3D acoustic modeling in layered overburden using multiple tip-wave superposition method with effective coefficients

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ABSTRACT

Three-dimensional seismic modeling is an important tool widely used in many areas of exploration. Geologically complex areas with strong reflectors, shadow zones and diffracting edges still impose challenges to conventional modeling techniques. To overcome some of the existing limitations, we have been developing a new approach to the theoretical description and numerical modeling of three-dimensional acoustic wavefields scattered in layered media. This approach belongs to the group of analytical approaches which merge the methods based on the surface integral representation and the wavenumber-domain decomposition. The approach is based on an explicit representation of the scattered wavefield as the superposition of events multiply reflected and transmitted in accordance with the wavecode, which allows modeling of selected events independently. Each event is formed by the sequential action of classical surface integral propagators and convolutional reflection and transmission operators. We use a high-frequency approximation of the propagators in the form of a multiple tip-wave superposition method. Also, we reduce the reflection and transmission operators to effective reflection and transmission coefficients. The effective coefficients represent a generalization of the plane-wave coefficients widely used in the conventional seismic modeling for curved reflectors, non-planar wavefronts and finite frequencies. As we demonstrate in the paper, the new method is capable of reproducing complex wave phenomena, such as caustics, edge diffractions and head waves. We believe that the proposed approach has a strong potential for improving the seismic image resolution, in particular by a better description of the Green’s function in geologically complex media.
INTRODUCTION

Three-dimensional seismic modeling is an important tool widely exploited in many areas of exploration (Sheriff and Geldart, 1995; Aki and Richards, 2002). It has been extensively used for general understanding of the subsurface structure, in illumination studies, survey design and data interpretation. Also, modeling-based techniques form the basis for various imaging and inversion schemes (Claerbout, 1971; Gazdag and Sguazzero, 1984; Goldin, 1992; Gray, 2001; Treitel and Lines, 2001; Ursin, 2004).

With the growing computing power, the demand for advanced modeling and processing techniques is growing (Ramsden and Bennett, 2005). Seismic modeling evolves towards rather fast, efficient and accurate methods, which allow for an accurate description of complex wave phenomena. With the development of quantitative seismic analysis and amplitude-versus-offset (AVO) studies, not only the phase, but also the amplitude information is considered to be of major importance (Hilterman, 2001; Roden and Forrest, 2005).

The current practice shows that seismic modeling based on the Kirchhoff integral and asymptotic ray theory is a sensible option for preserving the seismic image resolution for moderate-offset observation systems (Červený, 2001; Ursin and Tygel, 1997). These techniques are based on high-frequency approximations of the Green's function for macro-blocky models, and therefore neglect caustic effects caused by the curvature of intermediate interfaces, diffracted waves generated by their edges and vertices, head waves and other near-critical and post-critical effects. It is also known that the seismic image resolution experiences strong decrease in models with strong-contrast and steeply dipping reflectors.

To improve the conventional Kirchhoff approach, Klem-Musatov et al. (2005) introduced an explicit representation of the full scattered wavefield in layered media in the form of the superposition of wave events multiply reflected and transmitted in accordance with their wavecodes. Within the approach, each event is formed by a sequential action of surface integral propagators and reflection and transmission operators. The propagators describe wave propagation in layers with smooth velocity and density variations. The reflection and transmission operators describe reflection and transmission at curved interfaces. The new technique generates the scattered wavefield in a way that accounts for multiple transmissions, as well as head waves and associated near-critical and post-critical effects (Aizenberg et al., 2004).

To implement the propagators, we use a tip-wave superposition method (TWSM) originally proposed by Klem-Musatov and Aizenberg (1985) and improved later by
Klem-Musatov et al. (1993), Klem-Musatov (1994) and Aizenberg et al. (1996). The TWSM is based on the superposition of tip-diffracted waves emerging from small patches forming the reflectors. The patches act as secondary sources in accordance with Huygens’ principle, thus generating accurate traveltimes and amplitudes at the receivers. Later Aizenberg (1992), Aizenberg (1993a) and Aizenberg (1993b) showed that the tip-diffracted waves from a small patch can be represented by a tip-wave beam, which corresponds to an integrated Green’s function. This fact allows for the generalization of the TWSM for multilayered media.

For the purposes of numerical modeling, in an earlier publication (Ayzenberg et al., 2007), we introduced so-called effective reflection and transmission coefficients (ERC and ETC) and demonstrated their advantages in Kirchhoff-type modeling. ERC and ETC are designed to generalize classical plane-wave reflection and transmission coefficients (PWRC and PWTC) for wavefields from point sources at curved interfaces. In particular, Kirchhoff-type modeling with effective coefficients removes the artifacts generated by the points of discontinuous slope of plane-wave coefficients, which represent a drawback of conventional Kirchhoff modeling. Our experience of modeling with the TWSM suggests that the approach allows for preserving the kinematical and dynamic properties of the reflected wavefields. In particular, it is capable of accurate modeling of caustic triplications and head waves at curved interfaces.

Here, we develop the multiple version of the TWSM (MTWSM), which is designed for modeling of three-dimensional (3D) scattered wavefields in layered media with several curved reflecting interfaces. The paper contains two parts. In the first part of the paper, we provide a schematic explanation of the theoretical background for the wavefield representation in layered media. We rewrite the wavefield as the sum of separate events arriving from different reflectors in accordance with their own wavecodes. Also, in this part we introduce the surface integral propagators and reflection and transmission operators. We explain the MTWSM concept in detail, and schematically show how ERC and ETC can be derived from the reflection and transmission operators. The second part is devoted to numerical tests. We consider two models. One of them contains smooth curved reflectors, another contains a reflector with diffracting edges. Using the two models, we demonstrate the ability of MTWSM to accurately model multiply reflected and transmitted wavefields, as well as multipathing, diffractions and head waves.

Appendices provide a more detailed information about the theoretical background. In Appendix A, we derive the propagators from the Helmholtz-Kirchhoff integral. We show how the propagators inside layers can be reduced to the form of matrix multiplication. In Appendix B, we shortly introduce the reflection and transmission operators for multiply scattered wavefields, and rewrite the boundary conditions in the matrix
form. In Appendix C, we explain how the propagators and reflection and transmission operators can be combined in order to obtain the transmission-propagation operators. In Appendix D, we derive an approximation for these operators, which is convenient for computational purposes. Thus we validate the MTWSM-approach.
MULTIPLY SCATTERED WAVEFIELDS IN LAYERED MEDIA

Wave propagation in layered media

Wave propagation in 3D layered media (Figure 1) may be regarded as a sequence of two processes: propagation inside layers with smoothly varying velocity and density, and reflection and transmission at internal interfaces (formed by velocity and/or density jumps). We believe that multiply scattered wavefields should be essentially described by two corresponding operators, which mimic a realistic wave propagation. Therefore the ultimate goal of this paper is to find an appropriate way of describing and modeling the wavefields in accordance with the stratigraphy of the subsurface. We aim at finding the two (possibly approximate) operators, and showing that they reproduce acoustic wave propagation in a physical way.

Various approaches to the description of wave propagation in such media have been proposed by different authors. Asymptotic ray theory (ART) may be regarded a sensible and computationally efficient option for the wavefield description in the cases when the distances between the source, reflectors and receivers are relatively large (Červený, 2001; Gjøystdal et al., 2007). The methodology works well in laterally inhomogeneous media, which makes it highly attractive. A weak point of ART, an in particular ray tracing, is the inability of modeling caustic shadows and near-critical and post-critical phenomena. Another approach to the wave propagation is described by Kennett (1983). He proposed a generalized ray method, which is applicable in horizontally layered structures. The approach is precise for layered-cake structures, but can not be generalized to laterally inhomogeneous media. Klem-Musatov et al. (2005) have found that the Kirchhoff-Helmholtz integral is a convenient option for describing the wave propagation inside smoothly varying layers. In Appendix A, we derive these integrals for the wave propagation inside a layer \( D \) in the operator form:

\[
P(s, s') < ... > = \int \int_{S'} \frac{1}{\rho(s')} \left[ \frac{\partial g_m(s, s')}{\partial n(s')} < ... > - g_m(s, s') \frac{\partial}{\partial n(s')} < ... > \right] dS', \tag{1}
\]

where \( S' \) is the reflecting interface, \( s \) and \( s' \) are points of two neighboring interfaces \( S \) and \( S' \) (\( s \) can also stand for receiver), and \( g_m(s, s') \) is the Green’s function. We show also that the full set of operators \( P(s, s') \) for all the layers forms a sparse matrix \( P \), which allows to rewrite the wave equation as a vector propagation equation:
\[ \mathbf{p} = P \mathbf{p}, \]  

where \( \mathbf{p} \) is the vector of boundary values of the wavefields reflected and transmitted at the interfaces.

There exist various methods for describing the boundary data in the Helmholtz-Kirchhoff integral. Within the conventional Kirchhoff modeling, the boundary data are empirically represented as the product of the incident wavefield and plane-wave reflection or transmission coefficients, PWRC or PWTC (Schleicher et al., 2001). Many authors have also proposed linearized versions of these coefficients (Aki and Richards, 2002; Ursin and Tygel, 1997). Such reflection and transmission coefficients are approximate. It is known that they allow for computationally efficient algorithms. They also serve as a basis for inversion routines, where linearized PWRC and PWTC is preferable. However, PWRC and PWTC are developed for the description of plane waves, and do not reproduce all features of the waves generated by point sources. In particular, they become the reason for artificial diffractions on synthetic seismograms, which are generated by the discontinuous slope of PWRC and PWTC at the critical incidence angle.

For plane interfaces in homogeneous media there exist various approaches for taking the wavefront curvature into account. They are based on including plane-wave coefficients in the integrand of the Weyl-type decompositions (Berkhout, 1987; Wenzel et al., 1990; Sen and Frazer, 1991; Downton and Ursenbach, 2006; van der Baan and Smith, 2006). For curved interfaces, various authors attempted to introduce integral equations for finding reflection and transmission operators which would generalize plane-wave coefficients (Kennett, 1984). However, the explicit form of the reflection and transmission operators has been unknown until a few years ago.

To handle curved reflectors and point sources in heterogeneous media, Klem-Musatov et al. (2004) introduced a rigorous theory of reflection and transmission for interfaces of arbitrary shape in acoustic models. They showed that the boundary data in the acoustic Helmholtz-Kirchhoff integral can be represented by generalized plane-wave decompositions called the “reflection and transmission operators”. If the reflector is curved, this decomposition is local and has to be evaluated separately for each individual point at the interface. Ayzenberg et al. (2007) proved that the exact action of the reflection operator upon the incident wavefield may be approximately described by multiplication of the incident wavefield and the corresponding effective reflection or transmission coefficients (ERC and ETC) for each point at the interface. This formalism incorporates the local interface curvature into the reflection response, does not generate artificial diffractions on synthetic seismograms, and
allows for modeling of head waves. In Appendix B, we schematically re-derive the reflection and transmission operators for layered media in the form:

\[
R(s, s) = F^{-1}(s, q) \hat{R}(q) F(q, s), \\
T(s, s') = F^{-1}(s, q) \hat{T}(q) F(q, s'),
\]

where \(F(q, s)\) is the double Fourier transform from the interface \(S\) for the plane of slowness components \(q = (q_1, q_2)\), \(F^{-1}(s, q)\) is the inverse Fourier transform for the plane of slowness components to the referent point \(s\), and \(\hat{R}(q)\) and \(\hat{T}(q)\) stand for PWRC and PWTC. In Appendix C, we show also that the boundary conditions can then be rewritten as

\[
p = Tp + p^{(1)} - p^{(0)},
\]

where \(T\) is the matrix transmission operator, \(p^{(0)}\) is the vector of boundary values of the source wavefield, and \(p^{(1)}\) is the vector of boundary values of single reflected and transmitted wavefield.

By introducing a matrix operator \(L = TP\) in Appendix C, we show that the full scattered wavefield can be represented as a branching sequence:

\[
p = \sum_{n=1}^{\infty} p^{(n)},
\]

in which each term \(p^{(n)} = Lp^{(n-1)}\) is a \(n\)-ply reflected and transmitted wavefield in accordance with a particular wavecode. Therefore the new approach is, in a sense, event-oriented.

As we show in Appendix C, the sequence of Kirchhoff-Helmholtz integrals, which would have to be calculated for long wavecodes, is successfully reduced to a matrix multiplication. This property becomes a key point of numerical implementation, by making it accurate and efficient. The methodology may be considered a generalization of the ray tracing, because it also generates wavefields for the wavecodes of interest. However, compared to ray tracing, it allows for modeling of complex wave phenomena, for instance diffractions and head waves (Ayzenberg et al., 2007). In the rest of this Section, we discuss in detail the way the wavecodes of interest can be generated.

**Transmission-propagation layer matrices**

For notational convenience, we denote layers by \(D_m (m = 1, \ldots, M)\) and assume that the source is placed in the upper layer \(D_1\). Each layer \(D_m\) is bounded by two interfaces, \(S_{m1}\) on the top, and \(S_{m2}\) at the bottom, such that a real geological interface
is represented as a two-sided surface. The waves propagate inside layer $D_m$ either from one interface to another, or along one interface. For relatively simple interface geometries, the propagation along interfaces can be considered weak compared to the propagation between two neighboring reflectors. Hence, each layer $D_m$ is fully described by four transmission-propagation layer matrices $L_{(m-1)m}$ and $L_{m(m-1)}$ given by formulas C-13. For their numerical realization, we use the transmission-propagation layer matrices $L(\hat{s})$ described by formula D-55, whose scalar elements are represented by the tip-wave beams $\Delta L(\hat{s})$ from D-54:

$$\Delta L(\hat{s}) = \chi(\hat{s}) \Delta P(\hat{s}, s),$$

where $\chi(\hat{s})$ is the ERC or ETC, and $\Delta P(\hat{s}, s)$ is the propagator from point $s$ to point $\hat{s}$.

To evaluate the layer matrices numerically, we split both boundaries of a layer into $K$ rhombic elements, so that the elements are small enough compared with the dominant wavelength. Typically, we chose the element size of the order of 1/4 to 1/8 wavelength. Then each layer matrix can be represented as a matrix of a size $K \times K$. A pair of transmission-propagation matrices describes the transfer characteristics of each layer in upward or downward directions (Figure 2). In half-spaces $D_1$ and $D_M$, the transmission-propagation matrices are absent, because these half-spaces have only one boundary.

Here we would like to draw the reader's attention to the fact that the wave processes in layers are independent of each other. In addition, the evaluation of the layer matrices can be done independently on the source position and survey geometry. This gives us the possibility of evaluating the layer matrices prior to modeling of the multiply reflected and transmitted wavefields. Whenever the velocities and densities within a layer need to be updated, only the four layer matrices describing this particular layer should be re-evaluated. If the interface geometry needs to be changed, two sets of layer matrices describing two adjacent layers should be re-evaluated. Thus, whenever minor changes of the model take place, rest of the matrices stays the same, which saves most computational time. To evaluate the scattered wavefields, the newly updated matrices have to be multiplied, as indicated by the wavecode.

**Transmission source vector**

We assume a single point source placed in the upper half-space $D_1$ at point $x^S$. It generates the pressure wavefield $p_1^{(0)}(x) = \frac{\rho_1}{4\pi|x-x^S|} e^{i \frac{\omega}{c_1} |x-x^S|}$. As follows from B-14 and D-53, the boundary values of the singly reflected and transmitted wavefield can
be represented by $K$-component transmission source vectors as

$$
p_{12}^{(1)} = (p_{12}^{(1)}(s_1), p_{12}^{(1)}(s_2), ..., p_{12}^{(1)}(s_K)), \quad p_{21}^{(1)} = (p_{21}^{(1)}(s_1), p_{21}^{(1)}(s_2), ..., p_{21}^{(1)}(s_K)), \quad (7)
$$

and

$$
p_{12}^{(1)}(s_k) = R_{12,12}(s_k, s_k')p_{12}^{(0)}(s_k'), \quad p_{21}^{(1)}(s_k) = T_{21,12}(s_k, s_k')p_{12}^{(0)}(s_k'), \quad (8)
$$

where $s_k$ is the referent point at boundaries $S_{12}$ or $S_{21}$, respectively, $s_k'$ are virtual points at boundaries $S_{12}$ or $S_{21}$, respectively. These two vectors clearly depend only on the source position and the geometry of the interface between $D_1$ and $D_2$, and remain unchanged for various receiver positions.

### Propagation receiver matrices

The receiver array can be arbitrarily placed in any layer. We denote the receivers by points $x^R$. For example, a receiver array positioned in the upper half-space records reflections arriving from the subsurface, and a receiver array positioned in the lower half-space records the Green’s function, which can be used in data migration or inversion.

For numerical evaluation of the resulting wavefield, there are two receiver matrices $P_{m(x^R), m1}$ and $P_{m(x^R), m2}$ describing the propagation from both boundaries of the actual layer $D_m$ towards the receivers (Figure 2). In both half-spaces, there is one receiver matrix $P_{1(x^R), 12}$ or $P_{M(x^R), 11}$. These matrices are described by formula D-53. For $K_R$ receivers, the matrix has a size $K_R \times K$. The receiver matrix clearly depends only on the receiver positions, and remains unchanged for various source positions.

### Multiply reflected and transmitted wavefields

Once we have the transmission source vectors, all layer matrices and receiver matrices, we are able to compute multiply reflected and transmitted wavefields of interest based on formulas C-8 and C-5. The full wavefield is then the sum of all possible multiply reflected and transmitted wavefields:

$$
p = \sum_{n=1}^{N} p^{(n)}, \quad (9)
$$

and each $n$-ply reflected and transmitted wavefield depends on the previous $(n-1)$-ply reflected and transmitted wavefield:
\[ p^{(n)} = L p^{(n-1)}. \]  

To calculate \( p^{(n)} \), we introduce a wavecode which uniquely describes a particular wavefield. The wavecode contains information about the sequence of the layers and interfaces passed through by the wavefield. Following the wavecode, the appropriate transmission source vectors are sequentially multiplied with corresponding layer matrices, and finally with the receiver matrix. For other wavecodes, the multiplication has to be repeated independently. Therefore the interference wavefield at the receiver can be represented by the double sum

\[ p_m(x_R, t) = \sum_{n=1}^{N(T)} \sum_{i=1}^{I_n} p_{mi}^{(n)} \left( x_R, t - \tau_{mi}^{(n)} (x_R) \right), \]  

where \( N(T) \) is the maximal multiplicity of reflections and transmissions within a time window \( t \in [0, T] \), \( I_n \) is the number of various wavecodes of the \( n \)-th multiplicity, \( \tau_{mi}^{(n)} (x_R) \) is the traveltime (eikonal) along an individual wavecode.

We provide all formulas in the angular-frequency domain. Conversion to the time domain can be performed by the Fourier transform:

\[ p_{mi}^{(n)} \left( x_R, t - \tau_{mi}^{(n)} (x_R) \right) = \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} p_{mi}^{(n)} (x_R, \omega) e^{-i\omega t} d\omega, & t > \tau_{mi}^{(n)} (x_R); \\ 0, & t < \tau_{mi}^{(n)} (x_R). \end{cases} \]

Using formulas 7-8 and D-53-D-53, we obtain the stationary wavefield corresponding to the \( i \)-th wavecode of the \( n \)-th multiplicity:

\[ p_{mi}^{(n)} (x_R, \omega) = (P_{m(x_R),m1} \text{ or } P_{m(x_R),m2}) \left[ \prod_{j=1}^{n-1} L_{ij} \right] (p_{12}^{(1)} \text{ or } p_{21}^{(1)}). \]

where \( L_{ij} \) is the transmission-propagation layer matrix corresponding to the \( j \)-th step of the \( i \)-th wavecode.

An example of different wavecodes for a five-layer model is provided in Figure 6. The time window \( t \in [0, 4] \) s corresponds to a maximal multiplicity \( N = 7 \) for the receivers distributed at zero depth. The full set of possible wavecodes consists of C1 with \( n = 1 \), C2 with \( n = 3 \), C3 with \( n = 5 \), C4 with \( n = 7 \) and C5 with \( n = 7 \).

The wavecode C1 (Figure 11a) contains the one matrix multiplication:

\[ p_{1(C1)}^{(1)} (x_R, \omega) = P_{1(x_R),12}^{(1)} p_{12}^{(1)}. \]
The wavecode C2 (Figure 12a) corresponds to the sequence of matrix multiplications:

\[ p_{22}^{(2)} = L_{22,21} p_{21}^{(1)}, \]
\[ p_{12}^{(3)} = L_{12,22} p_{22}^{(2)}, \]
\[ p_{1(C2)}^{(3)} (x^R, \omega) = P_{1(x^R),12} p_{12}^{(3)}. \]

From 15 we obtain the resulting formula:

\[ p_{1(C2)}^{(3)} (x^R, \omega) = P_{1(x^R),12} [L_{12,22} L_{22,21}] p_{21}^{(1)}. \]  \hspace{1cm} (16)

The wavecode C3 (Figure 13a) corresponds to the sequence of matrix multiplications:

\[ p_{31}^{(2)} = L_{31,21} p_{21}^{(1)}, \]
\[ p_{32}^{(3)} = L_{32,31} p_{31}^{(2)}, \]
\[ p_{32}^{(4)} = L_{22,32} p_{32}^{(3)}, \]
\[ p_{12}^{(5)} = L_{12,22} p_{32}^{(4)}, \]
\[ p_{1(C3)}^{(5)} (x^R, \omega) = P_{1(x^R),12} p_{12}^{(5)}. \]

From 17 we obtain the resulting formula:

\[ p_{1(C3)}^{(5)} (x^R, \omega) = P_{1(x^R),12} [L_{12,22} L_{22,32} L_{32,31} L_{31,21}] p_{21}^{(1)}. \]  \hspace{1cm} (18)

The wavecode C4 (Figure 14a) corresponds to the sequence of matrix multiplications:

\[ p_{31}^{(2)} = L_{31,21} p_{21}^{(1)}, \]
\[ p_{41}^{(3)} = L_{41,31} p_{31}^{(2)}, \]
\[ p_{42}^{(4)} = L_{42,41} p_{41}^{(3)}, \]
\[ p_{32}^{(5)} = L_{32,42} p_{42}^{(4)}, \]
\[ p_{22}^{(6)} = L_{22,32} p_{32}^{(5)}, \]
\[ p_{12}^{(7)} = L_{12,22} p_{22}^{(6)}, \]
\[ p_{1(C4)}^{(7)} (x^R, \omega) = P_{1(x^R),12} p_{12}^{(7)}. \]

From 19 we obtain the resulting formula:

\[ p_{1(C4)}^{(7)} (x^R, \omega) = P_{1(x^R),12} [L_{12,22} L_{22,32} L_{32,42} L_{42,41} L_{41,31} L_{31,21}] p_{21}^{(1)}. \]  \hspace{1cm} (20)

The wavecode C5 (Figure 15a) corresponds to the sequence of matrix multiplications:
\begin{align}
p_{31}^{(2)} &= L_{31,21} p_{21}^{(1)}, \\
p_{32}^{(3)} &= L_{32,31} p_{31}^{(2)}, \\
p_{31}^{(4)} &= L_{31,32} p_{32}^{(3)}, \\
p_{32}^{(5)} &= L_{32,31} p_{31}^{(4)}, \\
p_{22}^{(6)} &= L_{22,32} p_{32}^{(5)}, \\
p_{12}^{(7)} &= L_{12,22} p_{22}^{(6)}, \\
p_{1\{C5\}}^{(7)}(x^R, \omega) &= P_{1(x^R),12} p_{12}^{(7)}.
\end{align}

From 21 we obtain the resulting formula:
\begin{align}
p_{1\{C5\}}^{(7)}(x^R, \omega) &= P_{1(x^R),12} [L_{12,22} L_{22,32} L_{32,31} L_{31,32} L_{32,31} L_{31,21}] p_{21}^{(1)}.
\end{align}

Formulas 13, 14, 16, 18, 20 and 22 are based on the tip-wave beam decomposition in the coordinate space. Each layer matrix combines two processes: propagation of tip-wave beams in the layer and their reflection and transmission at the interface. Propagation of an individual tip-wave beam accounts for the energy flux along the apparent wave tube and its transversal diffusion along the tip-wave beam front. Reflection and transmission accounts for the incidence angle and respective curvature of the front and interface. In the particular case of flat homogeneous layers, the formulas above are equivalent to the formulas of the generalized ray method based on the plane-wave decomposition (Kennett, 1983).

In the case of one interface between two half-spaces the layer matrices are absent. The method reduces to the multiplication of the transmission source vector and the propagation receiver matrix, as in the example for the wavecode C1 in formula 14. In the earlier version of the method, in the absence of a rigorous transmission theory at curved interfaces, the transmission source vectors were evaluated using heuristic PWRC and PWTC (Klem-Musatov et al., 1993). In the later version of the method, the transmission source vector accounts for the reflection and transmission operators in the form of the ERC and ETC (Ayzenberg et al., 2007). Simultaneously we accounted for the critical effects, including the head waves, and suppressed the artifacts intrinsical to all Kirchhoff techniques.

In the general case of layered media, we refer to the method as the “multiple tip-wave superposition method” (MTWSM). The method gives the possibility to model the wavefield corresponding to a particular wave code, as well as to collect the full seismogram within a finite time window. The complete wavefield represents the Green’s function for the layered medium. Subject to the receiver positions, we are able to
model the Green’s function for particular purposes. In particular, the Green’s function for layered overburden may be of interest for various inversion problems.
SYNTHETIC MODELING

In this Chapter, we comprise the results of the 3D modeling with the MTWSM. We have chosen two models. The first one is a model with smooth reflectors and the compressional velocity and density similar to those found in the Gulf of Mexico. Another model, known as the French model, is artificially built. It comprises two reflectors, with the upper one containing diffracting edges. The two models are designed to demonstrate that MTWSM is capable of modeling multiply scattered wavefields, as well as diffraction phenomena.

Modeling in a model with smooth reflectors

For the numerical simulations in this group of tests, we use the Puzyrev wavelet given by the formula:

\[ F(t) = -e^{-p^2/\pi^2} \sin p, \quad p = 2\pi \frac{t - t_0}{T}, \]

where \( t_0 = 0.064 \) s is the time shift to the wavelet central point, and \( T = 0.048 \) s is the wave period. The dominant frequency is \( f_d = 22 \) Hz. The source wavelet and modulus of its spectrum are shown in Figure 5.

We consider a model with five homogeneous layers. Sub-salt reflections present a serious challenge for conventional imaging and inversion routines due to high velocity contrasts. Therefore, for modeling purposes, we chose the velocity and density profiles similar to those often observed in the Gulf of Mexico, (Ogilvie and Purnell, 1996). The model has equal extent of \( 2.56 \) km in both \( x \)- and \( y \)-directions. The interfaces are given by the formulas:

\[
\begin{align*}
z_1 &= -1.1 - 0.1 \tanh[1.5\pi(x - 1.28)], \\
z_2 &= -2.8 + 0.3 \exp[-4(x - 1.7)^2 - 4(y - 1.28)^2], \\
z_3 &= -3.7 + 0.1 \exp[-8(x - 1.7)^2 - 8(y - 1.28)^2], \\
z_4 &= -4.0.
\end{align*}
\]

The source is positioned at point \( x^S = (1.0, 1.28, 0) \) km. The receiver arrays containing 101 equally spaced receivers with a step of 20 m are placed at different depths. Figure 6 shows the vertical section of the model in the \( x \)-direction for \( y = 1.28 \) km.
At first, we model primary transmissions in each of the layers $D_2$, $D_3$, $D_4$ and $D_5$. These wavecodes are of interest for us, because they represent the Green’s functions, which may be useful for imaging schemes. We place four receiver arrays inside the layers: P1 corresponding to $z = -2.0$ km; P2 corresponding to $z = -3.5$ km; P3 corresponding to $z = -3.95$ km; and P4 corresponding to $z = -4.8$ km. The survey geometry, wavecodes and corresponding synthetic seismograms modeled using the MTWSM with ETC are given in Figures 7-10. The seismograms modeled with ETC and PWTC do not exhibit large differences. However, the local curvature of some interfaces or strong velocity contrasts might generate considerable phase and amplitude differences. For example, for some of the wavecodes, the difference reaches 10% for the zero-offset traces. Therefore we may conclude that ETC give a smoother representation of the Green’s function compared with PWRC.

Another test is devoted to modeling of multiply scattered wavefields. We placed the receiver array at a level $z = 0$ km with the first receiver coinciding with the source. We use the same Puzyrev wavelet as the one in Figure 5. Our ultimate goal in this test is to collect all events, which arrive within the time window $t < 3.8$ s. Simple kinematic calculations show that there are five reflections that fit this time window. The wavecodes and corresponding synthetic seismograms modeled using the MTWSM with ERC and ETC are given in Figures 11-15. The survey geometry, full set of wave codes and complete seismogram are shown in Figure 16. We weight each event on the full seismogram with a scaling factor for better visibility. The weighting coefficients are: 0.5 for C1, 5.0 for C2 and C3, 30.0 for both C4 and C5.

The ability of modeling separate events is a valuable property in this particular model. Because the events corresponding to the wavecodes C4 and C5 have almost the same traveltimes, this makes the interpretation of the lower part of the full seismogram difficult. Thus we may conclude that MTWSM is a useful tool for interpretation of complex wave interferences.

**Modeling in a model with diffracting edges**

For this group of tests, we use a Ricker wavelet given by the formula:

$$F(t) = (1 - p^2)e^{-p^2/2}, \ p = \pi f_p(t - t_0),$$

(25)

where $f_p = 20$ Hz and $t_0 = 0.064$ s is the time shift to the wavelet central point. The dominant linear frequency is about $f_d = 16$ Hz, and the dominant period is 0.064 s. The source wavelet and modulus of its spectrum are shown in Figure 17.

We run synthetic modeling of the 3D Green’s function for a three-layer model
known as the French model. Because of the discontinuous geometry of the upper reflector, synthetic modeling simulates strong diffraction events, thus allowing to test the ability of the MTWSM to model complex wave phenomena.

The upper reflector has complex topography. Both reflectors are determined for $0 \leq x \leq 5.12$ km and $0 \leq y \leq 5.12$ km. Algorithmically, the upper reflector is described as a surface with five smooth pieces:

If $|y - x - 5.12| \leq 4.36$ then $z(x, y) = -1.1$;

If $|y - x - 5.12| > 4.36$ and $|y - x + 5.12| > 5.34$ then $z(x, y) = 0.593(y - x) - 1.55$;

If $\sqrt{(x - 2.58)^2 + (y - 1.3)^2} \leq 0.7$ then
\[ z(x, y) = -1.19 - 1.18 + \sqrt{1.392 - (x - 2.58)^2 - (y - 1.3)^2}; \]

If $\sqrt{(x - 3.84)^2 + (y - 2.56)^2} \leq 0.7$ then
\[ z(x, y) = -1.19 - 1.18 + \sqrt{1.392 - (x - 3.84)^2 - (y - 2.56)^2}; \]

If $|y - x + 5.12| \leq 5.34$ and $\sqrt{(x - 2.58)^2 + (y - 1.3)^2} > 0.7$ and $\sqrt{(x - 3.84)^2 + (y - 2.56)^2} > 0.7$ then $z(x, y) = -1.42$.

The lower reflector is a horizontal plane at a depth of $z = -2.1$ km.

The velocity in the two halfspaces above the upper reflector and below the lower reflector are chosen to be $c_1 = c_3 = 3.6$ km/s. The velocity in the intermediate layer is chosen to be 10% lower, which is $c_2 = 3.24$ km/s. The density is constant throughout the model, $\rho_1 = \rho_2 = \rho_3 = 2$ g/cm$^3$.

A source is positioned in the center of the model at point $x^S = (2.56, 2.56, 0.0)$ km. The receivers are equally distributed with a step of 40 m at the source level, as shown in the vertical cross-section for $y = 2.56$ km in Figure 18. We assume that the receivers with the $x$-coordinates smaller than the source’s $x$-coordinates correspond to positive offsets, and the receivers with the $x$-coordinates larger than the source’s $x$-coordinates correspond to negative offsets.

The synthetic data contain two primary reflections from the upper and lower reflectors and several peg-legs formed in the intermediate layer between the two reflectors. However, due to the geometrical spreading and weak reflection coefficients, only three wavecodes are clearly distinguishable on the data: two primary reflections corresponding to the wavecodes C1 and C2 and a first-order peg-leg corresponding to the wavecode C3. These three events are shown in Figures 19, 20 and 21. The full seismogram containing all three wavecodes in shown in Figure 22. The peg-leg event
is mainly formed by multiple diffraction, and therefore has quite weak amplitude, which makes it almost invisible on the total seismogram. For comparison we chose the rays corresponding to zero offset such that they are reflected and transmitted with zero angle. The traveltimes for both reflections are modeled precisely, which is the natural result for the method.

The primary reflection from the upper reflector, corresponding to the wavecode C1, arrives in a time window between approximately 0.8 s and 1.0 s. The wavefield is the superposition of the reflection itself and several diffractions. The five smooth parts of the discontinuous reflector generate the reflection itself. There are two straight and two circular diffracting edges with eight diffracted waves diverging from them. The primary reflection can be described by the asymptotic ray theory (Červený, 2001), and the associated diffractions can be described by the theory of edge and tip diffractions (Klem-Musatov, 1994; Klem-Musatov et al., 1994).

At the upper reflector, the reflected wave and diffracted waves generated at the lower edge interfere. The distance between the reflection point and the lower edge is 0.156 km. When neglecting a small time shift between the reflected and diffracted waves, we can estimate the amplitude of their interference by an asymptotic formula \( A_g[1 - W(w)] \), where \( w = \sqrt{\frac{4f_d(\tau_d - \tau_g)}{c^2}} \), \( \tau_d \) is the diffracted wave eikonal, and \( \tau_g \) is the geometrical-wave eikonal (Klem-Musatov et al., 1994). Taking into account the dominant linear frequency \( f_d = 16 \) Hz, the velocity 3.6 km/s, the depth of the interface 1.42 km and the distance between the reflection point and the lower edge, we obtain that \( w = 0.5 \) and \( W(0.5) = 0.28 + i0.11 \). The analytic amplitude of the reflected wave is equal to \( A_g = \frac{R_{12,12}}{L_{C1}} = -\frac{0.0526}{2.84} = -0.0185 \). Finally, we obtain the total amplitude \( |A_g[1 - W(w)]| = 0.0185 \times 0.728 = 0.0135 \). The synthetic amplitude at her peak is estimated as 0.013 with the respective error of less than 4%.

The wavefield arriving in a time window between 1.3 s and 1.8 s from the lower reflector and corresponding to the wavecode C2, is the superposition of the primary reflections from many illuminated zones, and multiple diffractions. The multiple diffractions contain first-order and second-order edge waves. The first-order edge waves contain reflections and transmissions followed by diffractions at the edges, and diffractions at the edges followed by reflections and transmissions.

At the lower reflector, the reflected wave from the plane and the diffracted wave generated at the lower edge of the upper interface interfere. The geometrical spreading is \( L_{C2} = 2[h_1 + h_2 \frac{c_d}{c_g}] \), where \( h_1 = 1.42 \) km and \( h_2 = 0.68 \) km are the layer thicknesses. The analytic amplitude of the primary reflected wave is equal to \( A_g = \frac{T_{12,21}R_{22,21}T_{21,12}}{L_{C2}^4} = \frac{0.05245}{4.064} = 0.0129 \). However, the synthetic amplitude at her peak estimated from the seismogram is 0.0092. There are two main reasons for this difference.
in the amplitudes. The first reason is that the asymptotic ray theory and the theory of edge and tip diffractions provide only an approximate high-frequent amplitude. Another reason is that the diffracted wavefield has a complex structure, which includes higher-order diffractions. It is natural to assume that the MTWSM amplitude is more precise than the predicted amplitude, because the method accounts for all orders of diffraction.

Because asymptotic methods cannot handle the complexity of the model, we tried to justify the correctness of the MTWSM codes using a full 3D elastic finite-difference (EFD). For some technical reasons, we were not able to fully match the two results. Therefore we provide a qualitative comparison only. The two seismograms look kinematically similar, if we neglect shear and converted waves. We were unable to compare the amplitudes, because of some differences in the input parameters. Therefore a more detailed comparison remains to be done. We plan on a trace-by-trace comparison in the nearest future. A preliminary comparison regarding the computational speed shows that for the French model, the MTWSM modeling is approximately 3 times faster than the EFD scheme.
CONCLUSIONS

We presented an analytical approach to the theoretical description and numerical modeling of the three-dimensional acoustic wavefields scattered in layered media. The approach is based on the representation of the full wavefield as the branching sequence of multiply scattered waves, which can be directly identified with the interfaces generating them. Separate events are formed by a sequential action of transmission-propagation operators. The composite operators describe wave propagation in layers with smooth velocity and density variations and reflection and transmission at curved interfaces.

We implemented the transmission-propagation operators using the multiple tip-wave superposition method (MTWSM). We showed that MTWSM generates the reflection response by the superposition of tip-diffracted waves excited at the reflector in accordance with Huygens’ principle. The numerical tests demonstrated that MTWSM is capable of simulating complex wave phenomena, such as caustic triplications and diffractions. For numerical simulations, we reduced the reflection and transmission operators to the form of effective reflection and transmission coefficients (ERC and ETC). ERC and ETC generalize plane-wave reflection and transmission coefficients (PWRC and PWTC) for wavefields from point sources at curved interfaces, and are not anyhow limited to small incidence angles and weak parameter contrasts across the reflector. Simple synthetic modeling showed that ERC and ETC accurately reproduce the head-wave amplitudes and associated near-critical and post-critical effects.

We provided the results of testing the MTWSM approach for synthetic modeling of the Green’s function and multiply scattered wavefields in realistic models. A comparison of the modeling results with the asymptotic ray theory and elastic finite differences showed that the approach enables preserving the kinematic and dynamic properties of acoustic scattered wavefields in layered overburden.
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APPENDIX A
SURFACE INTEGRAL PROPAGATORS

We consider models with several sharp reflectors. We assume that interfaces are
curved and smooth, and outside some bounded volume they become flat and parallel.
Let the layers be denoted $D_m$, where $m = 1, \ldots, M$. Each layer $D_m$ is bounded by
two interfaces, $S_{m1}$ on the top, and $S_{m2}$ at the bottom, such that a real geological
interface is represented as a two-sided surface. Let $n_{mj}$ be the normal to the interface
$S_{mj}$ directed towards $D_m$. Points belonging to layer $D_m$ are denoted by $x_m$, and
points belonging to interface $S_{mj}$ are denoted by $x_{mj}$. Each of the layers is described
by the wave propagation velocity $c_m$ and density $\rho_m$. For simplicity, we assume a
single source of the pressure wave positioned in the upper half-space at point $x^S$.
The receiver array can be arbitrary placed in any of the layers. The receivers are
denoted by points $x^R$. For example, the receiver array placed in the upper layer
records reflections arriving from the overburden; and the receiver array placed in the
lower half-space records the Green’s function, which can be used for data migration
or inversion.

Here we consider an inhomogeneous acoustic domain (halfspace or layer) $D$ with
a varying velocity $c(x)$ and density $\rho(x)$. We assume that the velocity and density
become constant outside some bounded part of the domain. If the domain is a
halfspace, its boundary $S$ is represented by an infinite curved interface, which becomes
a plane outside some part of the domain. If the domain is a layer, it is bounded by
two isolated infinite curved interfaces, which become two parallel planes outside of
some part of the domain.

The scattered wavefield $p(x)$ satisfies the acoustic wave equation at all points $x$
belonging to the domain (Aki and Richards, 2002):

$$\nabla \cdot \left[ \frac{1}{\rho(x)} \nabla p(x) \right] + \frac{\omega^2}{\rho(x)c^2(x)} p(x) = 0, \quad (A-1)$$

and the radiation condition in an infinitely distant region (Costabel and Dauge, 1997).

It is known that using the divergence theorem, the wavefield $p(s)$ can be represen-
ted as a decomposition to elementary waves emerging from a virtual point source,
often referred to as the fundamental solution (Costabel and Dauge, 1997; Červený,
2001). The fundamental solution is generally defined to an arbitrary solution of
the homogeneous acoustic wave equation, which depends on the shape of the domain (Friedlander, 1958; Kottler, 1965). Therefore it may contain components which would not be observed in realistic wavefields. In problems where the full wavefield is computed, the choice of the arbitrary component does not play an important role. For instance, a simpler fundamental solution, called the free-space Green’s function, can be chosen. However, when particular parts of the full wavefield are to be evaluated, the choice of the fundamental solution has a critical importance. Aizenberg and Ayzenberg (2008) suggested a rigorous formulation of the absorbing condition at the interface, which allows derivation of a so-called “feasible fundamental solution”. The feasible fundamental solution \( g(x, y) \) satisfies the acoustic wave equation at points \( x \) and \( y \):

\[
\nabla_x \cdot \left[ \frac{1}{\rho(x)} \nabla_x g(x, y) \right] + \frac{\omega^2}{\rho(x)c^2(x)} g(x, y) = -\delta(x - y),
\]

\[
\nabla_y \cdot \left[ \frac{1}{\rho(y)} \nabla_y g(x, y) \right] + \frac{\omega^2}{\rho(y)c^2(y)} g(x, y) = -\delta(x - y),
\]

(A-2)

the radiation condition at infinity (Costabel and Dauge, 1997) and an additional absorbing condition at the boundary of the domain. The feasible fundamental solution \( g(x, y) \) is a realistic wavefield which would be generated by a point source and observed in regions bounded by regular interfaces producing shadow zones. In the regions whose boundaries have simple geometrical forms and do not contain large shadow zones, the feasible fundamental solution may be approximated by the free-space Green’s function. In a homogeneous medium, where \( c \) and \( \rho \) are constant, the free-space Green’s function is a spherical wave of the form \( g(x, y) = \frac{\rho}{4\pi R} e^{i\omega R} \), where \( R = |x - y| \). In the regions with significant shadow zones, the feasible fundamental solution can be expressed in the form \( g(x, y) = a_g(x, y)e^{i\omega\tau_g(x, y)} \), where \( a_g(x, y) \) is the amplitude and \( \tau_g(x, y) \) is the eikonal or traveltime (Červený, 2001).

We apply the divergence theorem in order to represent the spatial wavefield through the surface Kirchhoff integral. Although the derivation was frequently published by many other authors, we briefly repeat it in order to use it further. We introduce a vector:

\[
v(x, y) = \left[ \frac{1}{\rho(y)} \nabla_y g(x, y) \right] p(y) - g(x, y) \left[ \frac{1}{\rho(y)} \nabla p(y) \right],
\]

(A-3)

which is a function of point \( y \), while point \( x \) is a parameter. The vector \( v(x, y) \) has the divergence:
\[ \nabla_y \cdot \mathbf{v}(x, y) = \nabla_y \cdot \left[ \frac{1}{\rho(y)} \nabla_y g(x, y) \right] p(y) - g(x, y) \nabla \cdot \left[ \frac{1}{\rho(y)} \nabla p(y) \right]. \quad (A-4) \]

Because vector \( A-3 \) has a singularity when \( y \rightarrow x \), we apply the divergence theorem in a regular area \( D \setminus B_x \), with an infinitesimal ball \( B_x \) which surrounds the singular point \( x \). The domain \( D \setminus B_x \) is bounded by a multiply-connected surface \( S \cup S^\infty \cup S_x \), where \( S^\infty \) is an infinitely distant part of the surface, and \( S_x \) is the boundary of the ball \( B_x \). The divergence theorem may be written as:

\[ -\int \int \int_{D \setminus B_x} \nabla_y \cdot \mathbf{v}(x, y) dV(y) = \int \int_{S \cup S^\infty \cup S_x} \mathbf{n}(s) \cdot \mathbf{v}(x, s) dS(s), \quad (A-5) \]

where \( s \) represents points of the surface \( S \cup S^\infty \cup S_x \), \( \mathbf{n}(s) \) is the normal to \( S \cup S^\infty \cup S_x \) directed into \( D \setminus B_x \). We will further assume that the radius of \( B_x \) tends to zero.

To evaluate integral \( A-5 \), we need the explicit form of the divergence \( A-4 \). Substitution of \( A-1 \) and \( A-2 \) into \( A-5 \) shows that the following holds true:

\[ \nabla_y \cdot \mathbf{v}(x, y) = 0, \quad x \neq y. \quad (A-6) \]

Substitution of \( A-6 \) into \( A-5 \) gives that the volume integral is zero. From the radiation condition it follows that the surface integral over the surface \( S^\infty \) is also equal to zero. Miranda (1970) showed that the surface integral over the infinitesimal sphere \( S_x \) tends to \(-p(x)\) when the radius of the sphere tends to zero. Thus we obtain the scattered wavefield at point \( x \) of the domain \( D \) can be represented by a surface Kirchhoff-type integral:

\[ p(x) = \int \int_{S} \mathbf{n}(s) \cdot \mathbf{v}(x, s) dS(s), \quad (A-7) \]

where the vector \( \mathbf{v}(x, s) \) has the form

\[ \mathbf{v}(x, s) = \left[ \frac{1}{\rho(s)} \nabla_s g(x, s) \right] p(s) - g(x, s) \left[ \frac{1}{\rho(s)} \nabla p(s) \right], \quad (A-8) \]

\( p(s) \) and \( \frac{1}{\rho(s)} \nabla p(s) \) are the boundary values of the wavefield and its weighted derivative. We assume that the kernels \( g(x, s) \) and \( \frac{1}{\rho(s)} \nabla_s g(x, s) \) are constructed based on the feasible fundamental solution for \( y \rightarrow s \).

Within the framework of mathematical wave theory, we need to define the pair of boundary data, \( p(s) \) and \( \frac{1}{\rho(s)} \mathbf{n}(s) \cdot p(s) \) by using standard methods for solving
the problem of boundary data (Costabel and Stephan, 1985). For layered media, these approaches lead to complex formulas. We show in Appendix D that a specific approximation of the propagators allows to limit the search for only one boundary value, \( p(s) \). Because this approximation is accurate enough, we will use it further in the paper. We write A-7 at point \( x \) of layer \( D_m \) with the upper and lower boundaries \( S_{m1} \) and \( S_{m2} \) in the operator form:

\[
p_m(x) = P_m(x, s_{m1})p_{m1} + P_m(x, s_{m2})p_{m2},
\]

where we introduced surface integral propagators

\[
P_m(x, s_{m1}) < ... > = \int \int_{S_{m_j}} \frac{1}{\rho(s_{m_j})} \left[ \frac{\partial g_m(x, s_{m1})}{\partial n(s_{m1})} < ... > \right] dS_{m_j}
- \int \int_{S_{m_j}} \frac{1}{\rho(s_{m_j})} \left[ g_m(x, s_{m1}) \frac{\partial}{\partial n(s_{m1})} < ... > \right] dS_{m_j},
\]

(A-10)

where \( p_{mj} \) are the unknown boundary data, \( \frac{\partial}{\partial n(s_{m_j})} = n(s_{m_j}) \cdot \nabla s_{m_j} \) is the operator of normal differentiation, and point \( s_{m_j} \) belongs to the interface \( S_{m_j} \).

To define the unknowns \( p(s_{m_j}) \), we use the limiting values of the scattered wavefield at the interface (Costabel and Stephan, 1990; Kleinman and Martin, 1988):

\[
p_{12} = P_{12,12}p_{12},
\]
\[
p_{mj} = P_{mj,mj}p_{mj} + P_{mj,m(3-j)}p_{m(3-j)},
\]
\[
p_{M1} = P_{M1,M1}p_{M1}.
\]

(A-11)

Here \( P_{mj,mj'} = P_m(s_{mj}, s_{mj'}) \) (where \( j' \) equals either \( j \) or \( 3-j \)) are the long-distance Kirchhoff-type propagators given by the formulas:

\[
P_m(s_{mj}, s_{mj'}) < ... > = \int \int_{s_{mj'}} \frac{1}{\rho(s_{mj'})} \left[ \frac{\partial g_m(s_{mj}, s_{mj'})}{\partial n(s_{mj'})} < ... > \right] dS_{mj'}
- \int \int_{s_{mj'}} \frac{1}{\rho(s_{mj'})} \left[ g_m(s_{mj}, s_{mj'}) \frac{\partial}{\partial n(s_{mj'})} < ... > \right] dS_{mj'}.
\]

(A-12)

We observe that the system A-11 is incomplete in strict sense, because completeness requires the same number of similar relations for the weighted normal derivatives.
However, as we pointed out earlier, the relations provided in A-11 are sufficient for further approximations.

The operator relations A-11 can formally be rewritten in the form of a vector propagation equation:

\[ \mathbf{p} = \mathbf{Pp}, \]  

(A-13)

where the vector of boundary values is defined as

\[ \mathbf{p} = \begin{pmatrix} p_{12} \\ p_{21} \\ p_{22} \\ \vdots \\ p_{(M-1)1} \\ p_{(M-1)2} \\ P_{M1} \end{pmatrix}, \]  

(A-14)

and the surface propagator matrix is

\[ \mathbf{P} = \begin{pmatrix} P_{12,12} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & P_{21,21} & P_{21,22} & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & P_{22,21} & P_{22,22} & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & P_{(M-1)1;(M-1)1} & P_{(M-1)1;(M-1)2} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & P_{(M-1)2;(M-1)1} & P_{(M-1)2;(M-1)2} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & P_{M1,M1} \end{pmatrix}. \]  

(A-15)

We notice that the surface propagators in each layer form 2x2 sub-matrices

\[ \begin{pmatrix} P_{m1,m1} & P_{m1,m2} \\ P_{m2,m1} & P_{m2,m2} \end{pmatrix}, \]  

which are aligned with the main diagonal of matrix A-15.

The surface propagators in each halfspace are formed by scalar elements \( P_{12,12} \) and \( P_{M1,M1} \).
APPENDIX B
REFLECTION AND TRANSMISSION OPERATORS

Conventional methods, which solve the boundary value problem, interconnect the boundary values in relations A-11 and A-13 through additional conditions, which imply the continuity of the pressure and normal particle velocity across the interface. These conditions at the upper interface can be written as:

\[ p_{12} + p_{12}^{(0)} = p_{21}, \]
\[
\frac{1}{\rho_1} \frac{\partial p_{12}}{\partial n_{12}} + \frac{1}{\rho_1} \frac{\partial p_{12}^{(0)}}{\partial n_{12}} = -\frac{1}{\rho_2} \frac{\partial p_{21}}{\partial n_{21}},
\]

and at other interfaces \((m = 2, ..., \ M - 1)\) as:

\[ p_{m2} = p_{(m+1)1}, \]
\[
\frac{1}{\rho_m} \frac{\partial p_{m2}}{\partial n_{m2}} = -\frac{1}{\rho_{m+1}} \frac{\partial p_{(m+1)1}}{\partial n_{(m+1)1}},
\]

The system of equations B-1 and B-2 can be written in the matrix form:

\[ \mathbf{p} = \mathbf{U} \mathbf{p} + \mathbf{U} \mathbf{p}^{(0)} - \mathbf{p}^{(0)}, \]

where the interface matrix is

\[
\mathbf{U} = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & \ldots & 1 & 0 \\
\end{pmatrix},
\]

the vector of boundary values of the source wavefield is
\[
p(0) = \begin{pmatrix}
p_{12}^{(0)} \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \
\end{pmatrix},
\]  
\( (B-5) \)

and the vector \( p \) is defined in A-14. Traditionally, the system of matrix equations B-3 is used to solve the boundary value problem. Because matrix B-4 does not account for the existence of waves propagating in the vicinity of interfaces, this approach is not capable of reproducing a complete set of reflection and transmission effects at interfaces.

Klem-Musatov et al. (2004) and later Aizenberg et al. (2005) showed that it is possible to account for the existence of the propagating waves along interfaces, if the interface conditions are represented in the form of the reflection-transmission transform. This transform was written for a single reflector. Here we extend the method to several reflecting interfaces. We start with rewriting the interface conditions B-1 and B-2 in the form of the reflection-transmission transform, following the work of Klem-Musatov et al. (2004): For \( m = 1 \):

\[
p_{12} + p_{12}^{(0)} = R_{12;12}(p_{12} + p_{12}^{(0)}) + T_{12;21}p_{21};
\]

\( (B-6) \)

for \( m = 2 \):

\[
p_{21} = R_{21;21}p_{21} + T_{21;12}(p_{12} + p_{12}^{(0)}),
p_{22} = R_{22;22}p_{22} + T_{22;31}p_{31};
\]

\( (B-7) \)

for \( m = 3, ..., M - 1 \):

\[
p_{m1} = R_{m1;m1}p_{m1} + T_{m1;(m-1)2}p_{(m-1)2},
p_{m2} = R_{m2;m2}p_{m2} + T_{m2;(m+1)1}p_{(m+1)1};
\]

\( (B-8) \)

for \( m = M \):

\[
p_{M1} = R_{M1;M1}p_{M1} + T_{M1;(M-1)2}p_{(M-1)2}.
\]

\( (B-9) \)

The operators \( R_{mj;mj} \) and \( T_{mj;mj'} \) (where the pair \( mj' \) equals either \((m-1)2\) or \((m+1)1\)) are the so-called “reflection and transmission” operators represented by the formulas:
\[
R_{mj; mj} = R_{mj; mj}(s_{mj}, s_{mj}) = F^{-1}(s_{mj}, q) \hat{R}_{mj; mj}(q) F(q, s_{mj}),
\]
\[
T_{mj; m'j'} = T_{mj; m'j'}(s_{mj}, s_{m'j'}) = F^{-1}(s_{mj}, q) \hat{T}_{mj; m'j'}(q) F(q, s_{m'j'}). 
\] (B-10)

Here \(F(q, s_{mj})\) is the double Fourier transform from the interface \(S_{mj}\) for the plane of slowness components \(q = (q_1, q_2)\), and \(F^{-1}(s_{mj}, q)\) is the inverse Fourier transform for the plane of slowness components to the referent point \(s_{mj}\). If the interface contains edges, then the double Fourier transform acts over the smooth part of the corresponding interface. Formulas B-10 can be considered as double convolutions over the curved interface with the kernels \(F^{-1}(s_{mj}, q) \hat{R}_{mj; mj}(q)\) and \(F^{-1}(s_{mj}, q) \hat{T}_{mj; m'j'}(q)\).

The spectra \(\hat{R}_{mj; mj}(q)\) and \(\hat{T}_{mj; m'j'}(q)\) are the reflection and transmission coefficients (PWRC and PWTC) for plane waves. They depend on the slowness components and acoustic properties at the reflection or transmission point (Aki and Richards, 2002),

\[
\hat{T}_{mj; m'j'} = \frac{2 \rho^{-1}_{m} \sqrt{c_{m}^{2} - q^{2}}}{\rho_{m}^{-1} \sqrt{c_{m}^{2} - q^{2}} + \rho_{m'}^{-1} \sqrt{c_{m'}^{2} - q^{2}}},
\]
\[
\hat{R}_{mj; mj} = 1 - \hat{T}_{mj; m'j'},
\] (B-11)

where \(q = \sqrt{q_1^2 + q_2^2}\) is the tangential to the interface component of the slowness vector. Note that in the case of horizontal plane interfaces the equations B-6-B-9 simplify to the form proposed by Berkhout (1987).

Relations B-6-B-9 can be rewritten in the matrix form:

\[
p = Tp + p^{(1)} - p^{(0)},
\] (B-12)

where the matrix transmission operator is:

\[
T = \begin{pmatrix}
R_{12; 12} & T_{12; 21} & 0 & 0 & \ldots & \ldots & 0 & 0 \\
T_{21; 12} & R_{21; 21} & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & R_{22; 22} & T_{22; 31} & \ldots & \ldots & 0 & 0 \\
0 & 0 & T_{31; 22} & R_{31; 31} & \ldots & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \ldots & R_{(M-1)2; (M-1)2} & T_{(M-1)2; M1} \\
0 & 0 & 0 & 0 & \ldots & \ldots & T_{M1; (M-1)2} & R_{M1; M1}
\end{pmatrix},
\] (B-13)

and the vector of boundary values of single reflected and transmitted wavefield is:
\[
\mathbf{p}^{(1)} = \mathbf{Tp}^{(0)} = \begin{pmatrix}
R_{12,12} p_{12}^{(0)} \\
T_{21,12} p_{12}^{(0)} \\
0 \\
\vdots \\
0 \\
0 \\
0
\end{pmatrix}.
\]  
(B-14)

One might notice that the reflection operators are distributed along the main diagonal of the matrix \(\mathbf{T}\), and the transmission operators are placed at the two neighboring diagonals.
Finding the unknown values $p_{mj}$ in operator representations A-9 can be done using methods for solving the boundary value problem. In order to obtain an analytic representation of the solution, one may use a direct boundary integral method (see for details Costabel and Stephan (1985) and Fokkema and van den Berg (1993)). The conventional approach consists in generation of the system of boundary integral equations, which combines interface conditions B-3 with limiting integral relationships A-13. This augmented system of vector equations can be written in our notation as

$$p = Pp,$$

$$p = Up + Up^{(0)} - p^{(0)}. \quad \text{(C-1)}$$

Although the augmented system can be reduced to one vector equation in various ways, a unique analytic solution of this equation can be obtained by the Neumann iterative technique. It is known that the technique generates an infinite series (Fokkema and van den Berg, 1993). None of the terms of this series possess a physical meaning, because the matrix does not reproduce the reflection and transmission effects at interfaces. As a consequence, particular terms cannot therefore be associated with a wave event, which would be observed in reality.

Our ultimate goal is to generate a Neumann-type series in which each term could be associated with the corresponding multiple reflected and transmitted event. This can be achieved by replacing the matrix-vector equation B-3 with the vector transmission equation B-12, following the results of (Klem-Musatov et al., 2004; Ayzenberg et al., 2007) for a single interface. The modified direct integral method gives a new representation of the solution, which allows to describe and model multiply reflected and transmitted wavefields according to their wavecodes. Under this requirement, a wavefield should evidently be written as a composition of propagation in layers and reflection and transmission at interfaces, as it is often done when modeling using asymptotic ray theory (Červený, 2001) or the generalized ray method introduced by Kennett (1983). We also attempt to avoid particular problems arising within the ray-theoretical approach, for instance the description of the wavefield in the vicinity of caustics and shadow zones. Below we introduce a modified system of vector boundary integral equations, which provides both smooth continuation of the wavefield inside layers, as well as accurate description of the reflection and transmission phenomena.
These results generalize the case of one interface between two halfspaces considered by Ayzenberg et al. (2007).

To generate the desired interference form of the solution, we replace matrix-vector equation B-3 in system C-1 with the vector transmission equation B-12. Then we obtain the modified augmented system of vector boundary integral equations:

\[ \mathbf{p} = \mathbf{Pp}, \]
\[ \mathbf{p} = \mathbf{Tp} + \mathbf{p}^{(1)} - \mathbf{p}^{(0)}. \]  

(C-2)

We substitute the first propagation equation into the right-hand side of the second transmission equation and obtain the modified vector boundary integral equation

\[ \mathbf{p} = \mathbf{Lp} + \mathbf{p}^{(1)} - \mathbf{p}^{(0)}, \]  

(C-3)

with the matrix operator

\[ \mathbf{L} = \mathbf{TP}. \]  

(C-4)

Because the matrix operator \( \mathbf{L} \) describes sequential action of the transmission operator at all interfaces on the matrix propagator in all layers, we name it the “transmission-propagation operator”.

To solve vector equation C-3, we introduce, so far formally, the vectors of boundary values of the \( n \)-ply reflected and transmitted wavefields by the formula:

\[ \mathbf{p}^{(n)} = \mathbf{Lp}^{(n-1)}, \quad n = 2, 3, \ldots \]  

(C-5)

Adding sequentially the quantity \( \mathbf{p}^{(n)} - \mathbf{Lp}^{(n-1)} = 0 \) to the right-hand side of equation C-3, we obtain:

\[ \mathbf{p} - \sum_{n=1}^{N} \mathbf{p}^{(n)} = \mathbf{L} \left[ \mathbf{p} - \sum_{n=1}^{N} \mathbf{p}^{(n)} \right] + \mathbf{p}^{N+1}, \]  

(C-6)

where \( \mathbf{p}^{N+1} = \mathbf{p}^{N+1} - \mathbf{p}^{(0)} \) is the remainder term. If the remainder term tends to zero when \( N \to \infty \), equation C-6 has a unique trivial solution \( \mathbf{p} - \sum_{n=1}^{\infty} \mathbf{p}^{(n)} = 0 \). Then we obtain the vector of boundary values of the scattered wavefields in the interference form:

\[ \mathbf{p} = \sum_{n=1}^{\infty} \mathbf{p}^{(n)}. \]  

(C-7)
We leave the proof of equality \( \lim_{N \to \infty} \tilde{p}^{(N+1)} \to 0 \) outside the scope of this article because of its complexity.

We will now concentrate on equation C-6. In practice we do not need to consider the large values of \( N \), because the remainder term does not contribute within a chosen seismic time window. The term \( p^{(0)} \) does not contribute to the scattered wavefield, because the Kirchhoff integral over the source wavefield is identically equal to zero. Therefore we can reduce C-7 to a partial sum:

\[
\mathbf{p} = \sum_{n=1}^{N} \mathbf{p}^{(n)}. \tag{C-8}
\]

Consider the explicit form of the terms in C-8. The first vector of this sum represented by formula B-14 describes single reflection and transmission. From this formula it follows that its first component denotes the singly reflected wavefield at the interface \( S_{12} \), and its second component denotes the singly transmitted wavefield at the interface \( S_{21} \) (see Figure 2). The explicit form of operator C-4 is

\[
\mathbf{L} = \begin{pmatrix}
R_{12,12}P_{12,12} & T_{12,21}P_{21,21} & T_{12,21}P_{21,22} & 0 & 0 & \ldots \\
T_{21,12}P_{12,12} & R_{21,21}P_{21,21} & R_{21,21}P_{21,22} & 0 & 0 & \ldots \\
0 & R_{22,22}P_{22,22} & T_{22,31}P_{31,31} & T_{22,31}P_{31,32} & \ldots \\
0 & 0 & T_{31,22}P_{22,22} & R_{31,31}P_{31,31} & R_{31,31}P_{31,32} & \ldots \\
0 & 0 & 0 & R_{32,32}P_{32,31} & R_{32,32}P_{32,32} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}. \tag{C-9}
\]

Substituting C-9 and \( \mathbf{p}^{(1)} \) from formula B-14 to C-5, we obtain the second vector from C-8:

\[
\mathbf{p}^{(2)} = \mathbf{L} \mathbf{p}^{(1)} = \begin{pmatrix}
p_{12}^{(2)} \\
p_{21}^{(2)} \\
p_{22}^{(2)} \\
p_{31}^{(2)} \\
0 \\
\ldots
\end{pmatrix} = \begin{pmatrix}
R_{12,12}P_{12,12}p_{12}^{(1)} + T_{12,21}P_{21,21}p_{21}^{(1)} \\
T_{21,12}P_{12,12}p_{12}^{(1)} + R_{21,21}P_{21,21}p_{21}^{(1)} \\
R_{22,22}P_{22,22}p_{22}^{(1)} \\
T_{31,22}P_{22,22}p_{22}^{(1)} \\
0 \\
\ldots
\end{pmatrix}. \tag{C-10}
\]

Vector C-10 describes double reflection and transmission. Its third component represents transmission at \( S_{21} \), propagation through layer \( D_2 \), and reflection at \( S_{22} \).
The fourth component represents transmission at $S_{21}$, propagation through layer $D_2$, and transmission at $S_{31}$. The first and second components describe the process of repeated scattering at the upper interface $S_{12}$. The repeated scattering at relatively simple interfaces can be quite weak.

We can thus continue the process of generation of higher-order terms $p^{(3)}, p^{(4)}, \ldots$. Each next term contains more components than the previous one. This corresponds to a branching system of wavecodes generating various sequences of reflections and transmissions. The example illustrates the fact that analytical representation C-8 extends the generalized ray method proposed by Kennett (1983) for arbitrary interface geometry. Therefore sum C-8 represents the boundary values in operator representations A-9 as the branching sequence of successive reflections and transmissions. Each term $p^{(n)}$ of sum C-8 is a physical event corresponding to the $n$-ply reflected and transmitted wavefield coming from a particular reflecting interface in accordance to its own wavecode. This provides the possibility to describe and evaluate the scattered wavefield as a set of multiply scattered wavefields, which can be identified with the interfaces generating them.

The four-diagonal matrix operator C-9 can be represented as a blocky matrix:

\[
L = \begin{pmatrix}
L_{11} & L_{12} & 0 & 0 & \cdots \\
L_{21} & L_{22} & L_{23} & 0 & \cdots \\
0 & L_{32} & L_{33} & L_{34} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

where the diagonal matrices are

\[
L_{mm} = \begin{pmatrix}
R_{m_2,m_2}P_{m_2,m_2} & T_{m_2,(m+1)_1}P_{(m+1)_1,1}P_{(m+1)_1,(m+1)_1} \\
T_{(m+1)_1,m_2}P_{m_2,m_2} & R_{(m+1)_1,1}P_{(m+1)_1,1}P_{(m+1)_1,(m+1)_1}
\end{pmatrix},
\]

\[
m = 1, 2, \ldots,
\]

the off-diagonal matrices are

\[
L_{(m-1)m} = \begin{pmatrix}
T_{(m-1)_2,m_1}P_{m_1,m_2} & 0 \\
R_{m_1,m_1}P_{m_1,m_2} & 0
\end{pmatrix},
\]

\[
L_{m(m-1)} = \begin{pmatrix}
0 & R_{m_2,m_2}P_{m_2,m_1} \\
0 & T_{(m+1)_1,m_2}P_{m_2,m_1}
\end{pmatrix},
\]

\[
m = 2, 3, \ldots,
\]

and $0$ is the null-matrix. Diagonal matrix operators C-12 contain propagators between points of the same interface and describe repeated scattering. For relatively simple
interface geometries, the role of operators C-12 can be negligible, because the repeated scattering is quite weak. The off-diagonal matrices C-13 describe multiply scattered wavefields, because they contain propagators between points of neighboring reflectors.

In the following, we neglect repeated scattering and concentrate on matrix operators C-13 acting inside layers. Each of them contains four layer transmission-propagation operators: 

\[
L_{(m-1)2,m2} = T_{(m-1)2,m1} P_{m1,m2}, \quad L_{m1,m2} = R_{m1,m1} P_{m1,m2};
\]

\[
L_{m2,m1} = R_{m2,m2} P_{m2,m1} \quad \text{and} \quad L_{(m+1)1,m1} = T_{(m+1)1,m2} P_{m2,m1}.
\]
APPENDIX D
APPROXIMATION OF TRANSMISSION-PROPAGATION OPERATORS

Direct implementation of the layer transmission-propagation operators from C-13 faces the problem of growing mathematical operations proportional to the order $n$ of these operators when computing $p(n)$ in sum C-8. Practically this means exponential increase in the number of operations with increasing order of the evaluated wavefield. Therefore we need to find an approximation of the layer transmission-propagation operators, which would allow an essential decrease in the computational time.

We observe that operators $L_{(m-1)2,m2}$, $L_{m1,m2}$, $L_{m2,m1}$ and $L_{(m+1)1,m1}$ from C-13 have the form $L = RP$ or $L = TP$. These operators decompose the boundary data, which are generally represented by numerical values, into analytical solutions of the acoustic wave equation. Indeed, the operator $P$ decomposes arbitrary wavefield at the “radiating” interface into the superposition of elementary waves (spherical waves, in the particular case of homogeneous media), which diverge from virtual sources towards the “recording” interface (Červený, 2001). The operator $R$ or $T$ reflects or transmits each elementary wave separately (Ayzenberg et al., 2007). The decomposition into elementary waves allows us to search for useful approximations of the operators $L = RP$ and $L = TP$. In this Appendix, we propose algorithmic approximations to both operators $P$ and $R$ and $T$, following the results of Ayzenberg et al. (2007).

We start off with approximating $P$. From A-6 it follows that A-3 describes a solenoidal wavefield, and has therefore a vector potential $w(x, y)$ in $D \setminus B_x$ such that

$$v(x, y) = \nabla_y \times w(x, y), \quad x \neq y. \quad (D-1)$$

The vector potential $w(x, y)$ can be defines to the gradient of a scalar function. Because $x \neq s$ when $y \to s$, we obtain a similar relationship for the trace $w(x, s)$ of $w(x, y)$ on $S$:

$$v(x, s) = \nabla_s \times w(x, s). \quad (D-2)$$

We represented the spatial wavefield through surface integral A-7 by using the divergence theorem, which allowed us to reduce the dimensionality of the problem
from third to second. Surface integrals like $A-7$ are routinely evaluated using approximate high-frequency methods (Červený, 2001; Schleicher et al., 2001). However, these methods require asymptotic description not only of the kernels $g(x, s)$ and $\frac{1}{\rho(s)}\nabla_s g(x, s)$, but also of the boundary data $p(s)$ and $\frac{1}{\rho(s)}\nabla p(s)$. The kernels do not cause computational difficulties, because they may be analytically represented by elementary waves from point sources. The boundary data represent the interference wavefield at the interface, which often can only be obtained numerically. This fact may cause difficulties when explicitly computing surface integrals like $A-7$. Therefore we are interested in such methods which allow explicit evaluation of the integral without having an analytic representation of the boundary data.

One of the methods is to make use of Stokes’ theorem, which connects surface and contour integrals. Thus we can reduce the dimensionality of the problem to the first. Using property D-2 of the integrand $A-8$, we apply Stoke’s theorem to integral $A-7$:

$$p(x) = \int \int_S \mathbf{n}(s) \cdot (\nabla \times \mathbf{w}(x, s)) dS(s) = \oint_E \mathbf{e}(\bar{s}) \cdot \mathbf{w}(x, \bar{s})dE(\bar{s}), \quad (D-3)$$

where $\mathbf{w}(x, s)$ is a vector potential defined in the vicinity of the surface $S$, $E$ is the contour surrounding the area of differentiability of $\mathbf{w}(x, s)$ on $S$, $\bar{s}$ is the radius-vector of points on $E$, $\mathbf{e}(\bar{s})$ is the unit vector tangential to $E$, and $dE(\bar{s})$ is the differential of the arc of $E$. The vector potential has to be defined in a thin layer surrounding $S$ in order to make the rotor operation valid.

It is difficult to find the explicit form of the potential $\mathbf{w}(x, s)$ in practice. It is known that this vector can be obtained for the simple problem: a spherical wave impinging on a smooth interface $S$ in a homogeneous medium (Baker and Copson, 1953; Hönl et al., 1961; Skudzyk, 1971). In this problem, the vector $\mathbf{w}(x, s)$ has a singularity at point $s_0(x)$ of $S$, which is defined as a stationary phase point depending on $x$. Stokes’ theorem requires that such points be excluded from the surface of integration. In the general case, the vector $\mathbf{w}(x, s)$ is unknown. However, it is evident that there will be a set of stationary points. Some of them will be isolated, and some will form a curve of an arbitrary shape (Aizenberg et al., 1996; Klem-Musatov et al., 2004). Therefore Stokes’ theorem will reduce the contour integration to an integration along the contour which surrounds the set of stationary points. Because the shape of this contour is difficult to define, application of Stokes’ theorem to the whole integral is inappropriate.

To avoid the problem, we split the surface $S$ into small elements $\Delta S$, such that $S = \sum \Delta S$. The splitting can be, in particular, realized using the Chebyshev net $s = (s_1, s_2)$, which forms rhombic elements on $S$. If the whole surface cannot be
covered by a single net, it can always be covered by several of them. Thus integral A-7 takes the form:

\[ p(x) = \sum_{\cup \Delta S} \Delta p(x), \]  

(D-4)

where the contribution of a particular element is

\[ \Delta p(x) = \int \int_{\Delta S} n(s) \cdot v(x, s) dS(s), \]  

(D-5)

the differential of the area is

\[ dS(s) = ds_1 ds_2 \sin \gamma_{12}(s), \]  

(D-6)

d\(s_1\) and d\(s_2\) are the differentials along the coordinate lines, and \(\gamma_{12}(s)\) is the angle between the coordinate lines.

We will now consider the properties of the contribution from an arbitrary element \(\Delta S\). To use Stokes’ theorem for calculation of contributions D-5, we need to find an explicit form of the potential \(w(x, s)\) from equation D-2. Because the vector \(v(x, s)\) contains the function \(g(x, s)\), we need an analytical description of the feasible fundamental solution. It is not known for a general inhomogeneous medium, and therefore we assume that we can use its asymptotic representation:

\[ g(x, y) = a_g(x, y) e^{i\omega \tau_g(x, y)}. \]

We choose a point of the element such that it belongs to the joint between two arbitrary edges (say, having numbers 1 and 2). Denote this point as \(s_{12}\). Consider a ray \(G(x, y)\) between points \(x\) and \(y\). Introduce a function:

\[ R(x, y) = c(s_{12}) \tau_g(x, y) \int_{G(x,y)} \frac{c(s_{12})}{c(z)} dl(z), \]  

(D-7)

where \(z\) is an arbitrary point along the ray, and \(dl(z)\) is the differential of the arc along the ray. This function accurately enough represents the geodesic distance along the ray in a small vicinity of point \(s_{12}\). In a homogeneous medium, the geodesic distance becomes conventional distance between points \(x\) and \(y\). In the case of caustics, there might be a set of rays \(\{G_1(x, y), G_2(x, y), \ldots\}\), such that a set of geodesic distances \(\{R_1(x, y), R_2(x, y), \ldots\}\) should be considered.

We represent the fundamental solution as

\[ g(x, y) = \alpha(x, y) \beta(x, s_{12}) \tilde{g}(x, y), \]  

(D-8)

where the apparent fundamental solution
$$\overline{g}(x, y) = \frac{\rho(s_{12})}{4\pi R(x, y)} e^{i\frac{\pi}{2} R(x, y)}$$ (D-9)

is a spherical wave in the vicinity of point $s_{12}$, and the coefficients are

$$\alpha(x, y) = \frac{4\pi R(x, y) a_g(x, y)}{\rho(s_{12})}, \quad \beta(x, y) = e^{-i\frac{\pi}{2} k_g(x, y)},$$ (D-10)

where $k_g(x, y)$ is the KMAH-index along the ray $G(x, y)$.

Substituting D-8-D-10 into A-3, we obtain:

$$v(x, y) = \alpha^2(x, y) \nabla \overline{g}(x, y),$$ (D-11)

where

$$\nabla \overline{g}(x, y) = \frac{1}{\rho(y)} \nabla_y \overline{g}(x, y) \overline{p}(y) - \overline{g}(x, y) \frac{1}{\rho(y)} \nabla \overline{p}(y),$$ (D-12)

and

$$\overline{p}(y) = \frac{\alpha(x, y)}{\beta(x, y)} p(y).$$ (D-13)

Vector D-12 contains the boundary data of vector D-13 and its gradient. We are looking for its description when the element size is small, and the range of frequencies lies within a typical seismic band. For the estimation of the linear size of an element, we use $\Delta L_{\text{max}} = 2 \Delta L$. To estimate the minimal wavelength, we use half of the dominant wavelength: $\lambda_{\text{min}} = \lambda_d/2$. The wavefield in the vicinity of $|y - s_{12}| < \Delta L_{\text{max}}$ of the point $s_{12}$ can be represented by the Taylor formula:

$$\overline{p}(y) = \overline{p}(s_{12}) + \nabla \overline{p}(s_{12})(y - s_{12}) + O(|y - s_{12}|^2),$$ (D-14)

where $O(|y - s_{12}|^2)$ is a remainder term. If the elements are chosen to be small enough ($\Delta L_{\text{max}}/\lambda_{\text{min}} < 1$), the remainder term can be neglected. Then representation D-14 describes a locally plane wave:

$$\overline{p}(y) = \overline{p}(s_{12}) e^{i\overline{k}(s_{12}) \cdot (y - s_{12})},$$ (D-15)

where $\overline{p}(s_{12}) = \overline{p}(s_{12}) = \frac{\alpha(x, s_{12})}{\beta(x, s_{12})} p(s_{12})$ is the apparent amplitude, $\overline{k}(s_{12}) = -\nabla \overline{p}(s_{12})/\overline{p}(s_{12})$ is the apparent wave vector. Because the coefficient $\beta(x, y)$ accounts for the KMAH-index and is constant in a small vicinity of the point $s_{12}$, the apparent vector has the form:

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\[ \overline{k}(s_{12}) = \nabla \text{Arg} p(s_{12}) - i [\nabla \ln |p(s_{12})| - \nabla \ln \alpha(x, s_{12})]. \quad (D-16) \]

For frequencies close to the dominant, the real part of the apparent wave vector is of the same order as the wavenumber \( \frac{\omega}{c(s_{12})} \). Therefore it describes the process of propagation of the apparent plane wave. The imaginary part is of a lower order and describes attenuation of the apparent plane wave.

We fix the slowly varying density at point \( s_{12} \) and thus use D-15 to write D-12 as

\[ \nabla (x, s) = \frac{1}{\rho(s_{12})} \nabla_s \tilde{g}(x, s) \tilde{p}(s) - \tilde{g}(x, s) \frac{1}{\rho(s_{12})} \nabla \tilde{p}(s). \quad (D-17) \]

Substitution of D-17 into D-5 gives

\[ \Delta p(x) = \int \int_{\Delta S} \mathbf{n}(s) \cdot \mathbf{v}(x, s) dS(s) = \int \int_{\Delta \overline{S}} \mathbf{n}(s) \cdot \overline{\mathbf{v}}(x, s) d\overline{S}(s), \quad (D-18) \]

with the apparent differential

\[ d\overline{S}(s) = \alpha^2(x, s) dS(s) \simeq \alpha^2(x, s_{12}) dS(s). \quad (D-19) \]

It follows from D-18 that the contribution of the actual element \( \Delta S \) is equal to the contribution of the apparent element \( \Delta \overline{S} \) at point \( x \). In the presence of caustics, the contribution is equal to the sum of contributions from the set of apparent elements \( \{\Delta \overline{S}_1, \Delta \overline{S}_2, \ldots\} \), which corresponds to the set of rays \( \{G_1(x, y), G_2(x, y), \ldots\} \) and functions \( \{R_1(x, y), R_2(x, y), \ldots\} \). Formulas D-7-D-11, D-15 and D-17-D-19 are the basis for the so-called “refraction transform” of the contribution from the element. A simplified version of this transform for high frequencies was earlier proposed by Aizenberg et al. (1996).

Let us rewrite D-2 with the apparent vector D-17 in the right-hand side:

\[ \nabla_s \times \overline{\mathbf{w}}(x, s) = \overline{\mathbf{v}}(x, s). \quad (D-20) \]

This equation in the vicinity of \( s_{12} \) can be solved in any convenient way, which is applicable for the boundary values of the plane wave D-15 and apparent fundamental solution D-9. Using D-20, we apply Stokes’ theorem to integral D-18 and obtain:

\[ \Delta p(x) = \int \int_{\Delta \overline{S}} \mathbf{n}(s) \cdot [\nabla_s \times \overline{\mathbf{w}}(x, s)] d\overline{S}(s) = \sigma_0 \Delta p_0(x) + \sum_{j=1}^{j=4} \Delta p_j(x), \quad (D-21) \]
where the contour integral over the contour $\Delta \mathbf{E}_0$, which bounds the set of singular points of the vector potential $\mathbf{w}(\mathbf{x}, \mathbf{s})$ at the element $\Delta \mathbf{S}$, is

$$\Delta p_0(\mathbf{x}) = \oint_{\Delta \mathbf{E}_0} \mathbf{e}_0(\mathbf{s}) \cdot \mathbf{w}(\mathbf{x}, \tilde{s}) d\mathbf{E}(\mathbf{s}), \quad (D-22)$$

$\sigma_0 = 1$ if there are singular points at the element, and $\sigma_0 = 0$, if there are none; $\Delta p_j(\mathbf{x})(j = 1, ..., 4)$ are the integrals along the regular edges $\Delta \mathbf{E}_j$ of the element $\Delta \mathbf{S}$:

$$\Delta p_j(\mathbf{x}) = \oint_{\Delta \mathbf{E}_j} \mathbf{e}_j(\mathbf{s}) \cdot \mathbf{w}(\mathbf{x}, \tilde{s}) d\mathbf{E}(\mathbf{s}). \quad (D-23)$$

Here we used the following notations: $\mathbf{e}_0(\mathbf{s})$ and $\mathbf{e}_j(\mathbf{s})$ are the unit vectors tangential to the contours $\Delta \mathbf{E}_0$ and edges $\Delta \mathbf{E}_j$, correspondingly. The unit vector $\mathbf{e}_0(\mathbf{s})$ is directed opposite to the vectors $\mathbf{e}_j(\mathbf{s})$.

Contour integrals D-22 and D-23 generalize the Maggi-Rubinowicz integral introduced for spherical waves in homogeneous media (Baker and Copson, 1953; Hönl et al., 1961; Skudrzyk, 1971). It becomes important in the following part of this Appendix that application of the refraction transform reduces the contribution of an element to the sum of contour integral whose form is indistinguishable from the classical Maggi-Rubinowicz integrals. It is necessary to fulfill two requirements: apply the explicit form of the vector potential for the Maggi-Rubinowicz integral; and use the parameters of the apparent plane wave D-15 instead of the parameters of the spherical wave along the contours $\Delta \mathbf{E}_0$ and $\Delta \mathbf{E}_j$ while assuming that the center of curvature is infinitely far.

Because the right-hand side of equation D-20 is approximate, even its rigorous solution will not be exact. Therefore application of the Maggi-Rubinowicz integral to evaluation of contour integrals D-22 and D-23 with an approximate $\mathbf{w}(\mathbf{x}, \tilde{s})$ does not make sense. Hence we solve equation D-20 in an approximate way. We assume that the parameter $\Delta L_{\text{max}}/\lambda_{\text{min}}$ is small, and the parameter $\frac{\omega}{c(\mathbf{s}_{12})} R(\mathbf{x}, \mathbf{s}_{12})$ is large. To preserve the quality of evaluation of the contour integrals, fulfillment of the two conditions is necessary: the normal to $\Delta \mathbf{S}$ component of the vector $\mathbf{v}(\mathbf{x}, \mathbf{s})$ and the tangential to the edges of $\Delta \mathbf{S}$ component of the vector $\mathbf{w}(\mathbf{x}, \mathbf{s})$ have to be approximated with a high accuracy. To fulfill the first condition, we multiply equation D-20 with the normal $\mathbf{n}(\mathbf{s})$ and solve the equation

$$\mathbf{n}(\mathbf{s}) \cdot [\nabla_s \times \mathbf{w}(\mathbf{x}, \mathbf{s})] = \mathbf{n}(\mathbf{s}) \cdot \nabla(\mathbf{x}, \mathbf{s}). \quad (D-24)$$

If we substitute D-9 and D-15 to the right-hand side of D-24 and omit the terms of lower orders than $k_{12}$, we obtain:
\[ \mathbf{n}(s) \cdot [\nabla_s \times \mathbf{w}(x, s)] = -\frac{ik_{12}}{2\pi} h(x, s)\mathbf{p}(s_{12}) e^{ik_0 l(x, s)}, \]  
(D-25)

where

\[ l(x, s) = R(x, s) + \frac{k}{k_{12}} \cdot (s - s_{12}), \]  
(D-26)

\[ k_{12} = \frac{\omega}{c(s_{12})}, \quad h(x, s) = \frac{\cos \theta(x, s) + \cos \theta(s)}{2R(x, s)}, \quad \cos \theta(s) = \mathbf{n}(s) \cdot \frac{k}{k_{12}}, \quad \cos \theta(x, s) = -\mathbf{n}(s) \cdot \nabla_s R(x, s). \]  

To fulfill the second condition, we introduce a vector \( \mathbf{r}(s) \), which is tangential to the surface of the element and represent the solution of D-25 as the vector potential:

\[ \mathbf{w}(x, s) = -\frac{1}{2\pi} h(x, s)\mathbf{p}(s_{12}) e^{ik_{12} l(x, s)} \frac{\mathbf{r}(s) \times \mathbf{n}(s)}{\mathbf{r}(s) \cdot \nabla_s l(x, s)}. \]  
(D-27)

where \( \nabla_s l(x, s) = \nabla_s R(x, s) + \nabla_s \left[ \frac{k}{k_{12}} \cdot (s - s_{12}) \right] \). In order for vector D-27 to always have a component orthogonal to the unit vectors \( \mathbf{e}_0(s) \) and \( \mathbf{e}_i(s) \), it is enough to define the vector \( \mathbf{r}(s) \) as \( \mathbf{r}(s) = s_1 \mathbf{i}_1 + s_2 \mathbf{i}_2 \), where \( s_1 \) and \( s_2 \) are the orthogonal Chebyshev coordinates at the surface of the element \( \Delta S \) is a small vicinity of the stationary phase point \( s_0 \), and \( \mathbf{i}_1 \) and \( \mathbf{i}_2 \) are the unit vectors of these coordinates. We represent function D-26 as a function of Chebyshev coordinates in a small vicinity of point \( s_0 \) by the Taylor formula:

\[ l(x, s) = l(x, s_0) + \mathbf{r}(s) \cdot \nabla_s l(x, s_0) + \frac{1}{2} [\mathbf{r}(s) \cdot \nabla_s]^2 l(x, s_0) + O(|\mathbf{r}(s)|^3), \]  
(D-28)

where \( O(|\mathbf{r}(s)|^3) \) is the remainder term of the third order. Because the element is small, there is only one stationary phase point \( s_0 \) of function D-26, and it can be defined by the equality \( \nabla_s l(x, s_0) = 0 \). We find that the following relations hold true:

\[ \nabla_s \left[ \frac{k}{k_{12}} \cdot (s - s_{12}) \right] = \frac{k}{k_{12}} = \cos \theta(s) \mathbf{n}(s) + \sin \theta(s) \mathbf{t}(s) \text{ and } \nabla_s R(x, s) = -\cos \theta(x, s) \mathbf{n}(s) - \cos \theta(x, s) \mathbf{t}(s). \]  

Therefore \( \nabla_s l(x, s_0) = \left[ \cos \theta(s_0) - \cos \theta(x, s_0) \right] \mathbf{n}(s) + \left[ \sin \theta(s_0) \mathbf{t}(s) - \sin \theta(x, s_0) \mathbf{t}(s) \right]. \) Because \( \theta(x, s_0) = \theta(s_0) \) and \( \mathbf{t}(s_0) = \mathbf{t}(s_0) \) at point \( s_0 \), we obtain the equation \( \nabla_s l(x, s_0) = 0 \), which defines the position of the stationary point. In the vicinity of point \( s_0 \), representation D-28 and the quantity \( \mathbf{r}(s) \cdot \nabla_s l(x, s) \) take the form:

\[ l(x, s) \approx l(x, s_0) + \frac{1}{2} l_{(2)}(x, s), \quad \mathbf{r}(s) \cdot \nabla_s l(x, s) \approx l_{(2)}(x, s), \]  
(D-29)
where \( l_{(2)}(x, s) = \left[ r(s) \cdot \nabla_s \right]^2 l(x, s_0) = l_{11}(x, s_0)s_1^2 + 2l_{12}(x, s_0)s_1s_2 + l_{22}(x, s_0)s_2^2 \). For the form of function D-26 it follows that point \( s_0 \) is the minimum of the elliptic paraboloid \( l_{(2)}(x, s) \). Therefore \( l_{11}(x, s_0), l_{22}(x, s_0) > 0 \) and \( l_{11}(x, s_0)l_{22}(x, s_0) - l_{12}^2(x, s_0) > 0 \). It is straightforward to show that

\[
  h(x, s) \simeq h(x, s_0) = \frac{\cos \theta(s_0)}{R(x, s_0)} = \sqrt{l_{11}(x, s_0)l_{22}(x, s_0) - l_{12}^2(x, s_0)}.
\]  

(D-30)

Relation D-29 shows that the denominator in D-27 is zero at point \( s_0 \). Therefore vector potential D-27 has singularity at this point.

We substitute D-27 into integral D-22 and account for D-29 and D-30. Thus we obtain the contribution of the stationary point:

\[
  \Delta p_0(x) = \bar{p}(s_{12})e^{ik_{12}(x, s_0)}I_0(x, s_0),
\]

where we introduced an additional quantity

\[
  I_0(x, s_0) = \frac{h(x, s_0)}{2\pi} \int_{\Delta p_0} e^0 \frac{\overline{r(s)} \cdot [\mathbf{n}(s) \times \mathbf{e}_0(s)]}{l_{(2)}(x, s)} d\bar{E}(s).
\]

(D-32)

We evaluate integral D-32 under the assumption that point \( s_0 \) belongs to the element. Let the contour be a circle with the radius \( r = |r(s)| \) spanning around point \( s_0 \). The differential can then be written as \( d\bar{E}(s) = r d\varphi \), and the quadratic form is \( l_{(2)}(x, s) = r^2\cos^2 \varphi [l_{11}(x, s_0) + 2l_{12}(x, s_0) \tan \varphi + l_{22}(x, s_0) \tan^2 \varphi] \). Because the quadratic form has a period of \( \pi \), we can rewrite the contour integral in D-32 as the sum of two standard integrals:

\[
  \int_{\Delta p_0} e^0 \frac{\overline{r(s)} \cdot [\mathbf{n}(s) \times \mathbf{e}_0(s)]}{l_{(2)}(x, s)} d\bar{E}(s) = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d(\tan \varphi)}{l_{11}(x, s_0)l_{22}(x, s_0) + 2l_{12}(x, s_0) \tan \varphi + l_{12}^2(x, s_0) \tan^2 \varphi}.
\]

(D-33)

We use the standard indefinite integral 160.01 from Dwight (1961) for \( l_{11}(x, s_0)l_{22}(x, s_0) - l_{12}^2(x, s_0) > 0 \) and D-30 to obtain that the integral in the right/ hand side of D-33 is equal to \( \frac{\pi}{2} \). Therefore \( I_0(x, s_0) = 1 \), and the contribution of the stationary phase point is D-31 is

\[
  \Delta p_0(x) = \bar{p}(s_{12})e^{ik_{12}(x, s_0)}.
\]

(D-34)

We direct the axis \( s_2 \) of the local Chebyshev coordinates along the edge \( \Delta \bar{E}_j \) and the axis \( s_1 \) orthogonal to the edge at point \( s_0 \). Then we write the differential
\[ dE(\mathbf{s}) = ds_2 \quad \text{and} \quad \mathbf{r}(\mathbf{s}) \cdot [\mathbf{n}(\mathbf{s}) \times \mathbf{e}_j(\mathbf{s})] = \sigma_j s_1, \] where \( s_1 \) is always positive and 
\( \sigma_j = \text{sign}(\mathbf{r}(\mathbf{s}) \cdot [\mathbf{n}(\mathbf{s}) \times \mathbf{e}_j(\mathbf{s})]) \) defines the sign of the function. We substitute vector potential D-27 to integral D-23 and account for D-29 and D-30 to obtain:

\[ \Delta p_j(x) = \bar{p}(s_{12}) e^{i k_{12l}(x,s_0)} \sigma_j I_j(x,s_0), \] (D-35)

with the integrals along the edges:

\[ I_j(x,s_0) = \frac{h(x,s_0)}{2\pi} \int_{s_{2,j1}}^{s_{2,j2}} e^{i k_{12l}(x,s_0)} \frac{s_1}{l_{2l}(x,s_0)} ds_2. \] (D-36)

Each edge corresponds to its own interval of integration \([s_{2,j1}, s_{2,j2}]\). Introduce the sign functions \( \sigma_{jk} = \text{sign}[s_{2,jk} + \frac{l_{12}(x,s_0)}{l_{22}(x,s_0)} s_1] \) and automodel variables:

\[ w_j = \sqrt{\frac{k_{12l_{22}}(x,s_0)}{\pi} h(x,s_0)} |s_1|, \quad u_j = \sqrt{\frac{k_{12l_{22}}(x,s_0)}{\pi}} \left| s_2 + \frac{l_{12}(x,s_0)}{l_{22}(x,s_0)} s_1 \right|, \] (D-37)

which satisfy the conditions \( w_j, u_j > 0 \). In this variables, the function \( l_{2l}(x,\mathbf{s}) \) takes the form \( l_{2l}(x,\mathbf{s}) = \frac{\pi}{k_{12l}} \xi^2 \), where \( \xi^2 = u_j^2 + u_j^2 \). The diffraction point at the edge is at point \( u_j = 0 \), i.e. \( s_2 = -\frac{l_{12}(x,s_0)}{l_{22}(x,s_0)} s_1 \). Taking into account our notations, we reduce integral D-36 to the form:

\[ I_j(x,s_0) = e^{i \frac{\pi}{2} w_j^2} \frac{\sigma_{j2} - \sigma_{j1}}{2} F(w_j) + \sigma_{j1} G(w_j, u_{j1}) - \sigma_{j2} G(w_j, u_{j2}), \] (D-39)

where \( F(w) \) is the classical Fresnel integral (Aizenberg, 1982):

\[ F(w) = \frac{e^{i \frac{\pi}{4} w^2}}{\pi} \int_0^{+\infty} \frac{w}{w^2 + u^2} e^{i \frac{\pi}{2} u^2} du, \] (D-40)

and \( G(w, u) \) is the generalized Fresnel integral (Aizenberg, 1982):

\[ G(w, u) = \frac{e^{i \frac{\pi}{4} w^2}}{2\pi} \int_u^{+\infty} \frac{w}{w^2 + u^2} e^{i \frac{\pi}{2} u^2} du. \] (D-41)
In is evident that $G(w, 0) = 2F(w)$. The generalized Fresnel integral was first derived and studied by Clemmow and Senior (1971).

We substitute contribution D-34 of the contour around the stationary point and contributions D-35 to D-21 and find that total contribution of the element is

$$\Delta p(x) = \bar{p}(s_{12}) \Delta P(x, s_0),$$

(D-42)

where the diffraction coefficient of amplitude attenuation has the form:

$$\Delta P(x, s_0) = e^{ik_{12}(x,s_0)} \left[ \sigma_0 + \sum_{j=1}^{4} \sigma_j \frac{\sigma_j - \sigma_j^1}{2} F(w_j) + \sum_{j=1}^{4} \sigma_j \sum_{k=1}^{2} (-1)^{k-1} \sigma_{jk} G(w_j, u_{jk}) \right].$$

(D-43)

The first term accounts for the energy propagation from the apparent plane wave. The second term describes the process of energy diffusion along the wavefronts of the edge waves diverging from the edges. The third term describes the process of eddy diffusion along the wavefronts of tip waves diverging from the vertices (Aizenberg, 1993a). Because the amount of elements for which $\sigma_0 \neq 0$ and $\sigma_{j2} \neq \sigma_{j1}$ is negligible compared to the total number of elements, we can drop all these elements. Hence the diffraction coefficient of the amplitude attenuation has the form of a beam containing tip waves:

$$\Delta P(x, s_0) = e^{ik_{12}(x,s_0)} \sum_{j=1}^{4} \sigma_j \sum_{k=1}^{2} (-1)^{k-1} \sigma_{jk} G(w_j, u_{jk}).$$

(D-44)

Therefore, when accounting for D-42, formula D-4 takes the form:

$$p(x) = \sum_{\cup \Delta S: \sigma_0 = 0, \sigma_{j2} = \sigma_{j1}} \bar{p}(s_{12}) \Delta P(x, s_0).$$

(D-45)

Formula D-45 together with the tip-wave beam D-44 explains why the approach is called the “tip-wave superposition method”.

Tip-wave beam D-44 contains Fresnel integrals, which are complex functions depending on frequency. This decreases the computational efficiency of its algorithmic implementation. Therefore we are interested in finding a simpler approximation of a tip-wave beam, which still preserves the accuracy. We follow the concept suggested by Aizenberg (1993b) and make use of the specific properties of a tip-wave beam. We assume that point $s_0$ lies close to element $\Delta S$ on its continuation. To obey the conditions $\sigma_0 \neq 0$ and $\sigma_{j2} \neq \sigma_{j1}$, we require that point $s_0$ lies between
the continuations of two neighboring edges closest to this point. Then the point of intersection of the two edges is closer to \( s_0 \) than the other three points of the element. For instance, we may assume that the point of intersection belongs to edges \( \Delta E_1 \) and \( \Delta E_2 \). Then following holds true: \( \sigma_1 = \sigma_2 = \sigma_{21} = \sigma_{22} = \sigma_{31} = \sigma_{32} = 1 \) and \( \sigma_3 = \sigma_4 = \sigma_{11} = \sigma_{12} = \sigma_{41} = \sigma_{42} = -1 \). Formula D-44 can then be written as

\[
\Delta P(x, s_0) = e^{ikz(x,s_0)}\{[G(w_3, u_{32}) - G(w_3, u_{31})] - [G(w_1, u_{11}) - G(w_1, u_{12})]
+ [G(w_4, u_{41}) - G(w_4, u_{42})] - [G(w_2, u_{22}) - G(w_2, u_{21})]\}. 
\] 

(D-46)

Assuming that the second arguments are not too different, we may substitute the difference between the two generalized Fresnel integrals by their differential:

\[
\Delta P(x, s_0) = e^{ikz(x,s_0)}\{[\partial_{u_3}G(w_3, u_{32}) - \partial_{u_1}G(w_1, u_{11})]\Delta u_1
+ [\partial_{u_4}G(w_4, u_{41}) - \partial_{u_2}G(w_2, u_{22})]\Delta u_2\}, 
\] 

where we accounted for equalities \( u_{32} - u_{31} \approx u_{11} - u_{12} = \Delta u_1 \) and \( u_{41} - u_{42} \approx u_{22} - u_{21} = \Delta u_2 \). We can also assume that the small rhombic element has parallel edges. Then the difference of the derivatives over the variables with odd numbers can be approximated by the first-order increment over the variable \( u_2 \) of even order when \( w_2 = \text{const} \). The same applies to the derivatives over the variables with even numbers. Then D-47 can be rewritten as

\[
\Delta P(x, s_0) = e^{ikz(x,s_0)}\partial_{u_3}\partial_{u_2}[G(w_1, u_{11}) + G(w_2, u_{22})]\Delta u_1\Delta u_2. 
\] 

(D-48)

We will now calculate the derivatives of the generalized Fresnel integral:

\[
\partial_{u_j}G(w_j, u_j) = -\frac{e^{i\pi \xi}w_j^2}{2\pi} \frac{w_j}{w_j^2 + u_j^2}. 
\] 

(D-49)

We introduce polar coordinates at the plane \((w_j, u_j)\) by the formulas: \( \xi^2 = w_j^2 + u_j^2 \) and \( \zeta_j = \arctan \frac{w_j}{u_j} \). We notice that \( \zeta_1 + \zeta_2 = \pi - \Omega_{12} \), where \( \Omega_{12} \) is the angle between the axes \( u_1 \) and \( u_2 \). Because \( \partial_{u_j}\zeta_j = \frac{w_j}{w_j^2 + u_j^2} \), we obtain: \( \partial_{u_j}G(w_j, u_j) = -\frac{e^{i\pi \xi}w_j^2}{2\pi} \partial_{u_j}\zeta_j \). Thus \( \partial_{u_{j-2}}\partial_{u_j}G(w_j, u_j) = -\frac{e^{i\pi \xi}w_j^2}{2\pi} [\partial_{u_{j-2}}\partial_{u_j}\zeta_j + i\pi \xi \partial_{u_{j-2}}\xi \partial_{u_j}\zeta_j] \). After summing up the two second derivatives, we obtain:

\[
\partial_{u_1}\partial_{u_2}[G(w_1, u_{11}) + G(w_2, u_{22})] = -\frac{e^{i\pi \xi}w_j^2}{2\pi} \\
\times [\partial_{u_1}\partial_{u_2}(\zeta_1 + \zeta_2) + i\pi \xi (\partial_{u_1}\xi \partial_{u_2}\zeta_1 + \partial_{u_2}\xi \partial_{u_1}\zeta_1)]. 
\]
Using the property \( \zeta_1 + \zeta_2 = \pi - \Omega_{12} \), we obtain the differential form of a tip-wave beam \( D-44 \):

\[
\Delta P(x, s_0) = -\frac{i}{2} e^{ik_{12} R(x, s_{12})} \Delta \Sigma(x, s_0).
\]  

(\text{D-50})

Here we introduced a quantity \( \Delta \Sigma(x, s_0) = \sin \Omega_{12} \Delta u_1 \Delta u_2 \), where we used \( \sin \Omega_{12} = \sin(\zeta_1 + \zeta_2) = \frac{u_1 u_2 + u_2 u_1}{\ell^2} \) and \( R(x, s_{12}) = l(x, s_{12}) = l(x, s_0) + \frac{\pi}{2k_{12}} \xi^2 \), which follows from \( D-29 \) and \( D-37 \). The quantity \( \Delta \Sigma(x, s_0) \) expresses the square of the cross-section of the beam written in automodel variables (Aizenberg, 1993b). It is evident that it depends on the direction of the apparent wavevector \( \vec{k} \) defines by formula \( D-15 \).

Representation \( D-50 \) of the tip-wave beam has the same accuracy as \( D-44 \). However, it contains only elementary functions of frequency, because \( \Delta \Sigma(x, s_0) \) is proportional to frequency.

We pay attention to the fact that tip-wave beam representation \( D-50 \) has a special property: its phase does not depend on the parameters of the boundary data \( \vec{p}(s_{12}) \). The quantity \( \Delta \Sigma(x, s_0) \) weakly depends on the angular displacement of the wavevector \( \vec{k} \) of the apparent plane wave. Therefore we can substitute vector \( \vec{k} \) by \( k_{12} \vec{e}(x, s_{12}) \), where \( \vec{e}(x, s_{12}) = -\nabla_s R(x, s_{12}) \). Point \( s_0 \) simultaneously shifts towards point \( s_{12} \), such that it does not fall onto the element. All the derivations above stay the same, however, the quantity \( \Delta \Sigma(x, s_0) \) approximately becomes \( \Delta \Sigma(x, s_{12}) = \sin \Omega_{12} \Delta u_1 \Delta u_2 \). Because the variables \( u_{12}, u_{21}, w_1, w_2 \) and \( \xi \) tend to zero simultaneously, the arguments take the form \( \Delta u_1 = u_{11} \) and \( \Delta u_2 = u_{22} \), and the quantity \( \sin \Omega_{12} \) becomes indefinite. Its limiting value can be obtained from the geometrical interpretation of the angle \( \Omega_{12} \), which is a dihedral angle of the ray tube from the plane wave emitted by the element along the vector \( \vec{e}(x, s_{12}) \). The dihedral angle \( \Omega_{12} \) is defined by the formula: \( \sin \Omega_{12} = \frac{\vec{e}(x, s_{12}) \cdot \vec{e}}{|\vec{e}(x, s_{12})|} \), where \( e_{j\perp} = e_j - (e_j \cdot e)e \).

Therefore \( \sin \Omega_{12} = \frac{e_{(e_1 \times e_2)}}{|e_{(e_1 \times e_2)|}} \). Because \( e_1 \times e_2 = \sin \gamma(s_{12}) n(s_{12}) \), we obtain: \( \sin \Omega_{12} = \frac{e \cdot n}{|e_1 - (e_1 \cdot e)e_1| \cdot |e_2 - (e_2 \cdot e)e_2|} \sin \gamma(s_{12}) \), where \( \gamma(s_{12}) \) is the known angle between the edges of the element. We observe that the application of the approximation allows us not to neglect the elements which do not obey the conditions \( \sigma_0 \neq 0 \) and \( \sigma_{j2} \neq \sigma_{j1} \). Then \( D-50 \) takes the form

\[
\Delta P(x, s_{12}) = -\frac{i}{2} e^{ik_{12} R(x, s_{12})} \Delta \Sigma(x, s_{12}).
\]  

(\text{D-51})

We use formula \( D-51 \) to reduce \( D-45 \) to the sum of tip-wave beams:

\[
p(x) = \sum_{\cup \Delta S} \vec{p}(s_{12}) \Delta P(x, s_{12}).
\]  

(\text{D-52})

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Representation D-51 has advantages compared to D-50, because it does not depend on the value of the boundary data $\overline{p}(s_{12})$. However, it also has a disadvantage, namely the error in approximations is proportional to the difference $|\Delta \Sigma(x, s_{12}) - \Delta \Sigma(x, s_0)|$, which grows as $|k(x, s_{12}) - \overline{k}|$ grows. This disadvantage is compensated by the fact that the beams with large values of $|k(x, s_{12}) - \overline{k}|$ do not belong to the Fresnel zone of a fixed observation point $x$. Such beams superimpose destructively and do not contribute much to formula D-52. Formulas D-52 and D-51 may be used for numerical implementation. If we consider a set of observation points $\{x(1), x(2), \ldots\}$, then D-52 may be regarded as a matrix product:

$$\begin{pmatrix} p(x(1)) \\ p(x(2)) \\ \vdots \end{pmatrix} = \begin{pmatrix} \Delta P(x(1), s_{12}(1)) & \Delta P(x(1), s_{12}(2)) & \ldots \\
\Delta P(x(2), s_{12}(1)) & \Delta P(x(2), s_{12}(2)) & \ldots \\
\vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \overline{p}(s_{12}(1)) \\ \overline{p}(s_{12}(2)) \\ \vdots \end{pmatrix}. \quad (D-53)$$

The matrix of tip-wave beams $B = [\Delta P(x_{(m)}, s_{12(n)})]$ is a matrix propagator.

Now we will consider approximation of the reflection and transmission operators $R$ and $T$ through effective reflection and transmission coefficients (ERC and ETC) $\chi(\hat{s}_{12})$. Our derivations here are based on the theory provided by Ayzenberg et al. (2007). ERC and ETC depend on two parameters: the incidence angle and relative curvature of the wavefront with respect to the interface. The two parameters can be easily extracted from the tip-wave beams D-51. Therefore a reflected or transmitted tip-wave beam can be written as

$$\Delta L(\hat{s}_{12}) = \chi(\hat{s}_{12}) \Delta P(\hat{s}_{12}, s_{12}). \quad (D-54)$$

Thus, by combining formulas D-53 and D-54, we obtain the operator $L = RP$ or $L = TP$:

$$L(\hat{s}_{12}) = \begin{pmatrix} \Delta L(\hat{s}_{12}(1), s_{12}(1)) & \Delta L(\hat{s}_{12}(1), s_{12}(2)) & \ldots \\
\Delta L(\hat{s}_{12}(2), s_{12}(1)) & \Delta L(\hat{s}_{12}(2), s_{12}(2)) & \ldots \\
\vdots & \vdots & \ddots \end{pmatrix}. \quad (D-55)$$

Formulas D-55 and D-54 are the main formulas for approximating the operators $L = RP$ and $L = TP$. Their application to evaluation of the operators $L_{(m-1),2,m2}$, $L_{m1,m2}$, $L_{m2,m1}$ and $L_{(m+1),1,m1}$ in each layer in the form of four “layer matrices” independently of their further application in modeling. If evaluation of the wavefields corresponding to particular wavecodes is required, the layer matrices need to be multiplied according to the wavecodes of interest. This implies geometrical increase in
operations when modeling wavefields of higher orders. Therefore the introduces ap-
proximation essentially reduces the amount of computational efforts. It is also natural
to call the approach “tip-wave superposition method”.
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<th>Layer</th>
<th>Velocity (m/s)</th>
<th>Density (g/cm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sea water</td>
<td>$c_1 = 1500$</td>
<td>$\rho_1 = 1$</td>
</tr>
<tr>
<td>Sandstone</td>
<td>$c_2 = 2400$</td>
<td>$\rho_2 = 2.1$</td>
</tr>
<tr>
<td>Salt</td>
<td>$c_3 = 4500$</td>
<td>$\rho_3 = 2.6$</td>
</tr>
<tr>
<td>Gas sand</td>
<td>$c_4 = 2000$, $\rho_4 = 2.0$</td>
<td></td>
</tr>
<tr>
<td>Shale</td>
<td>$c_5 = 2400$, $\rho_5 = 2.25$</td>
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