Lecture Notes

Theoretical Acoustic Anisotropy in Rocks

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NTNU, Trondheim, 2000
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1. **Definition and origin of anisotropy.**

By anisotropy we mean that certain material properties (such as sound velocities, permeability, electrical resistivity etc.) are directionally dependant. The origin of anisotropy is heterogeneity characterised by some degree of structural order at a length scale much shorter than that of the probe (e.g. the wavelength). As an example, the properties of single crystals exhibit macroscopic anisotropy resulting from the ordered lattice of atoms at the microscale. Clearly, at the atomic length scale, crystals are heterogeneous.

In rocks, the main causes of anisotropy can be classified as follows:

**Lithological (or intrinsic) anisotropy:**

This is anisotropy as a result of the rock texture. It can be caused by lamination or bedding. For example, when a river deposits sand grains, there is a tendency for more coarse grains to settle when the flow rate is high, and more fine grains to settle at low flow rates. Seasonal changes in the river flow thus result in a microlamination of deposited sand. Lithological anisotropy may also be caused by lamination at larger length scales, e.g. meter thick. Individually isotropic beds will contribute to seismic anisotropy, since the seismic wavelength is 100 m or so. When grains that are nonspherical are deposited, they will tend to deposit with their flat sides preferentially horizontal and facing each other. Inherent anisotropy in the mineral crystallites may also contribute to the overall lithological anisotropy of rocks. The latter mechanisms lead to a strong anisotropy in shales, which contain oriented, sheet like clay minerals.

Figure 1 illustrates some sources of lithological anisotropy.
Figure 1: Different lithological factors that may affect anisotropy.

**Stress-induced (extrinsic) anisotropy:**

External anisotropic stresses acting on a material lead to anisotropy for two reasons. First, the stress anisotropy leads to an effective anisotropy through the second order terms in the strain tensor. This so-called direct stress-induced anisotropy is very small. However, if stresses are sufficiently large to generate cracks, or to close pre-existing cracks, a usually very strong stress- (or crack-) induced anisotropy is induced.

Examples of stress-induced anisotropy is seen in laboratory tests, where velocity anisotropy increases strongly prior to rock failure. It is also observed in core specimens retrieved from the Earth. Then the anisotropy is a result of the stress release from *in situ* to atmospheric conditions. Such anisotropy can be removed (or at least strongly reduced) by reloading the core, which forms the basis for a core based technique to assess *in situ* stress directions (and potentially also magnitudes).

There is evidence for increased anisotropy prior to Earthquakes, and monitoring of seismic anisotropy may therefore be a possible way of predicting earthquakes. The possibility for detecting fractured reservoirs is an interesting and important application of stress-induced anisotropy.
2. Formal description of anisotropy.

Physical properties are usually response functions characterising the effect (e.g. strain) of an action (e.g. a stress). The response function in a stress - strain relationship is an elastic stiffness tensor, relating two second rank tensors. The stiffness tensor therefore must be a 4th rank tensor, i.e. Hooke's law may be written

\[ \sigma_{ij} = \sum_{k,l} C_{ijkl} \varepsilon_{kl} \]  

(2.1)

It is convenient here to use the so-called Einstein convention, which means that summation over repeated indices is understood; i.e. Eq. (2.1) reads

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]  

(2.2)

Similarly, the permeability tensor is 2nd rank, since it relates two vectors (1st rank tensors) (the flow rate and the pressure gradient). The symmetry of these tensors are uniquely given by the material symmetry, which means that all 4th rank response functions have the same symmetry in the same material, independent of the physical properties that they describe.

Since there are 3 independent spatial directions, i, j, k, and l may each take 3 different values (1, 2 and 3; equivalent to x, y and z in a Cartesian coordinate system). Thus, the elastic stiffness tensor has a total of \(3^4 = 81\) components. These are not all independent. Since i and j and also k and l can be interchanged in Eq. (2.1) and furthermore ij and kl may be interchanged (may be shown by energy considerations), the total number of independent components in the most general case (triclinic symmetry) is reduced to 21. This permits us to represent \(C_{ijkl}\) by a symmetrical 6x6 matrix \(C_{ij}\) where \(ij \rightarrow I\) and \(kl \rightarrow J\) according to the Voigt notation:

\[ 11 \rightarrow 1, \ 22 \rightarrow 2; \ 33 \rightarrow 3; \ 23 \rightarrow 4; \ 13 \rightarrow 5; \ 12 \rightarrow 6. \]

Material symmetry further reduces the number of components. Clearly, combinations of lithological and stress induced anisotropy, or changes in stress regimes throughout tectonic history may lead to complex symmetries. However, for practical purposes such symmetries are of little interest since a large number of parameters is required in order to apply the theory. If the symmetry is relatively complex, one should try to simplify and describe the material by an approximate higher symmetry. We will therefore focus only on symmetries higher than or equal to the orthorhombic. Orthorhombic means that the x-, y- and z-directions are different, but that mirror symmetry exists about the origin of the coordinate system along each principal direction. Orthorhombic symmetry may be expected e.g. from a 3D stress state with 3 independent principal stresses. Because of invariance of the stress - strain relation under symmetry operations, the following 9 elements of the stiffness tensor are non-zero:
In sedimentary rocks, the simplest non-isotropic symmetry is the transversely isotropic case, when there is a unique symmetry axis (say the z-axis), and everything is isotropic in the xy-plane. Thus, indices 1 and 2 are equivalent in the above scheme, leading to the following 5 independent elements of the stiffness tensor in this case:

\[
\begin{align*}
C_{1111} &= C_{2222} \rightarrow C_{11} \\
C_{3333} &\rightarrow C_{33} \\
C_{1122} &\rightarrow C_{12} \\
C_{2233} &= C_{1133} \rightarrow C_{13} \\
C_{2323} &= C_{1313} \rightarrow C_{44} \\
C_{1212} &\rightarrow C_{66}; \\
C_{11} - C_{12} &= 2C_{66}
\end{align*}
\]

For an isotropic solid, there are only 2 independent elements; e.g. the Lamé coefficients \( \lambda \) and \( G \). Then \( C_{11} = C_{33} = \lambda + 2G; C_{12} = C_{13} = \lambda \) and \( C_{44} = C_{66} = G \).

The wave equation for an anisotropic solid is given by:

\[
\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} = C_{ijkl} \frac{\partial}{\partial x_j} \left( \frac{1}{2} \left( \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right) \right) = C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_j} \tag{2.3}
\]

The latter identity occurs because \( k \) and \( l \) may be interchanged in \( C_{ijkl} \) and since summation is over \( k \) and \( l \).

Assume a propagating wave solution:

\[
u_i = u_i^0 e^{i(\omega t - \mathbf{q} \cdot \mathbf{x})} \tag{2.4}
\]

Here \( \omega \) is the angular frequency, and \( \mathbf{q} \) is the wavevector. Then, inserting for the directional cosines

\[
n_i = \frac{q_i}{|\mathbf{q}|} \tag{2.5}
\]

Eq. (2.3) may be rewritten as
\[ C_{ijkl} n_i n_j u_k^0 - \rho \left( \frac{\omega}{q} \right)^2 u^0_i = 0 \]

\[
\Rightarrow \quad \left[ C_{ijkl} n_i n_j - \rho \nu^2 \delta_{ik} \right] u^0_k = 0
\]

\( \delta_{ik} \) is the Kronecker delta (= 1 when \( i = k \); 0 otherwise). The phase velocity \( v = \omega / q \) is inserted.

Eq. (2.6) is known as the Christoffel equation. Wave solutions are non-trivial solutions of this equation, i.e. when the determinant of the matrix on the left side is zero. For the Transversely Isotropic (TI) medium described above, the Christoffel equation may be written:

\[
\begin{bmatrix}
    C_{11} n_i^2 + C_{12} n_i^2 + C_{44} n_i^2 - \rho \nu^2 \\
    (C_{12} - C_{11}) n_i n_j \\
    (C_{12} + C_{44}) n_i n_j \\
    (C_{13} + C_{44}) n_i n_j \\
    (C_{13} + C_{44}) n_i n_j \\
    (C_{13} + C_{44}) n_i n_j \\
    (C_{13} + C_{44}) n_i n_j \\
    C_{44} (n_i^2 + n_j^2) + C_{33} n_i^2 - \rho \nu^2
\end{bmatrix}
\begin{bmatrix}
    u_1^0 \\
    u_2^0 \\
    u_3^0 \\
\end{bmatrix} = 0
\]  \( (2.7) \)

For wave propagation parallel to the symmetry axis \( n_3 = 1 \), \( n_1 = n_2 = 0 \). In this case, Eq. (2.7) simplifies to

\[
\begin{bmatrix}
    C_{44} - \rho \nu^2 & 0 & 0 \\
    0 & C_{44} - \rho \nu^2 & 0 \\
    0 & 0 & C_{33} - \rho \nu^2
\end{bmatrix}
\begin{bmatrix}
    u_1^0 \\
    u_2^0 \\
    u_3^0 \\
\end{bmatrix} = 0
\]  \( (2.8) \)

This equation has three solutions. One is a degenerate solution corresponding to

\[ v_{s;zx} = v_{s;zy} = \sqrt{\frac{C_{44}}{\rho}} \quad (u \parallel x \text{ or } y; \ q \parallel z) \]  \( (2.9) \)

By subscript \( s;zx \) we indicate that this is an S-wave, propagating in the z-direction, and with polarisation in the x-direction. The third solution is a P-wave (i.e. propagation and polarisation in the z-direction):

\[ v_{p;z} = \sqrt{\frac{C_{33}}{\rho}} \quad (u \parallel z; \ q \parallel z) \]  \( (2.10) \)

Wave propagation in the symmetry plane can be considered by choosing e.g. \( n_1 = 1 \), and \( n_2 = n_3 = 0 \) (any direction in the xy-plane may be considered, since they are all equivalent). In this case we find three different solutions:

\[ v_{p;x} = \sqrt{\frac{C_{11}}{\rho}} \quad (u \parallel x; \ q \parallel x) \]  \( (2.11) \)
The fact that \( v_{s;zx} = v_{s;xz} \) is a consequence of the underlying symmetry of the problem. The difference between \( v_{s;xy} \) and \( v_{s;xz} \) represents a phenomenon known as shear wave splitting (or birefringence), i.e. that two S-waves propagating the same path but with different polarisation will travel at different speeds. If the z-axis is parallel to the vertical direction, then the s,xz mode is called an “SV”-wave, while the s,xy mode is termed “SH”. This symmetry is then referred to as azimuthal anisotropy, or TIV.

The angular dependence when considering a wave propagating at an angle \( \theta \) to the z-axis is evaluated by inserting \( n_1 = \sin \theta \), \( n_2 = 0 \), and \( n_3 = \cos \theta \). The three solutions are in this case:

\[
v_s = \sqrt{\frac{C_{66} \sin^2 \theta + C_{44} \cos^2 \theta}{\rho}}
\]

\[
v = \sqrt{\frac{C_{11} \sin^2 \theta + C_{33} \cos^2 \theta + C_{44} \pm \sqrt{\Delta}}{2\rho}}
\]

where

\[
\Delta = \left[ (C_{11} - C_{44}) \sin^2 \theta - (C_{33} - C_{44}) \cos^2 \theta \right]^2 + 4[C_{13} + C_{44}]^2 \sin^2 \theta \cos^2 \theta
\]

Eq. (2.14) represents a shear wave propagating in the xz-plane with a polarisation in the y-direction ("SH"-wave), whereas Eq. (2.15) gives two waves with propagation and polarisation directions in the xz-plane. Only along the symmetry directions do these waves become pure P- or S-waves. In general, they are quasi-P (the + solution) and quasi-S (the - solution) waves. Notice in particular that in non-symmetry directions the phase velocity and the group velocity will have different propagation directions.

Thomsen introduced three so-called anisotropy parameters, which all are zero in the isotropic case. These are:

\[
\varepsilon = \frac{C_{11} - C_{33}}{2C_{33}}
\]

\[
\gamma = \frac{C_{66} - C_{44}}{2C_{44}}
\]

\[
\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})}
\]
For cases of weak anisotropy ($\varepsilon, \delta$ and $\gamma$ small), Eqs. (2.14) and (2.15) may be simplified to

\[ v_p(\theta) = v_p(0) \left[ 1 + \delta \sin^2 \theta \cos^2 \theta + \varepsilon \sin^4 \theta \right] \]  
(2.20)

\[ v_{sv} = v_s(0) \left[ 1 + \frac{v^2_p(0)}{v^2_s(0)} (\varepsilon - \delta) \sin^2 \theta \cos^2 \theta \right] \]  
(2.21)

\[ v_{sh} = v_s(0) \left[ 1 + \gamma \sin^2 \theta \right] \]  
(2.22)

One may see that $\varepsilon$ represents the P-wave anisotropy, $\gamma$ represents the S-wave anisotropy, while $(\delta - \varepsilon)$ represents the deviation from elliptical anisotropy (elliptical anisotropy means that P-wave-fronts emanating from a point source are elliptical).
3. Anisotropy in layered media (Backus-model).

Let us assume that we have a layered solid medium composed of two sets of alternating plane and parallel layers. For simplicity, we shall perform our calculations by assuming that the two layer materials in themselves are isotropic. The layer planes as illustrated in Figure 2 below have normals in the z-direction.

![Figure 2: Layers in a periodically layered medium.](image)

Layer type 1 has Lamé coefficients $\lambda_1$ and $G_1$, while layer type 2 have $\lambda_2$ and $G_2$. The concentration of each type is $\phi_j$; i.e. such that here $\phi_1 + \phi_2 = 1$. Furthermore, the total density is $\rho = \phi_1 \rho_1 + \phi_2 \rho_2$.

Consider first the case where a stress $\sigma_{zz}$ is applied in the z-direction in such a way that zero lateral strain results. The total resulting strain $\varepsilon_{zz}$ is then related to the stiffness $C_{33}$ of the effective medium; i.e.

$$\sigma_{zz} = C_{33} \varepsilon_{zz} \tag{3.1}$$

The two types of layers are both exposed to the same stress, so that the resulting strain also can be calculated from the Reuss average (iso-stress model):

$$\frac{1}{C_{33}} = \frac{\phi_1}{\lambda_1 + 2G_1} + \frac{\phi_2}{\lambda_2 + 2G_2} \Rightarrow C_{33} = \frac{(\lambda_1 + 2G_1)(\lambda_2 + 2G_2)}{\phi_1(\lambda_1 + 2G_1) + \phi_2(\lambda_2 + 2G_2)} \tag{3.2}$$

$C_{11}$ can be found from

$$\sigma_{xx} = C_{11} \varepsilon_{xx} \tag{3.3}$$

by imposing a stress $\sigma_{xx}$ so that the total lateral strain $\varepsilon_{yy} = \varepsilon_{zz} = 0$. It is straightforward to see that the strain in the y-direction must be zero in both layers. The strain in the z-direction will however be different in the two layers, with a total strain given by the Reuss average; i.e.
The strain in the x-direction is the same in both layers, while the stresses are different, as for the Voigt (iso-strain) average:

\[ \varepsilon_{xx} = \varepsilon_{xx1} = \varepsilon_{xx2} \]

\[ \sigma_{xx} = \phi_1 \sigma_{xx1} + \phi_2 \sigma_{xx2} \]  

(3.5)

Using these equations, one finds:

\[ C_{11} = C_{33} \left( 1 + 4 \phi_1 \phi_2 \frac{(G_1 - G_2)(\lambda_1 + G_1 - \lambda_2 - G_2)}{(\lambda_1 + 2G_1)(\lambda_2 + 2G_2)} \right) \]  

(3.6)

By similar considerations (see e.g. White's book "Underground Sound") the three remaining stiffness tensor elements can be found:

\[ C_{44} = \frac{1}{\phi_1 \frac{1}{G_1} + \phi_2 \frac{1}{G_2}} \]  

(3.7)

\[ C_{66} = \phi_1 G_1 + \phi_2 G_2 \]  

(3.8)

\[ C_{13} = \left( \frac{\phi_1 \lambda_1}{\lambda_1 + 2G_1} + \frac{\phi_2 \lambda_2}{\lambda_2 + 2G_2} \right) \frac{1}{\phi_1 \frac{1}{\lambda_1 + 2G_1} + \phi_2 \frac{1}{\lambda_2 + 2G_2}} \]  

(3.9)

This model can be generalised to the case of several different layers, which in themselves may be anisotropic. The result may be written:

\[ C_{11} = \begin{pmatrix} C_{11} - \frac{C_{13}^2}{C_{33}} \end{pmatrix} + \begin{pmatrix} C_{13} \end{pmatrix}^{-1} \begin{pmatrix} C_{13} \end{pmatrix}^2 \]  

(3.10)

\[ C_{33} = \begin{pmatrix} C_{33} \end{pmatrix}^{-1} \]  

(3.11)

\[ C_{13} = \begin{pmatrix} C_{33} \end{pmatrix}^{-1} \begin{pmatrix} C_{13} \end{pmatrix} \]  

(3.12)

\[ C_{44} = \begin{pmatrix} C_{44} \end{pmatrix}^{-1} \]  

(3.13)

\[ C_{66} = \begin{pmatrix} C_{66} \end{pmatrix} \]  

(3.14)
Hence; it also follows that

\[ C_{12} = \left( C_{12} - \frac{C_{13}^2}{C_{33}} \right) + \left( C_{33}^{-1} \right)^{-1} \left( \frac{C_{13}}{C_{33}} \right)^2 \]  

(3.15)

Notice that the Backus theory assumes the wavelength to be much larger than the layer thickness. In the opposite case, when the wavelength is much shorter than the layers, the wave velocities perpendicular to the layers are found by adding the travel times in each layer (time average), whereas waves propagating in the layer plane will be split into individually propagating waves in each type of layer. If a first arrival is used to pick velocity, one would measure the velocity of the fastest layer.

Figure 3 shows the P- and S-wave velocities vs. angle of incidence for a layered medium; according to the Backus theory.

![Figure 3: P- and S-wave velocities for a periodically layered medium according to the Backus theory. The parameters are $\lambda_1=\rho_1=5$ GPa; $\rho_1=2.25$ g/cm$^3$; $\lambda_2=\rho_2=1$ GPa; $\rho_2=2.0$ g/cm$^3$.](image)
4. Anisotropy in fractured media (Schoenberg & Douma’s model)

The elastic properties of a rock mass with penetrating, parallel fractures can be modelled in a manner quite analogous to the layered media above. Schoenberg & Douma considered the fractures as compliant layers. The total thickness of the rock mass is $H$ (in the z-direction), out of which a height $H_{fr}$ is constituted by the fractures. The relative fracture thickness is defined as

$$h_{fr} = \frac{H_{fr}}{H}$$

(4.1)

The vertical fracture strain is

$$\varepsilon_{zz}^{fr} = \frac{\Delta u_z}{H_{fr}} = \frac{1}{\lambda_{fr} + 2G_{fr}} \sigma_{zz}$$

(4.2)

The total strain is

$$\varepsilon_{zz} = \frac{\Delta u_z}{H} = \frac{h_{fr}}{\lambda_{fr} + 2G_{fr}} \sigma_{zz} \equiv Z_N \sigma_{zz}$$

(4.3)

where $Z_N$ is the normal fracture compliance, defined in the limit when $h_{fr}$ and

$$\lambda_{fr} + 2G_{fr} \to 0$$. The shear strain $\varepsilon_{xz}^{fr}$ for the fracture is similarly given by the shear modulus

$$\varepsilon_{xz}^{fr} = \frac{\Delta u_x}{H_{fr}} = \frac{1}{G_{fr}} \sigma_{xz}$$

(4.4)

and the total strain is

$$\varepsilon_{xz} = \frac{\Delta u_x}{H} = \frac{h_{fr}}{G_{fr}} \sigma_{xz} \equiv Z_T \sigma_{xz}$$

(4.5)

This defines the transverse fracture compliance $Z_T$. Schoenberg & Douma further introduced the relative compliances $E_N$ and $E_T$:

$$E_N = Z_N (\lambda_b + 2G_b)$$

(4.6)

$$E_T = Z_T G_b$$

(4.7)

The subscript b refers to the background medium (intact rock).

Using the results of the Backus theory above, the stiffness tensor elements of the fractured medium can be written as follows:
\[ C_{11} = (\lambda_b + 2G_b) - \frac{\lambda_b^2}{\lambda_b + 2G_b} \frac{E_N}{1 + E_N} \]  

(4.8)

\[ C_{33} = \frac{\lambda_b + 2G_b}{1 + E_N} \]  

(4.9)

\[ C_{13} = \frac{\lambda_b}{1 + E_N} \]  

(4.10)

\[ C_{44} = \frac{G_b}{1 + E_T} \]  

(4.11)

\[ C_{66} = G_T \]  

(4.12)
5. **Isolated parallel cracks (Hudson theory).**

![Figure 4: Aligned microcracks.](image)

Isolated and oriented crack inclusions are a source of rock anisotropy. Two different non-self consistent models exist whereby the effects of parallel cracks can be calculated. These are Hudson's theory, which uses a scattering approach and finds the stiffness tensor elements as

\[
C_{ij} = C_{ij}^0 (1 - Q_{ij} \xi + O(\xi^2) + ...)
\]  

(5.1)

\(C_{ij}^0\) is the stiffness tensor elements for the (in principle anisotropic) background material. Here we shall consider the background to be an isotropic solid, with Lamé coefficients \(\lambda_s\) and \(G_s\) and Poisson's ratio \(\nu_s\). Notice that summation over repeated indices is not performed here.

\(\xi\) is the crack density, which for penny-shaped thin cracks is

\[
\xi = n \left\langle a^3 \right\rangle
\]  

(5.2)

Here \(n\) is the number of cracks per unit volume, and \(a\) is the crack radius. \(<\>\) denotes an average value.

Garbin & Knopoff calculated the effect of cracks using a static approach, in which the crack density appears as a perturbation to the compliance rather than to the stiffness (as in Hudson's theory):

\[
C_{ij} = \frac{C_{ij}^0}{1 + Q_{ij} \xi}
\]  

(5.3)
Notice that to the lowest order in the crack density the two models give the same result. Notice also that both models assume a spatially homogeneous distribution of cracks. In rocks that are close to failure, localisation of cracks will occur, and these models can not be expected to be valid in that case.

The Q-matrix in Eqs. (5.1) and (5.3) is, for cracks oriented with their normal in the z-direction:

\[
Q = \frac{16}{3} \begin{bmatrix}
\frac{\nu_s^2}{1-2\nu_s} & \nu_s(1-\nu_s) & (1-\nu_s)^2 & 0 & 0 & 0 \\
\nu_s(1-\nu_s) & \frac{\nu_s^2}{1-2\nu_s} & (1-\nu_s)^2 & 0 & 0 & 0 \\
1-2\nu_s & 1-2\nu_s & 1-2\nu_s & 0 & 0 & 0 \\
\end{bmatrix}
\]

Thus, the stiffness tensor elements in the case of Hudson's theory can be written as:

\[
C_{11} = (\lambda_s + 2G_s)(1-\frac{16}{3} \frac{\nu_s^2}{1-2\nu_s} \xi)
\]

\[
C_{33} = (\lambda_s + 2G_s)(1-\frac{16}{3} (1-\nu_s)^2 \frac{\nu_s^2}{1-2\nu_s} \xi)
\]

\[
C_{13} = \lambda_s (1-\frac{16}{3} \frac{(1-\nu_s)^2}{1-2\nu_s} \xi)
\]

\[
C_{44} = G_s (1-\frac{16}{3} \frac{1-\nu_s}{1-2\nu_s} \xi)
\]

\[
C_{66} = G_s
\]

For low crack densities, different orientational distributions may be analysed by simply adding crack sets with different orientations.

Figure 5 shows P- and S-wave velocities vs. angle of incidence according to Hudson’s theory, for a selected crack density of 0.10.
Figure 5: P- and S-wave velocities for a solid containing parallel cracks with their normals in the z-direction (angle of incidence = 0), according to the Hudson theory. The parameters are $\lambda_s=G_s=3\,\text{GPa}$; $\rho=2.0\,\text{g/cm}^3$ and crack density $\xi=0.10$. 
References and Suggested readings


