Stabilized least-squares imaging condition

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ABSTRACT

The classical deconvolution imaging condition consists of dividing the upgoing wavefield by the downgoing wavefield and summing over all frequencies and sources. The least-squares imaging condition consists of summing the cross-correlation of the upgoing and downgoing wavefields over all frequencies and sources, and dividing the result by the total energy of
the downgoing wavefield. This procedure is more stable than using the classical imaging condition, but it still requires stabilization in zones where the energy of the downgoing wavefield is small. To stabilize the least-squares imaging condition the energy of the down-going wavefield is replaced by its average value computed in a horizontal plane in poorly illuminated regions. Application to the Marmousi and Sigsbee2A data sets shows that the stabilized least-square imaging condition produces better images than the least-squares and cross-correlation imaging conditions.
INTRODUCTION

The classic Claerbout’s (1971) imaging condition is a deconvolution imaging condition where the ratio of the upgoing and downgoing wave fields is used to get a direct estimate of the reflection coefficient. Different stabilization techniques has been proposed to avoid the division by zero or near to zero values of the downgoing wave field (Valenciano and Biondi (2003); Guitton et al. (2006); Vivas and Pestana (2007)).

The reflection coefficient can be estimated by a local least-squares procedure. The resulting imaging condition has previously been given by Shin et al. (2001) and Plessix and Mulder (2004). In Appendix A it is shown that division and least-squares both procedure unbiased estimates of reflectivity, but that the least-squares estimate always has less variance. In fact, a least-squares estimate is a minimum-variance estimate under fairly general conditions (Tarantola (1987)).

A least-squares imaging condition, that uses all the shots simultaneously, computes the migration imaging weight as the summation of energy of the downgoing wave field by frequency and shot, also called the total illumination function (Plessix and Mulder (2004)). Theoretically the least-square imaging condition avoids instabilities, but the finite and nonuniform source and receiver coverage produces instabilities in zones of strong defocused energy (Kiyashchenko et al. (2007)).

It is possible to stabilize the least-squares imaging condition by adding a positive condition constant to the illumination function in order to avoid division by zero or a very small number. This corresponds to a damped least-squares procedure (Lines and Treitel (1984)). By analysing the illumination of the downgoing wave field at different depth extrapolation levels it is possible to identify regions which are poorly illuminated in comparison with the
mean value of the transversal illumination. Vivas and Pestana (2007) proposed a stabilized
deconvolution-type imaging condition by replacing the illumination function by its mean
transverse value wherever the illumination function becomes too small. Here we use the
same approach in the least-squares imaging condition by replacing the illumination function
in the sum over frequency and sources by its transverse average value wherever it becomes
too small.

We show the advantage of this new criterion for seismic imaging applied to the Sigsbee2A
and the Marmousi synthetic data sets. We compare the imaging results obtained by cross-
correlation and least-squares without stabilization with the least-squares imaging condition
using the stabilization criterion.

IMAGING USING UPGOING AND DOWNGOING WAVEFIELDS

Claerbout (1971) introduced seismic migration using upgoing and downgoing wavefields.
The wavefield from the source is downward propagated, and denoted by \( d(x, t) \) where
\( x = (x, y, z) \). The recorded primary reflected wavefield is downward retropropagated, and
denoted by \( u(x, t) \). At a specific depth point a reflector exists where the first arrival of the
dwangoing is time-coincident with an upgoing wave. A signal-and-noise model describing
this is

\[
u(x, t_k) = R(x)d(x, t_k) + n(x, t_k).
\]

(1)

In the frequency domain this equation becomes

\[
U(x, \omega_m) = R(x)D(x, \omega_m) + N(x, \omega_m).
\]

(2)
The migrated image is a crude estimate of the reflectivity

\[ R_D(x) = \frac{1}{M} \sum_{\omega} \frac{U(x, \omega)}{D(x, \omega)} \]  

(3)

where the summation is over \( M \) discrete frequencies.

As shown in Appendix A, a least-squares estimate of the reflectivity in equation (1) or (2) is

\[ R_{LS}(x) = \frac{\sum_k u(x, t_k) d(x, t_k)}{\sum_k d(x, t_k)^2} = \frac{\sum_m U(x, \omega_m) D^*(x, \omega_m)}{\sum_m |D(x, \omega_m)|^2}. \]  

(4)

These expressions are, of course, identical, and they are stable unless the downgoing waveform is zero. This estimation of reflectivity was proposed by Shin et al. (2001)

**DECONVOLUTION AND LEAST-SQUARES IMAGING CONDITION**

The deconvolution imaging condition for shot-profile migration uses the quotient of the downward continued wavefields from the sources at \( x^s, D(x^s; x, \omega) \) divided by the upgoing wavefield \( U(x^s; x, \omega) \) retropropagated from the receivers to obtain a measurement of the image strength. For several shot records, equation (3) is replaced by

\[ R(x) = \frac{1}{N M} \sum_{x^s} \sum_{\omega} \frac{U(x^s; x, \omega)}{D(x^s; x, \omega)}, \]  

(5)

where \( N \) is the number of shots and \( M \) is the number of discrete frequencies in the sum. We may write (5) as

\[ R(x) = \frac{1}{N M} \sum_{x^s} \sum_{\omega} \frac{U(x^s; x, \omega) D^*(x^s; x, \omega)}{|D(x^s; x, \omega)|^2}, \]  

(6)

where \(^*\) denotes complex conjugate.
Equation (6) is unstable for small values of the illumination function or the energy of the downgoing wave field

\[ I(x^s; x, \omega) = |D(x^s; x, \omega)|^2. \]  

To avoid division by zero, the imaging condition can be modified to:

\[ R(x) = \sum_{x^s} \sum_{\omega} \frac{U(x^s; x, \omega) D^*(x^s; x, \omega)}{I(x^s; x, \omega) + V}, \]  

where \( V \) is a constant or a slowly varying function of frequency and space (Claerbout (1971)). A possible choice for the function \( V \) is the average of the energy of the downgoing wave field along frequency multiplied by a damping parameter \( \lambda \) (Valenciano and Biondi (2003))

\[ V(x) = \frac{\lambda}{N,M} \sum_{x^s} \sum_{\omega} I(x^s; x, \omega). \]  

However, at the points where the energy of the downgoing wave is small for all frequencies very noisy images are obtained, independent of the value of \( \lambda \).

Smoothing of the downgoing wave field in the transversal coordinate (Guitton et al. (2006)) is another stabilization criterion used. In this way the zeros in the energy of the downgoing wavefield are filled with the energy of the neighbouring points, and the imaging condition, equation (6), is applied in the following form:

\[ R(x) = \frac{1}{N,M} \sum_{x^s} \sum_{\omega} \frac{U(x^s; x, \omega) D^*(x^s; x, \omega)}{\langle I(x^s; x, \omega) \rangle}, \]  

where \( \langle \cdot \rangle \) represents a spatial smoothing filter. For a long length smoothing filter, the spectral density amplitude of the downgoing wave field is strongly affected and as a consequence the reflector amplitudes are also affected.

In order to avoid division by a small number, an image is often formed by cross-correlating the two wavefields.
\[ R(x) = \sum_{x^s} \sum_{\omega} U(x^s; x, \omega) \ D^*(x^s; x, \omega). \] (11)

This will, however, not give the correct amplitudes.

Instead of using the deconvolution or division imaging condition, one can use the least-squares imaging condition (Shin et al. (2001)). For several shot records, equation (4) is extended to:

\[ R(x) = \frac{\sum_{x^s} \sum_{\omega} U(x^s; x, \omega) \ D^*(x^s; x, \omega)}{\sum_{x^s} \sum_{\omega} I(x^s; x, \omega)} \] (12)

Although, the least-squares imaging condition (12) is more stable than the deconvolution imaging condition in equation (6), some stabilizing is needed in places where the energy of the downgoing wavefield is small. This may be done using a damped least-squares procedure (Lines and Treitel (1984)) which gives

\[ R(x) = \frac{\sum_{x^s} \sum_{\omega} U(x^s; x, \omega) \ D^*(x^s; x, \omega)}{\sum_{x^s} \sum_{\omega} I(x^s; x, \omega) + V} \] (13)

where again \( V \) is a constant or a slowly varying function of space.

Here we use a data-adaptive approach by first computing the average of the downgoing energy at a depth \( z \):

\[ I_{AV}(x^s; z, \omega) = \frac{1}{N_x} \frac{1}{N_y} \sum_x \sum_y I(x^s; x, \omega). \] (14)

Then we replace the downgoing energy flux in equation (12) by the stabilized value
\[ I_{ST}(x^8; x; \omega) = \begin{cases} 
I(x^8; x, \omega) \text{ if } I(x^8; x, \omega) > \epsilon I_{AV}(x^8; z, \omega), \\
\epsilon I_{AV}(x^8; z, \omega) \text{ if } I(x^8; x, \omega) \leq \epsilon I_{AV}(x^8, z, \omega). 
\end{cases} \] (15)

NUMERICAL TEST

In practice, the application of different imaging conditions produces different migrated images. We have compared seismic images produced with three different imaging conditions. The cross-correlation imaging condition in equation (11), the least-squares imaging condition in equation (12) and the stabilized least-squares imaging condition using the equation (15) with \( \epsilon = 1.0 \) in equation (12). Figure 1 shows the three images obtained for the Marmousi synthetic data set. Figures 1a and 1b correspond to the migrated images obtained through the correlation and least-squares imaging condition, respectively. In Figure 1b using the least-squares imaging condition a significant amount of high amplitude noise is seen. This is most serious in the shallow part of the image. Using stabilized least-squares imaging condition Figure 1c shows a better depth image and most of the instabilities due to poor illumination are avoided.

Figure 2 shows the three images obtained for the Sigsbee2A synthetic data set distributed by SMAART JV. Figures 2a and 2b we show the migrated images obtained through the correlation and least-squares imaging condition, respectively. In Figure 2b the amplitudes in the deepest reflectors below the salt body are better recovered than in the imaging condition using correlation (Figure 2a). However, instabilities show up in the regions of defocused energy and on the surface. These latter ones are associated with the poor source directivity of the one-way wave equations. Figure 2c shows the migrated image obtained using the
stabilized least-squares imaging condition. The amplitudes of the deepest reflectors are improved in the poorly illuminated zones, and the noise presented in Figure 2b is attenuated.

CONCLUSIONS

We have presented with success a criterion to avoid instability problems in the deconvolution type imaging condition applied to least-squares imaging condition. The result is a new data-adaptive stabilized least-square imaging condition where the energy of the downgoing wavefield is replaced by its transverse average value in regions of poor illumination. It was compared to the cross-correlation imaging condition and the standard least-squares imaging condition for the Marmousi and Sigsbee2A synthetic data sets. The results show that the images produced with the new imaging condition have less noise and improved amplitudes as compared with the other imaging conditions.

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We consider the very simple signal-plus-noise method of discrete data

\[ d_k = a_k R + n_k, \quad k = 1, \ldots, K \]  (A-1)

where the unknown parameter is \( R \). The amplitude coefficients \( a_k \) are known, and the noise terms \( n_k \) are assumed to be independent with zero mean and variance \( \sigma^2 \). The least-squares estimate is

\[ \hat{R}_{LS} = \frac{\sum_k d_k a_k}{\sum_k a_k^2} \]  (A-2)

where the summation (also in the following) is from \( k = 1 \) to \( k = K \).

An alternative estimate, which is commonly used in migration, is obtained by division:

\[ \hat{R}_{D} = \frac{1}{K} \sum_k \frac{d_k}{a_k} \]  (A-3)

With the assumption about the noise, it is easily seen that both estimates are unbiased, so that

\[ E\{\hat{R}_{LS}\} = E\{\hat{R}_{D}\} = R \]  (A-4)

where \( E\{\} \) denotes the expectation operator. The variance of the two estimates in equation A-2 and A-3 are

\[ \sigma^2_{LS} = E\{(\hat{R}_{LS} - R)^2\} = \frac{\sigma^2}{\sum_k a_k^2} \]  (A-5)

and

\[ \sigma^2_D = E\{(\hat{R}_{D} - R)^2\} = \frac{\sigma^2}{K^2} \sum_k \frac{1}{a_k^2}. \]  (A-6)

We want to show that \( \sigma^2_{LS} \leq \sigma^2_D \). This implies that

\[ \frac{1}{\sum_k a_k^2} \leq \frac{1}{K^2} \sum_j \frac{1}{a_j^2} \]  (A-7)
or

\[ \sum_k a_k^2 \sum_j \frac{1}{a_j^2} \geq K^2. \]  \hspace{1cm} (A-8)

In the expression on the left-hand side there are \( K^2 \) terms, both sums are from index 1 to \( K \). For \( k = j \) there are \( K \) terms equal to one, and for \( k \neq j \) there are two terms

\[ \frac{a_k^2}{a_j^2} + \frac{a_j^2}{a_k^2} \geq 2 \]  \hspace{1cm} (A-9)

since

\[ (a_k^2 - a_j^2)^2 \geq 0. \]  \hspace{1cm} (A-10)

Thus there are \( K(K - 1)/2 \) terms greater than or equal to two and \( K \) terms equal to one, so that the \( K^2 \) terms must be greater than or equal to \( K^2 \). It is seen that equality only occurs if \( a_k = 1 \) for \( k = 1, \ldots, K \), and in that case the two estimates are identical and equal to the arithmetic mean of the data. It is well known that, under fairly general conditions, the least-squares estimate is also a minimum-variance estimate (Tarantola, 1987).

A large signal (large value of \( a_k \)) is given a large weight in the least-square estimate in equation A-2, while it is given a small weight in the division estimate in equation A-3. The opposite is the case for a small signal (small value of \( a_k \)). This will give a large contribution to the variance of the division estimate, because much noise is being added to this estimate from terms with small amplitudes. Therefore the least-squares estimate is more stable than the estimate by division.

Note that in the Fourier domain the data and the amplitudes are complex, and the least-squares estimate is

\[ \hat{R}_{LS} = \frac{\sum_k d_k a_k^*}{\sum_k |a_k|^2} \]  \hspace{1cm} (A-11)

where * denotes complex conjugate.
REFERENCES


LIST OF FIGURES

1  Migrated image of the Marmousi data set using: a) The correlation imaging condition, (b) The least-squares imaging condition, (c) The stabilized least-squares imaging condition

2  Migrated image of the Sigsbee2A data set using: a) The correlation imaging condition, (b) The least-squares imaging condition, (c) The stabilized least-squares imaging condition
Figure 1: Migrated image of the Marmousi data set using: a) The correlation imaging condition, (b) The least-squares imaging condition, (c) The stabilized least-squares imaging condition.

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Figure 2: Migrated image of the Sigsbee2A data set using: a) The correlation imaging condition, (b) The least-squares imaging condition, (c) The stabilized least-squares imaging condition

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