gas in porous media. Because of the similarity of diffusion problems and gas flow problems a discussion of the writer's experience in applying ADEP (or Saul'ev)[1] and ADIP[2] methods to the solution of reservoir-type gas flow problems seems appropriate. In summary, this experience indicates: (1) that ADEP extends the range of conditions for which explicit methods may be used in practice to obtain gas flow solutions; (2) ADIP is a method of more general utility than ADEP; and (3) the additional storage and computer time required by ADIP over ADEP should normally be of little concern with present day computers. These latter two conclusions are at variance with an earlier speculation by Carter[1] regarding the relative merits of ADIP and ADEP for two-dimensional gas reservoir calculations.

The gas flow equation can be put in a form which is the same as the diffusion equation except for a coefficient of the time derivative which varies slowly with the dependent variable (Eq. 5-A, Ref. 1). If this coefficient is evaluated at the beginning of each time step in a numerical solution, a non-iterative diffusion equation approach can be employed to obtain solutions to gas flow problems.

Coats and Terhune point out that the ADEP method does not preserve material balance in diffusion equation, closed boundary, reservoir-type problems. The writer has obtained several solutions to two-dimensional gas reservoir problems using an ADEP method (Eqs. 10-A and 11-A, Ref. 1). In none of these solutions was material balance strictly preserved, but in many of these solutions the material balance error was sufficiently small (less than or equal to about 0.3 per cent of the original gas content after 75 per cent of the gas had been removed) that the results could be of practical value. The writer has found that the material balance error tends to be reduced as the combination of reservoir conditions, time step size and mesh spacing approach that in which a conventional explicit-type difference equation could be employed. Useful ADEP solutions to gas flow problems have been obtained for heterogeneous reservoir problems in which much of the reservoir has a sufficiently large critical time step size that a conventional explicit difference equation could be used, but that also contained some areas of high permeability in which the conventional explicit equation (Eq. 1-A, Ref. 1) could not be employed because the critical time step for the high permeability area was too small to be practical.

Some problems of considerable practical interest have been encountered for which the ADEP method has proven unusable. These are problems that are characterized in general by a requirement of small grid point spacing coupled with large areas of high permeability and/or reservoir pressure. This results in a very small critical time step size for these large areas. Use of ADEP with practical time step sizes in such cases has resulted in unacceptably high material balance errors. For example, a material balance error of about 1 per cent of the original gas content has resulted after 12.5 per cent of the gas had been produced in a case in which the ratio of critical time step size to practical time step size was about 0.02 over the entire region. A non-iterative ADIP method has been applied to some of these problems with success, and ADIP material balance errors have been negligible. This ADIP method has also been successfully applied to some of the problems for

[1]References given at end of discussion.

which usable ADEP solutions were obtained, and could probably have been applied to all of the problems for which usable ADEP solutions were obtained.

The ADIP equations referred to above are,

$$\Big[M_{m+1/2,n}(\Phi_{m+1,n}-\Phi_{m,n})-M_{m-1/2,n}(\Phi_{m,n}-\Phi_{m-1,n})\Big]_i+\Big[M_{m,n+1/2}(\Phi_{m,n+1}-\Phi_{m,n})-M_{m,n-1/2}(\Phi_{m,n}-\Phi_{m,n-1})\Big]_{i+1}$$

$$=C_{m,n}\left(\frac{d\left(\frac{p}{z}\right)}{d\Phi}\right)_{m,n,i}(\Phi_{i+1}-\Phi_i)_{m,n}\quad\ldots\quad(1\text{-A})$$

$$\Big[M_{m+1/2,n}(\Phi_{m+1,n}-\Phi_{m,n})-M_{m-1/2,n}(\Phi_{m,n}-\Phi_{m-1,n})\Big]_{i+1}+\Big[M_{m,n+1/2}(\Phi_{m,n+1}-\Phi_{m,n})-M_{m,n+1/2}(\Phi_{m,n}-\Phi_{m,n-1})\Big]_i$$

$$=C_{m,n}\left(\frac{d\left(\frac{p}{z}\right)}{d\Phi}\right)_{m,n,i}(\Phi_{i+1}-\Phi_i)_{m,n}\quad\ldots\quad(1\text{-B})$$

Eqs. 1-A and 1-B are employed alternately. These equations are an obvious extension of the ADIP equations first presented by Douglas, Peaceman and Rachford[2] for the solution of a problem in ideal gas flow.

Because the non-constant coefficient of the time derivative $\left(\frac{d\left(\frac{p}{z}\right)}{d\Phi}\right)$ is evaluated at the beginning of the time step, a source of material balance error exists in Eqs. 1-A and 1-B (as well as in the corresponding ADEP equations) that does not exist for the diffusion case. However, as previously stated, material balance errors using these ADIP equations have thus far proven to be negligible, the largest ADIP error encountered thus far being less than 0.04 per cent of the original gas content for a case in which the ratio of critical time step size to actual time step size was about 0.02 and more than 31 per cent of the original gas had been produced.

This discusser has found that, with the ADIP method, it is convenient to maintain storage of an additional dependent variable array over that required by ADEP. With present day computers, this will not normally constitute an objection to the use of ADIP. The ADIP program required about five fourths as much computing time per time step as the ADEP program, but neither the ADEP nor the ADIP program had been optimized. Both programs make repeated use of interpolation subroutines. All of the problems discussed above were for regions which had irregular (non-rectangular) boundaries.

NOMENCLATURE

$\Phi(p)$ = flow potential defined as $\int_{pc}^{p}\frac{\lambda\,d\lambda}{\mu(\lambda)\,z(\lambda)}$

μ = gas viscosity, function of pressure

z = gas compressibility factor

M = function of position proportional to permeability-thickness product

C = function of position proportional to porosity-thickness product, square of mesh spacing and reciprocal of time interval

p = pressure

λ = variable of integration corresponding to pressure

m, n, i = subscripts denoting spatial and time position

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