IMPROVED GEOMETRICAL SPREADING APPROXIMATION IN LAYERED
TRANSVERSELY ISOTROPIC MEDIA

by

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September 04, 2008
ABSTRACT

To take into account the geometrical spreading is important for prestack Kirchhoff migration and AVO analysis. It is convenient to define the geometrical spreading correction in terms of offset and traveltime parameters obtained in non-hyperbolic velocity analysis (moveout-based geometrical spreading).

We present a new geometrical spreading approximation for qP-waves in a layered transversely isotropic medium which is very accurate for a large offset spread. In our algorithm we take into account the radiation pattern.
INTRODUCTION

The geometrical spreading approximations for a layered medium are developed for isotropic case (Ursin, 1990). The extension of this approach for a layered transversely isotropic medium was done by Ursin and Hokstad (2003). In their algorithm the effect from radiation pattern was neglected. For azimuthally anisotropic medium the geometrical spreading correction was derived by Xu et al. (2005). Recently, the geometrical spreading algorithm was developed for converted waves in VTI medium (Xu and Tsvankin, 2008). All these methods are so-called moveout-based corrections which means that the parameters for geometrical spreading are derived in velocity analysis of reflection traveltime. The non-hyperbolic traveltime approximation which has been mostly used for that purpose was derived by Alkhalifah and Tsvankin (1995).

We derive new approximation (direct approximation) for the relative geometrical spreading in a layered transversely isotropic medium using three traveltime parameters: two-way vertical traveltime, the normal moveout velocity and the heterogeneity coefficient. The new approximation is based on the acoustic approximation for a single layer VTI medium (Alkhalifah, 1998) with coefficients defined from Taylor expansion and asymptotical behavior at infinite offset. The quasi-acoustic approximation is used to reduce the number of parameters. The independent approximation for radiation pattern has also been introduced. Two numerical tests (single layer and multi-layered VTI models) show that the new approximation is more accurate than the standard approximations based on traveltime approximations (shifted hyperbola approximation and two fractional approximations).
RELATIVE GEOMETRICAL SPREADING

The relative geometrical spreading in a horizontally layered VTI medium is given by (Ursin and Hokstad, 2003)

\[ \mathcal{L} = \mathcal{L}^\parallel \mathcal{L}^\perp \]  

(1)

with in-plane and out-of-plane geometrical spreading factors are given by

\[ \mathcal{L}^\parallel = \left( \cos \alpha' \cos \alpha^s \right)^{1/2} \left( \frac{d^2 T}{dx^2} \right)^{1/2} \]

\[ \mathcal{L}^\perp = \left( \frac{1}{x} \frac{dT}{dx} \right)^{1/2} \]

(2)

where \( \alpha' \) and \( \alpha^s \) are the angles the ray (group velocity vector) makes with the vertical axis at the receiver and source, respectively, \( T(x) \) is traveltime and \( x \) is offset. These expressions are valid for reflected and transmitted qP- and qSV-waves or SH-waves. The factor \( \Omega = \left( \cos \alpha' \cos \alpha^s \right)^{1/2} \) is called the radiation pattern.

RADIATION PATTERN

The terms \( \cos \alpha' \) and \( \cos \alpha^s \) in equation 2 are difficult to compute without performing ray-tracing. Assuming that source and receiver are placed in the same layer at the same depth, and the emitted and received wave-type is qP-wave, we obtain the expression for radiation pattern

\[ \Omega = \cos \alpha = \left[ 1 + \frac{x_i^2}{v_{01}T_i(0)} \right]^{-1/2} \]

(3)

where \( x_i \) being offset for reflected wave propagating in the first layer with group angle \( \alpha \) (Figure 1), \( T_i(0) \) and \( v_{01} \) are the two-way vertical traveltime and vertical velocity in this layer. Next step is to establish the link between the offset \( x_i \) and the overall offset \( x \) taking into account that both rays correspond to the same slowness \( p \). We consider two different
cases: 1) the traveltime approximation is available; 2) the traveltime approximation is not available.

Assuming hyperbolic reflection from the bottom of the first layer, the offset squared is given by (see equation B-4 with heterogeneity coefficient being one)

\[ x_1^2 = v_{NMO1}^2 T_1^2 (0) \frac{p^2 v_{NMO1}^2}{1 - p^2 v_{NMO1}^2}, \] (4)

where \( v_{n1} \) is the normal moveout velocity estimated from the bottom of the first layer and \( p \) is the horizontal slowness or ray parameter which can be defined from overall traveltime-offset dependence

\[ p = \frac{dT}{dx}, \] (5)

where \( x \) and \( T \) being the overall offset and traveltime. Within the frames of traveltime-based approach, the equation 5 gives the link between slowness and overall offset by using the traveltime approximation for \( T(x) \). The traveltime parameters are estimated in velocity analysis.

Substituting equations 4 and 5 into equation 3 results in

\[ \cos \alpha = \left[ 1 + \gamma^2 \frac{v_{NMO1}^2}{1 - v_{NMO1}^2} \left( \frac{dT}{dx} \right)^2 \right]^{-1/2}, \] (6)

where \( \gamma = v_{NMO1}/v_{b1} \) is the ratio between the normal moveout velocity and the vertical velocity for the first layer. Equation 6 gives approximations for the radiation pattern computed from the any given traveltime approximation \( T(x) \) and any wave-type and data acquisition in a layered VTI medium.

Another approach is to define equation similar to equation 4 for overall offset (see equation B-4) taking into account non-hyperbolicity
\[ x^2 = v_{NMO}^2 T^2 (0) \frac{P^2 v_{NMO}^2}{1 - S_2 P^2 v_{NMO}^2}, \quad (7) \]

where \( S_2 \) is the heterogeneity coefficient responsible for nonhyperbolicity of the overall reflection. Despite of the approximation 7 derived from a single layer VTI medium, it has good accuracy for arbitrary multilayered VTI medium. Excluding slowness from equations 4 and 7 result in

\[ x_i^2 = \frac{\gamma_n^2 x_i^2 T_i^2 (0)}{T^2 (0)} \frac{T_i^2 (0)}{T^2 (0)}, \quad (8) \]

where \( \gamma_n = v_{NMO1}/v_{NMO} \) is the ratio between the normal moveout velocity from the first layer and the overall normal moveout velocity. Substituting equation 8 into equation 3 results

\[
\cos \alpha = \left[ 1 + \frac{\gamma_n^2 x^2}{v_{NMO}^2 T^2 (0)} \frac{x^2}{T^2 (0)} \right]^{-1/2} \quad \left[ 1 + \left( S_2 - \gamma_n^2 \right) \frac{x^2}{v_{NMO}^2 T^2 (0)} \right]. \quad (9)
\]

### APPROXIMATIONS DERIVED FROM TRAVELTIME

The standard approach to compute the relative geometrical spreading is to use traveltime approximations (Ursin, 1990; Ursin and Hokstad, 2003).

A continuous fraction approximation is given by (Ursin and Stovas, 2006)

\[ T^2 (x) = T^2 (0) + \frac{x^2}{v_{NMO}^4 T^2 (0)} + \frac{Ax^4}{1 + B \frac{x^2}{v_{NMO}^2 T^2 (0)}} \quad , \quad (10) \]

with \( A = (1 - S_2)/4 \). The second–order heterogeneity coefficient \( S_2 \) (Ursin and Stovas, 2006) can be expressed by an effective parameter \( \eta \) (Siliqi and Bousquie, 2000)

\[ S_2 = 1 + 8 \eta . \quad (11) \]
Alkhalifah and Tsvankin (1995) and Alkhalifah (1997) choose the coefficient $B$ such that to preserve the qP-wave horizontal velocity in a single VTI layer

$$B = 1 + 2\eta = \frac{3 + S_3}{4}.$$  \hfill (12)

Ursin and Stovas (2006) chose the coefficient $B$ to preserve the sixth order coefficient in the Taylor expansion of the traveltime squared. This involves $S_3$, heterogeneous coefficient of third order. For a single VTI layer, the so-called acoustic approximation (Alkhalifah, 1998) gives $S_3 \approx \left( S_3^2 + 1 \right)/2$ (Appendix A) and

$$B = \frac{3S_3 + 1}{4} = 1 + 6\eta.$$ \hfill (13)

The traveltime approximation in 10 with equations 1 and 2 results in the geometrical spreading approximation

$$\left( \frac{1}{x} \frac{dT}{dx} \frac{d^2T}{dx^2} \right)^{-1/2} = \frac{v_{NMO}^2 T^2(x)}{T(0) \sqrt{\left[ 1 + F_1(x) \right] \left[ 1 + F_2(x) \right]}}, \hfill (14)$$

where

$$F_1(x) = \frac{Ax^2}{v_{NMO}^2 T^2(0)} \left[ 1 + B \frac{x^2}{v_{NMO}^2 T^2(0)} \right]^2 \left[ 2 + B \frac{x^2}{v_{NMO}^2 T^2(0)} \right]$$

and

$$F_2(\bar{x}) = \frac{Ax^2}{v_{NMO}^2 T^2(0)} \left[ 1 + B \frac{x^2}{v_{NMO}^2 T^2(0)} \right]^4 \left[ 6 + 3(1 + 3B) \frac{x^2}{v_{NMO}^2 T^2(0)} \right] + 2 \left( A + B + 2B^2 \right) \frac{x^4}{v_{NMO}^6 T^4(0)} - B \left( A + B - B^2 \right) \frac{x^6}{v_{NMO}^6 T^6(0)}.$$ \hfill (15)

To compute the radiation pattern for fractional approximation we have to take the derivative $dT/dx$ from equation 10, substitute it into equation 4 to obtain $x_1$ and then use this result in equation 3. The result is
Another type of traveltime approximation is the shifted hyperbola approach (Malovichko, 1978; de Bazelaire, 1988; Castle, 1988)

\[
T(x) = T(0) + \frac{T(0)}{S_2} \left[ 1 + \frac{S_2 x^2}{v_{NMO}^2 T(0)^2} \right]^{-1},
\]

which for \( S_2 = 1 \) reduces to the standard hyperbolic approximation.

Proceeding as above, equations 1 and 2 give the new approximation

\[
\left( \frac{1}{x} \frac{dT}{dx} \frac{d^2 T}{dx^2} \right)^{-1/2} = v_{NMO}^2 T(0) \left[ 1 + \frac{S_2 x^2}{v_{NMO}^2 T(0)^2} \right]^{-1/2},
\]

with radiation pattern

\[
\Omega_{SH} = \left[ 1 + \gamma^2 \frac{v_{NMO}^2 T(0)^2 \left( 1 + S_2 \frac{x^2}{v_{NMO}^2 T(0)^2} \right)}{1 - \gamma^2 \frac{v_{NMO}^2 T(0)^2 \left( 1 + S_2 \frac{x^2}{v_{NMO}^2 T(0)^2} \right)}{1 + S_2 \frac{x^2}{v_{NMO}^2 T(0)^2}}} \right]^{-1/2}.
\]

**DIRECT APPROXIMATION**

The direct approximation, we propose in this paper, has the form of the rational approximation with parameters defined from a single layer VTI medium in acoustic approximation. The reason we choose this approximation is that it has the same number of parameters that we are going to use. With the help of acoustic approximation we define the heterogeneity coefficient of the third order and the infinite offset asymptotic for the relative
geometrical spreading. For arbitrary multilayered VTI medium these dependences are no
longer valid, but even the marginal dependences improve the accuracy of our approximation.
To define the Taylor series and asymptotic behavior for geometrical spreading, the equations
1 and 2 can be rewritten with respect to slowness
\[
\mathcal{L} = \mathcal{L}^\perp \mathcal{L}^\parallel = \Omega \left( \frac{x}{p} \frac{dx}{dp} \right)^{1/2}
\]
\[
\mathcal{L}^\parallel = \left( \cos \alpha' \cos \alpha' \right) \left( \frac{d^2 t}{dx^2} \right)^{-1/2} = \cos \alpha \left( \frac{dx}{dp} \right)^{1/2}.
\]
(20)
\[
\mathcal{L}^\perp = \left( \frac{x}{p} \right)^{1/2}
\]
For a single VTI layer in acoustic approximation, the second term in the relative geometrical
spreading 20 takes the form (Appendix B)
\[
\left( \frac{x}{p} \frac{dx}{dp} \right)^{1/2} = v_{NMO}^2 T (0) \frac{1 + \frac{1}{2} \left( S_2 - 1 \right) p^2 v_{NMO}^2 + \frac{3}{16} \left( S_2 - 1 \right) \left( S_2 + 3 \right) p^4 v_{NMO}^4}{1 - \frac{1}{4} \left( S_2 - 1 \right) p^2 v_{NMO}^2} \left[ 1 - \frac{1}{4} \left( S_2 + 3 \right) p^2 v_{NMO}^2 \right].
\]
(21)
The Taylor series for this term with respect to normalized offset is given by (see Appendix B)
\[
\left( \frac{x}{p} \frac{dx}{dp} \right)^{1/2} = v_{NMO}^2 T (0) \left[ 1 + \frac{S_2 x^2}{v_{NMO}^2 T^2 (0)} - \frac{9 \left( S_2 - 1 \right) x^4}{16 v_{NMO}^4 T^4 (0)} + \ldots \right].
\]
(22)
We are going to use the fractional approximation of the form
\[
\left( \frac{x}{p} \frac{dx}{dp} \right)^{1/2} \approx v_{NMO}^2 T (0) \left[ 1 + S_2 \frac{x^2}{v_{NMO}^2 T^2 (0)} + \frac{C x^4}{v_{NMO}^4 T^4 (0)} \left( 1 + D \frac{x^2}{v_{NMO}^2 T^2 (0)} \right) \right]
\]
(23)
with coefficient \( C \) defined from the fourth-order coefficient in series 22, and coefficient \( D \)
defined from the asymptotic behavior of geometrical spreading at infinite offset (Appendix B)
\[
\left( \frac{x}{p} \frac{dx}{dp} \right)^{1/2} \sim v_{NMO}^2 T (0) \left[ \frac{\left( S_2 + 3 \right)^{3/2} \left( 3S_2 + 1 \right)}{32} + \frac{2}{\sqrt{S_2 + 3}} \frac{x^2}{v_{NMO}^2 T^2 (0)} \right].
\]
(24)
The first term in 24 corresponds to the asymptotic intercept or the semi-minor axis of asymptotic hyperbola, while the second term in 24 corresponds to the asymptotic slope.

Taking the infinite offset limit from the equation 23 and fitting with the limit 24 results in the following algebraic equation

\[ S_2 + \frac{C}{D} = \frac{2}{\sqrt{S_2 + 3}} \]  (25)

With the help of simple algebraic manipulations, we obtain

\[ C = -\frac{9}{16} (S_2^2 - 1) \]

\[ D = \frac{9(S_2 + 1)\sqrt{S_2 + 3}(S_2\sqrt{S_2 + 3} + 2)}{16(S_2 + 2)^2} \]  (26)

For isotropic and elliptically isotropic homogeneous media \( S_2 = 1 \) and \( C = 0 \) and \( D = 1 \). In case of weak anisotropy, i.e. \( |S_2 - 1| \ll 1 \), the coefficients \( C \) and \( D \) can be given by the first terms in the corresponding Taylor series

\[ C \approx -\frac{9}{8} (S_2 - 1) \]

\[ D \approx 1 + \frac{25}{48} (S_2 - 1) \]  (27)

For the radiation pattern we use approximation 9

\[ \Omega = \left[ 1 + \frac{\gamma^2 n^2 x^2}{v_{NMO}^2 T^2(0)} \right]^{-1/2} \left[ 1 + (S_2 - \gamma^2 n^2) \frac{x^2}{v_{NMO}^2 T^2(0)} \right]^{-1/2} \]  (28)
COMPARISON BETWEEN APPROXIMATIONS
IN A SINGLE LAYER VTI MEDIUM

For a single VTI layer we can compare the traveltime-based approximations 14 and 18 with
the direct approximation 23 for a single VTI layer. The Taylor series of \( \left( \frac{1}{x} \frac{dT}{dx} \frac{d^2T}{dx^2} \right)^{-1/2} \) from
Alkhalifah-Tsvankin traveltime approximation 10 (parameter \( B \) is given in equation 12)
\[
\left[ \frac{1}{x} \frac{dT}{dx} \frac{d^2T}{dx^2} \right]^{-1/2} = v_{NMO}^2 T(0) \left[ 1 + S_2 \frac{x^2}{v_{NMO}^2 T^2(0)} + \frac{9}{16} (S_2 - 1)(S_2 - 3) \frac{x^4}{v_{NMO}^4 T^4(0)} + ... \right]. \tag{29}
\]
The fourth order term in series 29 is \(- \frac{9}{8} (S_2 - 1) \left( 1 - \frac{S_2 - 1}{2} \right)\), while the correct one see Ursin-
Stovas traveltime approximation 10 (parameter \( B \) is given in equation 13) and direct
approximation is \(- \frac{9}{8} (S_2 - 1) \left( 1 + \frac{S_2 - 1}{2} \right)\). The shifted hyperbola traveltime approximation
results in zero value for this term.

At infinite offset the Alkhalifah-Tsvankin approximation (equations 12 and 14) gives
\[
\left[ \frac{1}{x} \frac{dT}{dx} \frac{d^2T}{dx^2} \right]^{-1/2} \sim v_{NMO}^2 T(0) \left[ \frac{16S_2}{\sqrt{5 + 10S_2 + S_2^2}} + \frac{3 + S_2}{\sqrt{5 + 10S_2 + S_2^2}} \frac{x^2}{v_{NMO}^2 T^2(0)} \right]. \tag{30}
\]
By comparing 30 with asymptotic behaviour given in equation 24 one can see that the slopes
of asymptotic are of the same order and intercepts are quite different.

At infinite offset the Ursin-Stovas approximation (equations 13 and 14) results in
\[
\left[ \frac{1}{x} \frac{dT}{dx} \frac{d^2T}{dx^2} \right]^{-1/2} \sim v_{NMO}^2 T(0) \left[ \frac{6S_2}{\sqrt{-3 + 10S_2 + 9S_2^2}} + \frac{1 + 3S_2}{\sqrt{-3 + 10S_2 + 9S_2^2}} \frac{x^2}{v_{NMO}^2 T^2(0)} \right] \tag{31}
\]
with approximately same intercept and slope as the Alkhalifah-Tsvankin approximation.
The shifted hyperbola has identical form in infinite offset asymptotic

\[
\left[ \frac{1}{x} \frac{d T}{d x} \frac{d^2 T}{d x^2} \right]^{1/2} \sim v_{NMO}^2 T(0) \left[ 1 + S_2 \frac{x^2}{v_{NMO}^2 T^2(0)} \right] \\
\sim v_{NMO}^2 T(0) \left[ 1 + \left( 1 + (S_2 - 1) \right) \frac{x^2}{v_{NMO}^2 T^2(0)} \right]
\]  

(32)

with more correct intercept but extremely different slope.

Despite of the Alkhalifah-Tsvankin traveltime approximation is based on the asymptotic for infinite offset in a single layer VTI medium, the geometrical spreading approximation based on the traveltime approximation does not result in correct asymptotic at infinite offset. For other velocity models the comparison of geometrical spreading approximations will be different.

**NUMERICAL EXAMPLES**

To test the approximations for the relative geometrical spreading we use the single layer VTI model with the range of parameter $\eta$ from -0.1 to 0.1 and one multilayered VTI model (Ursin and Hokstad, 2003) with parameters given in Table 1.

We do comparison for the relative geometrical spreading between the approximations derived from the fractional traveltime approximations given in equations 14 and 16 with $B$ defined in equation 12 (Alkhalifah-Tsvankin) and 13 (Stovas-Ursin), the shifted hyperbola approximation (equations 18 and 19) and the direct approximation given in equation 23 and 28 with parameters $C$ and $D$ defined in equation 26.

In Figure 2 we show the contour plots for the relative error in $L$ versus normalized offset and parameter $\eta$ for single VTI layer. One can see that the direct approximation performs extremely well for the entire range of parameter $\eta$. The traveltime-based approximations (from both Alkhalifah-Tsvankin and Stovas-Ursin approximations) result in approximately the same errors. The Stovas-Ursin type approximation is slightly better, while the Alkhalifah-
Tsvankin type approximation is more stable for negative values of \( \eta \). The shifted hyperbola type approximation is the worst and applicable for short offset spread only.

In Figure 3 one can see the contour plots for the relative geometrical spreading in the layered VTI medium plotted versus offset and the reflector number (from 1 to 13). The direct approximation gives the best results despite of the fact that the for the arbitrary layered medium, the relation between \( S_3 \) and \( S_2 \) (used to define parameter \( C \)) and relation between intercept of asymptotic and \( S_2 \) (used to define parameter \( D \)) are different from those in a single VTI layer. The traveltime based approximations (Alkhalifah-Tsvankin and Stovas-Ursin) result in similar error contour plots with contour lines approximately following the constant values for the normalized offset. The traveltime based approximation from shifted hyperbola performs well enough, especially for reflector 10. One can see that for the quality of the relative geometrical spreading approximations non-systematically varies for different reflectors with depth. Therefore, it seems to be impossible to develop approximation which performs uniform for arbitrary layered medium.
CONCLUSION

We derive new approximation for the relative geometrical spreading in a layered transversely isotropic medium. The proposed approximation is using the higher order heterogeneity coefficient and asymptotic behaviour for geometrical spreading derived from a single layer VTI medium. The proposed approximation requires three traveltime parameters to be estimated for a given reflection and two velocities, vertical and normal moveout for the first reflector in the model. The numerical tests performed both for a single VTI layer and a stack of VTI layers show that the new approximation performs better than the common used ones.

ACKNOWLEDGEMENTS

We would like to acknowledge StatoilHydro via the VISTA project and the Norwegian Research Council via the ROSE project for financial support.
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layered, azimuthally anisotropic media: *Geophysics*, 70, D43-D53.

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**Figure 1.** The sketch for definition of the radiation pattern in a multi-layered medium.

**Figure 2.** Contour plots for the relative error in $\mathcal{L}$ versus normalized offset and range of parameter $\eta$ from -0.1 to 0.1 for a single VTI layer. The approximation based on the shifted hyperbola is at top left, the one based on Alkhalifah-Tsvankin approximation is at top right, the one based on Stovas-Ursin approximation is at bottom left and the direct approximation is at the bottom right.

**Figure 3.** Contour plots for the relative error in $\mathcal{L}$ versus offset and reflector number for a multi-layered VTI model. The approximation based on the shifted hyperbola is at top left, the one based on Alkhalifah-Tsvankin approximation is at top right, the one based on Stovas-Ursin approximation is at bottom left and the direct approximation is at the bottom right.

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**Table 1.** Parameters of the layered VTI medium used as a model for the results in Figure 3. The symbols mean: $\Delta z$ is the layer thickness, $\alpha_0$ and $\beta_0$ are vertical velocities for P- and S-waves, respectively, $\varepsilon$ and $\delta$ are anisotropy parameters in Thomsen’s notation.
APPENDIX A

HETEROGENEITY COEFFICIENTS IN ACOUSTIC APPROXIMATION

The heterogeneity coefficients are defined in terms of the Taylor series coefficients for the vertical slowness squared (Ursin and Stovas, 2006)

\[
q^2_p = \frac{1}{\alpha_p^2} \left[ 1 - \rho^2 \alpha_p^2 \left( 1 + \sum_{n=0}^{\infty} a_n \left( \rho \alpha_p \right)^{2n} \right) \right], \quad (A-1)
\]

where coefficients \( a_n \) are given by Thomsen anisotropic parameters \( \alpha, \delta \) and \( \gamma = \beta_0 / \alpha_0 \)

the vertical velocities ratio for S- and P-waves

\[
a_0 = 2\delta \\
a_1 = \frac{2(\alpha - \delta)}{1 - \gamma_0^2} (1 - \gamma_0^2 + 2\delta) \\
a_2 = \frac{4(\alpha - \delta)}{(1 - \gamma_0^2)^2} (1 - \gamma_0^2 + 2\delta) \left[ \alpha - \delta \left( 1 + \gamma_0^2 \right) \right] \\
a_3 = \frac{8(\alpha - \delta)}{(1 - \gamma_0^2)^3} (1 - \gamma_0^2 + 2\delta) \left[ \alpha - \delta \left( 1 + \gamma_0^2 \right) \right]^2 - 2(\alpha - \delta) \gamma_0^2 \left( 1 - \gamma_0^2 + 2\delta \right) \right]
\]

According to Ursin and Stovas (2006), the heterogeneity coefficients are

\[
S_2 = 1 + \frac{4a_1}{(1 + a_0)^2} \\
S_3 = 1 + \frac{4a_1}{(1 + a_0)^2} + \frac{8a_2}{(1 + a_0)^3} \\
S_4 = 1 + \frac{24}{5} \frac{a_1}{(1 + a_0)^2} + \frac{32}{5} \frac{a_2}{(1 + a_0)^3} + \frac{16}{5} \frac{a_1^2 + 4a_3}{(1 + a_0)^4}
\]

In the quasi-acoustic approximation (Alkhalifah, 1998), the heterogeneity coefficients are reduced to
\[ S_2 = 1 + 8\eta \]
\[ S_3 = 1 + 8\eta + 32\eta^2 \]
\[ S_4 = 1 + \frac{48}{5}\eta + \frac{192}{5}\eta^2 + \frac{512}{5}\eta^3 \]  \hspace{1cm} \text{(A-4)}

... where anisotropic parameter $\eta$ is defined by (Alkhalifah and Tsvankin, 1995)

\[ \eta = \frac{\varepsilon - \delta}{1 + 2\delta} \]  \hspace{1cm} \text{(A-5)}

Therefore,

\[ S_3 = 1 + (S_2 - 1) + \frac{1}{2}(S_2 - 1)^2 = \frac{1 + S_2^3}{2} \]
\[ S_4 = 1 + \frac{6}{5}(S_2 - 1) + \frac{3}{5}(S_2 - 1)^2 + \frac{1}{5}(S_2 - 1)^3 = \frac{1 + 3S_2 + S_2^3}{5} \]  \hspace{1cm} \text{(A-6)}

...
APPENDIX B

THE GEOMETRICAL SPREADING FOR A SINGLE VTI LAYER

For a qP-wave in a single VTI layer the offset versus slowness can be written as (Ursin and Stovas, 2006)

\[ x(p) = -z \frac{dq_a(p)}{dp}, \]  

(B-1)

where \( z \) is layer thickness and \( q_a(p) \) is vertical slowness. In quasi-acoustic approximation, the vertical slowness is given by (Alkhalifah, 1998)

\[ q_a(p) = \frac{1}{\alpha_0} \sqrt{\frac{1 - \frac{S_2 + 3}{4} p^2 v_{NMO}^2}{1 - \frac{S_2 - 1}{4} p^2 v_{NMO}^2}}, \]  

(B-2)

Taking derivative from (B-2), substituting it into (B-1) and introducing the vertical traveltime \( T(0) = z/\alpha_0 \) results in

\[ x = \frac{p v_{NMO}^2 T(0)}{\left[1 - \frac{1}{4} (S_2 - 1) p^2 v_{NMO}^2\right]^{3/2} \sqrt{1 - \frac{1}{4} (S_2 + 3) p^2 v_{NMO}^2}}, \]  

(B-3)

Taking a square of (B-3) and expanding the expression in denominator into Taylor series gives an approximation

\[ x^2 \approx \frac{p^2 v_{NMO}^4 T^2(0)}{1 - S_2 p^2 v_{NMO}^2}. \]  

(B-4)

Substituting B-3 into equation 20 results in

\[ \left( \frac{x}{p} \frac{dx}{dp} \right)^{1/2} = v_{NMO}^2 T(0) \sqrt{\frac{1 + \frac{1}{2} (S_2 - 1) p^2 v_{NMO}^2 - \frac{3}{16} (S_2 - 1)(S_2 + 3) p^4 v_{NMO}^4}{\left[1 - \frac{1}{4} (S_2 - 1) p^2 v_{NMO}^2\right]^2 \left[1 - \frac{1}{4} (S_2 + 3) p^2 v_{NMO}^2\right]}}. \]  

(B-5)

The Taylor series for this term with respect to slowness is given by
\[
\left( \frac{x \, dx}{p \, dp} \right)^{1/2} = v_{NMO}^2 T(0) \left[ 1 + S_2 p^2 v_{NMO}^2 + \frac{1}{16} \left( \frac{7S_2^2}{9} + 9 \right)p^4 v_{NMO}^4 + \ldots \right] \quad (B-6)
\]

The Taylor series for the offset squared is

\[
x^2 = p^2 v_{NMO}^4 T^2(0) \left[ 1 + S_2 p^2 v_{NMO}^2 + \frac{1}{8} \left( 5S_2^2 + 3 \right)p^4 v_{NMO}^4 + \ldots \right] \quad (B-7)
\]

Composing of series B-6 and B-7 results in

\[
\left( \frac{x \, dx}{p \, dp} \right)^{1/2} = v_{NMO}^2 T(0) \left[ 1 + \frac{S_2 x^2}{v_{NMO}^2 T^2(0)} - \frac{9}{16} \frac{\left(S_2^2 - 1\right)x^4}{v_{NMO}^4 T^4(0)} + \ldots \right] \quad (B-8)
\]

To compute the gradient for infinite offset asymptotic we take the limit of (B-5)

\[
\lim_{p \to v_{NMO} \sqrt{S_2 + 3}} \frac{1}{2} \left( \frac{x \, dx}{p \, dp} \right)^{1/2} = \frac{2}{T(0) \sqrt{S_2 + 3}} \quad (B-9)
\]
Figure 1. The sketch for definition of the radiation pattern in a multi-layered medium.
Figure 2. Contour plots for the relative error in $\mathcal{L}$ versus normalized offset and range of parameter $\eta$ from -0.1 to 0.1 for a single VTI layer.

The approximation based on the shifted hyperbola is at top left, the one based on Alkhalifah-Tsvankin approximation is at top right, the one based on Stovas-Ursin approximation is at bottom left and the direct approximation is at the bottom right.
Figure 3. Contour plots for the relative error in $\mathcal{L}$ versus offset and reflector number for a multi-layered VTI model. The approximation based on the shifted hyperbola is at top left, the one based on Alkhalifah-Tsvankin approximation is at top right, the one based on Stovas-Ursin approximation is at bottom left and the direct approximation is at the bottom right.
Table 1. Parameters of the layered VTI medium used as a model for the results in Figure 3.

The symbols mean: $\Delta z$ is the layer thickness, $\alpha_0$ and $\beta_0$ are vertical velocities for P- and S-waves, respectively, $\varepsilon$ and $\delta$ are anisotropy parameters in Thomsen’s notation.