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THE PLANE-WAVE REFLECTION AND TRANSMISSION RESPONSE OF A VERTICALLY INHOMOGENEOUS ACOUSTIC MEDIUM

BJØRN URSIN

INTRODUCTION

A number of results are known for the normal-incidence plane-wave reflection and transmission response of a stack of homogeneous fluid layers (Goupillaud 1961; Kunetz and d'Erceville 1962; Sherwood and Trorey 1965; Claerbout 1968; O'Doherty and Anstey 1971; Koehler and Taner 1977; Robinson and Treitel 1977, 1978). Other results have been derived for normal-incidence plane waves in a vertically inhomogeneous fluid with continuous reflectivity function (Foster 1975; Gjevik, Nilsen and Høyen 1976; Resnick, Lerche, and Shuey 1985). Different inverse methods for the 1D wave equation were compared by Ursin and Berteussen (1986). Ursin (1983) extended many of the results for the forward problem to elastic and electromagnetic waves in general layered media where the parameters are continuous functions of depth in each layer. A number of simplifications can be made for acoustic waves, since the reflection and transmission responses are scalars, in contrast to the general case where they are square matrices.

Acoustic wave propagation is considered in a horizontally layered medium consisting of a stack of inhomogeneous layers bounded by horizontal interfaces. The density and propagation velocity are continuous functions of depth in each layer. The acoustic wave equation is transformed into an equation for up- and downgoing waves with proper boundary conditions at the interfaces between the layers (Claerbout 1976; Ursin 1984). By neglecting the interaction between the up- and downgoing waves a zero-order WKB approximation is obtained (Bremmer 1951). A first-order WKB approximation of the upgoing wavefield is then obtained by using the zero-order WKB approximation for the downgoing wavefield. These WKB approximations have been applied to migration problems by Ursin (1984) and Robinson (1986).

Two propagation invariant forms are used to derive a number of useful relations between the reflection and transmission response of a stack of inhomogeneous layers bounded by two half-spaces and by a free surface and a half-space. These propagation invariants have been derived by Kennett, Kerry and Woodhouse (1978) for elastic *P–SV* waves and Ursin (1983) for elastic and electromagnetic waves. For a layered medium bounded by two half-spaces and evanescent waves in both half-spaces the reflection and transmission responses are real. They can therefore be evaluated by finite integrals as shown by Tygel and Hubral (1986) for the reflection response of a stack of homogeneous layers. With a propagating wave in the top half-space and an evanescent wave in the lower half-space, the downward reflection response has magnitude one. This result was derived by Fokkema and Ziolkowski (1985) and termed the critical reflection theorem. It was also derived by Ursin (1983) for elastic and electromagnetic waves.

The spectral function of a vertically inhomogeneous medium is the downward energy flux due to an impulsive source at the top of the first layer. Robinson and Treitel (1977, 1978) defined the spectral function for plane waves at normal incidence in a medium consisting of homogeneous layers of equal travel time. Ferber (1987) then derived the spectral function for plane *P–SV* waves at non-normal incidence in a medium consisting of homogeneous layers, using the *z*-transform approach of Frasier (1970). Ursin (1987) derived this and other results for elastic and electromagnetic waves in vertically inhomogeneous media.

When the first layer is bounded by a free surface, the expression for the spectral function reduces to an identity given by Kunetz and d'Erceville (1962) and also by Claerbout (1968) for plane waves at normal incidence. This result was extended by Mendel (1980) to non-impulsive sources at normal incidence. Frasier (1970) derived an expression for plane *P–SV* waves at non-normal incidence in a medium consisting of homogeneous layers. Ursin (1983) derived this and other results for elastic and electromagnetic waves in media consisting of vertically inhomogeneous layers.

The solution of the transformed wave equation can be found by computing the propagator matrix (Gilbert and Backus 1966) from a set of linear differential equations or by computing the upward and downward transmission and reflection responses from a set of equations including a Riccati equation (Schelkunoff 1951; Brekhovskikh 1960; Reid 1972; Ursin 1983). The propagator matrix and its inverse will always exist, and the elements are causal functions of time (for a given wave number). When the transmission and reflection responses exist they are causal functions of time. By considering the relationship between the propagator matrix and the reflection and transmission response it is seen that the inverse of transmission responses always exist and are causal functions of time, while the reflection responses are not generally invertible. This result was derived by Sherwood and Trorey (1965) for vertically travelling plane waves in a stack of homogeneous layers of equal travel time.

At an interface between two inhomogeneous layers (where there is a discontinuity in the layer parameters) the boundary conditions require that the pressure and the vertical displacement velocity are continuous. The interface conditions for the propagator matrix for up- and downgoing waves are given as a matrix product. The interface conditions for the reflection and transmission responses are given by the Redheffer star product (Redheffer 1961; Kennett 1974; Ursin 1983). By neglecting the quadratic term in the Riccati equation and simplifying the interface conditions, an approximate solution of the reflection and transmission responses can be obtained. For the 1D wave equation with continuous reflectivity function the expression for the transmision response can be further simplified to the generalized O'Doherty–Anstey (1971) approximation derived by Resnick, Lerche and Shuey (1985).

The Riccati equation has been applied to the 1D inverse problem by Gjevik, Nilsen and Høyen (1976) and to the inversion at acoustic waves at non-normal incidence by

Nilsen and Gjevik (1978). In some inverse problems (Ursin and Berteussen 1986) it is of interest to remove the effect of the top layers (layer stripping). This can be done by solving a set of equations including a Riccati equation with proper interface conditions to remove the effect of the interfaces.

Decomposition of an Acoustic Wavefield into Up- and Downgoing Waves

We consider acoustic waves in a horizontally layered medium where the parameters are functions of depth only. The equations of linear acoustics are (Pierce 1981, Equation 1–5.3)

$$\nabla p = -\rho \frac{\partial v}{\partial t} + f \tag{11.1a}$$

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho \nabla v = 0 \tag{11.1b}$$

where v is the displacement velocity, p is the pressure, f is a volume force, ρ is the density and c is the wave propagation velocity.

We shall apply the 3D Fourier transform

$$P(\omega, k_1, k_2, x_3) = \iiint_{-\infty}^{\infty} p(t, x_1, x_2, x_3)$$

$$\exp(\mathrm{i}\omega t - \mathrm{i}k_1 x_1 - \mathrm{i}k_2 x_2)\,\mathrm{d}t\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{11.2a}$$

with inverse transform

$$p(t, x_1, x_2, x_3) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} P(\omega, k_1, k_2, x_3)$$

$$\exp(-\mathrm{i}\omega t + \mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2)\,\mathrm{d}\omega\,\mathrm{d}k_1\,\mathrm{d}k_2. \tag{11.2b}$$

This Fourier transform is applied to Equation (11.1) which gives

$$\frac{\partial}{\partial x_3}\begin{bmatrix} P \\ V_3 \end{bmatrix} = \mathrm{i}\omega \begin{bmatrix} 0, & \rho \\ \frac{1}{\rho}\left(\frac{1}{c^2} - \frac{k_r^2}{\omega^2}\right), & 0 \end{bmatrix} \begin{bmatrix} P \\ V_3 \end{bmatrix} + \begin{bmatrix} F_3 \\ \frac{k_1 F_1 + k_2 F_2}{\rho\omega} \end{bmatrix} \tag{11.3}$$

where $k_r^2 = k_1^2 + k_2^2$.

Now we drop the source term and write equation (11.3) as

$$\frac{\partial B}{\partial x_3} = \mathrm{i}\omega A B \tag{11.4}$$

where

$$A=\begin{bmatrix} 0, & \rho \\ \frac{1}{\rho}\left(\frac{1}{c^2}-\frac{k_r^2}{\omega^2}\right), & 0 \end{bmatrix} \tag{11.5}$$

and $B=(P,V_3)^{\mathrm{T}}$ (T denotes transpose). The wavefield is decomposed into up- and downgoing waves, denoted by U and D, by the linear transformation $W=L^{-1}B$ where $W=(U,D)^{\mathrm{T}}$,

$$L^{-1}=\frac{1}{2}\begin{bmatrix} 1, & -Z \\ 1, & Z \end{bmatrix} \tag{11.6a}$$

and

$$L=\begin{bmatrix} 1, & 1 \\ -Z^{-1}, & Z^{-1} \end{bmatrix} \tag{11.6b}$$

with

$$Z=\frac{\rho\omega}{k_3} \tag{11.7}$$

and

$$k_3=\begin{cases} \sqrt{\left[\left(\frac{\omega}{c}\right)^2-k_r^2\right]} & |k_r|\leq\frac{\omega}{c}, \text{ propagating waves} \\ \sqrt{\left[k_r^2-\left(\frac{\omega}{c}\right)^2\right]} & |k_r|>\frac{\omega}{c}, \text{ evanescent waves.} \end{cases} \tag{11.8}$$

(The positive square root is always taken.)

From Equation (11.4) we obtain

$$\frac{\partial W}{\partial x_3}=\begin{bmatrix} -\mathrm{i}k_3, & 0 \\ 0, & \mathrm{i}k_3 \end{bmatrix}W+\gamma(x_3)\begin{bmatrix} 1, & -1 \\ -1, & 1 \end{bmatrix}W \tag{11.9}$$

where

$$\gamma(x_3)=\frac{1}{2}\frac{\partial}{\partial x_3}\log Z(x_3) \tag{11.10}$$

is the reflectivity function.

We consider a stack of inhomogeneous layers where ρ and c are continuous functions of x_3 within each layer. At an interface between two layers the boundary conditions require the wave vector B to be continuous. For an interface at $x_3=x_{3k}$ we must have $L_+W_+=L_-W_-$ were $L_-=L(x_{3k-})$ is evaluated above the interface, and $L_+=L(x_{3k+})$ is evaluated beneath the interface (the x_3-axis is pointing vertically downwards). We therefore have

$$W_+=L_+^{-1}L_-W_-. \tag{11.11}$$

Equation (11.6) gives

$$L_+^{-1}L_- = \frac{1}{2}\begin{bmatrix} 1+\dfrac{Z_+}{Z_-}, & 1-\dfrac{Z_+}{Z_-} \\ 1-\dfrac{Z_+}{Z_-}, & 1+\dfrac{Z_+}{Z_-} \end{bmatrix} \tag{11.12}$$

which can be written (Ursin 1983, Equation 33)

$$L_+^{-1}L_- = \begin{bmatrix} T_U^{-1}, & R_U T_U^{-1} \\ R_U T_U^{-1}, & T_U^{-1} \end{bmatrix} \tag{11.13}$$

where T_U and R_U are the transmission and reflection coefficients for an upward travelling incident wave at the interface.

Combining Equations (11.12) and (11.13) gives for a single interface

$$R_U = \frac{Z_- - Z_+}{Z_- + Z_+} \tag{11.14a}$$

$$T_U = \frac{2Z_-}{Z_- + Z_+}. \tag{11.14b}$$

Let T_D and R_D be the transmission and reflection coefficients for a downward travelling wave at the interface. By interchanging + and − we obtain

$$R_D = \frac{Z_+ - Z_1}{Z_+ + Z_-} = -R_U \tag{11.14c}$$

$$T_D = \frac{2Z_+}{Z_+ + Z_-}. \tag{11.14d}$$

Note that

$$T_D T_U - R_D R_U = 1. \tag{11.15}$$

We also note that both the reflectivity function $\gamma(x_3)$ and the reflection and transmission coefficients are functions only of $Z(x_3)$. For a plane wave with wavenumber $k=(k_1, k_2, k_3)^T$ and direction $m=(m_1, m_2, m_3)^T$ we have $k=m\omega/c$ and $m^T=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$ where θ is the dip angle and φ is the azimuth angle. We see that

$$Z = \frac{\rho\omega}{k_3} = \frac{\rho c}{\cos\theta} \tag{11.16}$$

where $\cos\theta = \sqrt{(1-q^2c^2)}$ and $q=k_r/\omega$ is the ray parameter. When Equation (11.16) is used in equation (11.10) we obtain

$$\gamma(x_3) = \frac{1}{2}\left[\frac{1}{\rho}\frac{\partial\rho}{\partial x_3} + \frac{1}{\cos^2\theta}\frac{1}{c}\frac{\partial c}{\partial x_3}\right]. \tag{11.17}$$

The elements of the matrix $L_+^{-1}L_-$ in Equation (11.12) are functions of

$$\frac{Z_+}{Z_-}=\frac{(\rho c)_+}{(\rho c)_-}\frac{\cos\theta_-}{\cos\theta_-}. \tag{11.18}$$

When the source function in Equation (11.1) is independent of azimuth angle, the pressure and displacement velocity depend only on depth x_3 and radial distance $r=(x_1^2+x_2^2)^{\frac{1}{2}}$. In this case we may apply a Fourier transform with respect to time and a Hankel transform with respect to radius (Sneddon 1972) to obtain the transformed variable

$$U(\omega, k_r, x_3)=\int_{-\infty}^{\infty}\int_{0}^{\infty} u(t, r, x_3)\,J_n(k_r r)\,r\exp(\mathrm{i}\omega t)\,\mathrm{d}r\,\mathrm{d}t. \tag{11.19a}$$

The inverse transform is

$$u(t, r, x_3)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} U(\omega, k_r, x_3)J_n(k_r r)k_r\exp(-\mathrm{i}\omega t)\mathrm{d}k_r\mathrm{d}\omega. \tag{11.19b}$$

In Equation (11.1) we apply this transform with $n=0$ to the variables p and v_3, and with $n=1$ to v_r (the radial component of the displacement velocity), and we obtain again Equation (11.3) with the modified source term

$$\left[F_3, \frac{k_r F_r}{\mathrm{i}\rho\omega}\right]^T. \tag{11.20}$$

This result can also be obtained directly from Equation (11.3) by using the relationship between the 2D Fourier transform and the Hankel transform (Bracewell 1978).

The initial downgoing wavefield from a point source can be derived from the Sommerfield integral (Ursin 1983, Equation E13):

$$D(\omega, k_r, 0)=-\frac{G(\omega)}{\mathrm{i}k_3} \tag{11.21}$$

were $G(\omega)$ is the spectrum of the source signature. The measured response is often assumed to be the upcoming wavefield $u(t, r, x_3=0)$ with Fourier–Hankel transform $U(\omega, k_r, 0)$.

The zero-order WKB approximation corresponds to neglecting the interaction between the up- and downgoing waves. This gives the one-way equations

$$\frac{\partial}{\partial x_3}\begin{bmatrix}U\\D\end{bmatrix}=\begin{bmatrix}-\mathrm{i}k_3+\gamma, & 0\\0, & \mathrm{i}k_3+\gamma\end{bmatrix}\begin{bmatrix}U\\D\end{bmatrix} \tag{11.22}$$

with boundary conditions (see Equations 11.11 and 11.13)

$$\begin{bmatrix}U\\D\end{bmatrix}_+=T_{\mathrm{U}}^{-1}\begin{bmatrix}U\\D\end{bmatrix}_-. \tag{11.23}$$

The solution of these equations is the zero-order WKB approximation.

$$U_0(x_3)=U(0)T(x_3)\exp\left[-\mathrm{i}\int_0^{x_3}k_3(z)\mathrm{d}z\right] \tag{11.24a}$$

and

$$D_0(x_3)=D(0)T(x_3)\exp\left[\mathrm{i}\int_0^{x_3}k_3(z)\mathrm{d}z\right] \tag{11.24b}$$

where $D(0)$ is given in Equation (11.21) and $U(0)$ is the transformed recorded wavefield.

The factor

$$T(x_3)=Z^{-\frac{1}{2}}(0)Z^{\frac{1}{2}}(x_3)\prod_{0<x_{3k}<x_3}T_{Uk}^{-1}Z^{\frac{1}{2}}(x_{3k-})Z^{-\frac{1}{2}}(x_{3k+}) \tag{11.25}$$

is due to the interfaces between the inhomogeneous layers. Here T_{Uk} is the upward transmission coefficient at the interface at x_{3k}.

We have just computed the up- and downgoing wavefield neglecting the interaction between the two wavefields. A more accurate approximation can be obtained by using the downgoing wavefield computed by the zero-order approximation, when the upgoing wavefield is downward continued. The first-order WKB approximation of the upgoing wavefield satisfies the equation

$$\frac{\partial U_1}{\partial x_3}=-\mathrm{i}k_3U_1+\gamma(x_3)(U_1-D_0) \tag{11.26}$$

with interface condition

$$U_1(x_{3k+})=T_{Uk}^{-1}[U_1(x_{3k-})+R_{Uk}D_0(x_{3k-})]. \tag{11.27}$$

The solution of these equations is found to be

$$\begin{aligned}U_1(x_3)&=T(x_3)\exp\left[-\mathrm{i}\int_0^{x_3}k_3(z)\mathrm{d}z\right]\left\{U(0)-D(0)\int_0^{x_3}\gamma_1(z)\exp\left[\mathrm{i}2\int_0^{z}k_3(\xi)\mathrm{d}\xi\right]\mathrm{d}z\right\}\\&=U_0(x_3)-D_0(x_3)\int_0^{x_3}\gamma_1(z)\exp\left[-\mathrm{i}2\int_z^{x_3}k_3(\xi)\mathrm{d}\xi\right]\mathrm{d}z\end{aligned} \tag{11.28}$$

where the generalized reflectivity function is

$$\gamma_1(x_3)=\gamma(x_3)-\sum_{0<x_{3k}<x_3}R_{Uk}\delta(x_3-x_{3k}). \tag{11.29}$$

We see that we have obtained an additional term due to the effect of the generalized reflectivity function. This term corrects partly for the influence of the downgoing wavefield on the upgoing wavefield.

Propagation Invariants

Let $B_1=[P_1, V_1]^{\mathrm{T}}$ and $B_2=[P_2, V_2]^{\mathrm{T}}$ be two solutions of Equation (11.4). The form

$$G=P_1V_2-V_1P_2 \tag{11.30}$$

is then a constant not depending on x_3. This is easily seen by taking the derivative of G with respect to x_3 and using Equation (11.4).

For the transformed wavevectors $W_k = L^{-1} B_k$ we obtain that

$$\begin{aligned} G &= \frac{2}{Z}[U_1 D_2 - U_2 D_1] \\ &= \frac{2}{Z}[U_1 D_1]\begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix}\begin{bmatrix} U_2 \\ D_2 \end{bmatrix} \end{aligned} \tag{11.31}$$

is constant.

For lossless media the parameters ρ and c are real. Then the form

$$H = \tfrac{1}{4}[P_1^* V_2 + V_1^* P_2] \tag{11.32}$$

is constant (the star denotes complex conjugate).

For equal wave vectors, $B_1 = B_2$, the form H is the downward energy flux (see Foster 1975; Brekhovskikh 1960, p. 114).

For propagating waves (k_3 real) the matrix L is real, and we obtain that

$$\begin{aligned} H &= \frac{1}{2Z}[D_1^* D_2 - U_1^* U_2] \\ &= \frac{1}{2Z}[U_1^* \quad D_1^*]\begin{bmatrix} -1, & 0 \\ 0, & 1 \end{bmatrix}\begin{bmatrix} U_2 \\ D_2 \end{bmatrix} \end{aligned} \tag{11.33}$$

with Z real. For evanescent waves (k_3 purely imaginary) we obtain that

$$\begin{aligned} H &= \frac{1}{2Z}[U_1^* D_2 - D_1^* U_2] \\ &= \frac{1}{2Z}[U_1^* D_1^*]\begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix}\begin{bmatrix} U_2 \\ D_2 \end{bmatrix} \end{aligned} \tag{11.34}$$

and Z is now purely imaginary.

Layered Medium Bounded by Two Half-spaces

We consider a sequence of inhomogeneous layers bounded by two homogeneous half-spaces as shown in Fig. 11.1. An incident wave of strength 1 produces a reflected upcoming wave R_D and a transmitted downgoing wave T_D. An upgoing wave of strength 1 produces a reflected downgoing wave R_U and a transmitted upcoming wave T_U. Two propagation invariant forms G and H will be used to derive relations between the reflection and transmission responses.

With W_1 as shown in Fig. 11.1a and W_2 as shown in Fig. 11.1b the form G has the same value at the top and the bottom of the layers. This gives

$$\frac{T_U}{Z_0} = \frac{T_D}{Z_N} \tag{11.35}$$

where Z_0 is $Z(x_3)$ evaluated at the top of the layers, and Z_N is $Z(x_3)$ evaluated at the bottom of the layers.

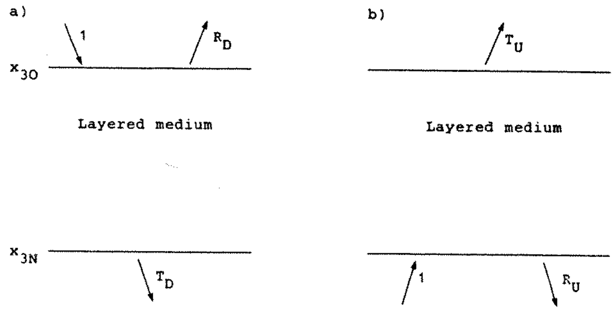

Fig. 11.1. Reflection and transmission response of a layered medium bounded by two half-spaces.

For a lossless medium and propagating waves in both half-spaces the form H in Equation (11.33) is also constant. This gives

$$\frac{R_D^* T_U}{Z_0} + \frac{T_D^* R_U}{Z_N} = 0. \tag{11.36}$$

Combining the two last equations gives

$$R_D^* T_D + T_D^* R_U = 0. \tag{11.37}$$

With $W_1 = W_2$ as shown in Fig. 11.1a and propagating waves, H=constant gives

$$\frac{1}{Z_0}[1 - |R_D|^2] = \frac{1}{Z_N}|T_D|^2. \tag{11.38}$$

Similarly we obtain

$$\frac{1}{Z_N}[1 - |R_U|^2] = \frac{1}{Z_0}|T_U|^2. \tag{11.39}$$

The last two equations express that the energy is constant.

For evanescent waves in both half-spaces and a lossless medium, H in Equation (11.34) is constant at the top and the bottom of the layers. Then all the reflection and transmission coefficients are real. Tygel and Hubral (1986) used the fact that R_D is real for evanescent waves in both half-spaces to compute a finite integral solution for the point-source response of a stack of homogeneous layers. The same procedure may therefore also be applied to compute the reflection and transmission response of a stack of inhomogeneous layers.

For propagating waves at the top of the layers and evanescent waves at the bottom of the layers the following relations are obtained:

$$\begin{aligned} &R_D^* T_D = T_D^* \\ &|R_D|^2 = 1 \\ &R_U - R_U^* = -\frac{Z_N}{Z_0}|T_U|^2 = -T_U^* T_D. \end{aligned} \tag{11.40}$$

The middle identity was derived by Fokkema and Ziolkowski (1985) for a stack of homogeneous layers and termed the critical reflection theorem. Ursin (1983, Equation 86) derived a general version of Equation (11.40) for the case of elastic and electromagnetic waves.

Layered Medium Bounded by a Free Surface and a Half-Space

Next we consider a layered medium bounded by a free surface and a half-space as shown in Fig. 11.2. The upward reflection coefficient at the free surface is $-R_0$. ($R_0=1$ is obtained from the boundary condition $P=0$ at the free surface.)

We consider an incident wave of strength 1 at the top of the layers as shown in Fig. 11.2a. This results in a reflected upcoming wave $\tilde{R}_D$ (which subsequently produces a downgoing wave $-R_0\tilde{R}_D$) and a transmitted downgoing wave $\tilde{T}_D$ at the bottom of the layers. We also consider an incident upcoming wave of strength 1 at the bottom of the layers as shown in Fig. 11.2b. This results in a downgoing wave $\tilde{R}_U$ and an upgoing wave $\tilde{T}_U$ (at the top of the layers) which subsequently results in a downgoing wave $-R_0\tilde{T}_U$.

By equating the form G at the top and the bottom of the layers for W_1 as shown in Fig. 11.2a and W_2 as shown in Fig. 11.2b we obtain

$$\frac{\tilde{T}_U}{Z_0}=\frac{\tilde{T}_D}{Z_N}. \tag{11.41}$$

This result does not depend on the value of R_0.

For a lossless medium and propagating waves at the top and the bottom of the layers the form H in Equation (11.33) is constant. This gives

$$\begin{aligned}
&\frac{1}{Z_0}[-R_0\tilde{T}_U+(|R_0|^2-1)\tilde{R}_D^*\tilde{T}_U]=\frac{1}{Z_N}\tilde{T}_D^*\tilde{R}_U\\
&\frac{1}{Z_0}(|R_0|^2-1)|\tilde{T}_U|^2=\frac{1}{Z_N}(|\tilde{R}_U|^2-1)\\
&\frac{1}{Z_0}[1-R_0^*\tilde{R}_D^*-R_0\tilde{R}_D+(|R_0|^2-1)|\tilde{R}_D|^2]=\frac{1}{Z_N}|\tilde{T}_D|^2.
\end{aligned} \tag{11.42}$$

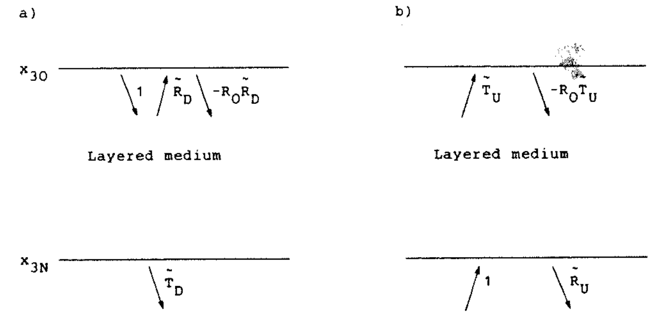

Fig. 11.2. Reflection and transmission response of a layered medium bounded by a free surface and a half-space.

For $W_1 = W_2$ the form H in Equation (11.33) is equal to the downward energy flux. For an impulsive source this is equal to the spectral function of the vertically inhomogeneous medium as defined by Robinson and Treitel (1977, 1978) who derived the last equation in (11.42). Ferber (1987) extended this result to P-SV waves in a stack of homogeneous elastic layers using the z-transform approach of Frasier (1970). Ursin (1987) extended the set of equations (11.42) to elastic and electromagnetic waves in vertically inhomogeneous media.

For a free surface $R_0 = 1$ and

$$\begin{aligned} &\tilde{T}_D + \tilde{T}_D^* \tilde{R}_U = 0 \\ &|\tilde{R}_U|^2 = 1 \\ &1 - \tilde{R}_D^* - \tilde{R}_D = \frac{Z_0}{Z_N} |\tilde{T}_D|^2 = \tilde{T}_U \tilde{T}_D^*. \end{aligned} \tag{11.43}$$

The last equation was derived by Kunetz and d'Erceville (1962) and also by Claerbout (1968) for normal-incidence plane waves in a stack of homogeneous layers. Frasier (1970) extended this expression to plane P–SV waves at non-normal incidence in a stack of homogeneous elastic layers. Ursin (1983, Equation 134) derived the results in Equation (11.43) for elastic and electromagnetic waves in vertically inhomogeneous media.

For propagating waves at the top of the layers and evanescent waves in the half-space the form H in Equation (11.33) evaluated at the top of the layers is equal to the form H in equation (11.34) evaluated at the top of the half-space. This gives

$$\begin{aligned} &\frac{1}{Z_0}[(|R_0|^2 - 1)\tilde{R}_D^* \tilde{T}_U - R_0 \tilde{T}_U] = -\frac{\tilde{T}_D^*}{Z_N} \\ &\frac{1}{Z_0}(1 - |R_0|^2)|\tilde{T}_U|^2 = \frac{1}{Z_N}(\tilde{R}_U - \tilde{R}_U^*) \\ &(|R_0|^2 - 1)|\tilde{R}_D|^2 - R_0^* \tilde{R}_D^* - R_0 \tilde{R}_D + 1 = 0. \end{aligned} \tag{11.44}$$

For a free surface $R_0 = 1$ and Equation (11.44) gives that $\tilde{R}_U$ is real, $\tilde{T}_D$ is real (using Equation 11.41), $\tilde{T}_U$ is purely imaginary (using Equation 11.41, and the fact that Z_0 is real and Z_N is purely imaginary), and the real part of $\tilde{R}_D$ is equal to 0.5.

In the case that the waves are evanescent at the top and the bottom of the layers the form H in Equation (11.34) is constant. This gives

$$\begin{aligned} &\frac{1}{Z_0}[R_0^* \tilde{R}_D^* \tilde{T}_U - R_0 \tilde{R}_D^* \tilde{T}_U - \tilde{T}_U] = -\frac{\tilde{T}_D^*}{Z_N} \\ &\frac{1}{Z_0}(R_0^* - R_0)|\tilde{T}_U|^2 = \frac{1}{Z_N}(\tilde{R}_U - \tilde{R}_U^*) \\ &(R_0^* - R_0)|\tilde{R}_D|^2 + \tilde{R}_D^* - \tilde{R}_D = 0. \end{aligned} \tag{11.45}$$

For a free surface $R_0 = 1$ and all transmission and reflection coefficients are real.

The two types of reflection and transmission responses are related. It can be shown

(Ursin 1983, Equation 149) that

$$R_D=\frac{\tilde{R}_D}{1-R_0\tilde{R}_D}$$
$$T_D=\frac{\tilde{T}_D}{1-R_0\tilde{R}_D} \tag{11.46}$$
$$R_U=\tilde{R}_U+R_0\frac{\tilde{T}_D\tilde{T}_U}{1-R_0\tilde{R}_D}$$
$$T_U=\frac{\tilde{T}_U}{1-R_0\tilde{R}_D}.$$

The first equation was also derived by Koehler and Taner (1977) for normal-incidence plane waves in a stack of homogeneous layers.

The Propagator Matrix and the Reflection and Transmission Responses

The general solution of the inhomogeneous wave Equation (11.9) is found by computing the 2×2 propagator matrix (Gilbert and Backus 1966) $Q(x_3, x_{30})$ from

$$\frac{\partial Q}{\partial x_3}=\begin{bmatrix}-ik_3+\gamma, & -\gamma\\ -\gamma, & ik_3+\gamma\end{bmatrix}Q \tag{11.47}$$

with $Q(x_{30}, x_{30})=I$. The elements Q_{ij}, $i,j=1,2$, of the matrix Q exist and are causal functions of time (Pierce, 1981, p. 43). The solution of the wave equation may also be expressed in terms of the reflection and transmission responses if these exist. These do not exist for channel waves and interface waves which cannot be decomposed into up- and downgoing waves. The propagator matrix and its inverse are (Ursin 1983, Equations 88 and 90)

$$Q(x_{3N}, x_{30})=\begin{bmatrix}T_U^{-1}, & -T_U^{-1}R_D\\ R_UT_U^{-1}, & T_D-R_UT_U^{-1}R_D\end{bmatrix} \tag{11.48}$$

and

$$Q(x_{30}, x_{3N})=\begin{bmatrix}T_U-R_DT_D^{-1}R_U, & R_DT_D^{-1}\\ -T_D^{-1}R_U, & T_D^{-1}\end{bmatrix}. \tag{11.49}$$

The reflection and transmission responses are causal functions of time. From equations (11.48) and (11.49) it is seen that the transmission responses are also invertible. The reflection responses are not, in general, invertible. These results were derived by Sherwood and Trorey (1965) for the 1D wave equation.

For an acoustic medium bounded by a free surface there exist similar equations which relate the propagator matrix to the modified reflection and transmission responses (Ursin 1983, Equations 136 and 138). From these it follows that also the modified reflection and transmission responses are causal functions of time, and that the modified transmission responses are invertible.

The composition of two propagator matrices $Q(x_3, x_{30})$ and $Q(x_{3N}, x_3)$ are given by ordinary matrix multiplication:

$$Q(x_{3N}, x_{30}) = Q(x_{3N}, x_3)Q(x_3, x_{30}). \tag{11.50}$$

The boundary conditions (11.11) must be taken into account at an interface between two inhomogeneous layers:

$$Q(x_{3k+}, x_{30}) = \begin{bmatrix} T_{Uk}^{-1}, & R_{Uk}T_{Uk}^{-1} \\ R_{Uk}T_{Uk}^{-1}, & T_{Uk}^{-1} \end{bmatrix} Q(x_{3k-}, x_{30}) \tag{11.51}$$

The S matrix defined by

$$S = \begin{bmatrix} T_D, & R_U \\ R_D, & T_U \end{bmatrix} \tag{11.52}$$

relates the outgoing (scattered) waves to the incoming waves

$$\begin{bmatrix} D(x_{3N}) \\ U(x_{30}) \end{bmatrix} = S(x_{30}, x_{3N}) \begin{bmatrix} D(x_{30}) \\ U(x_{3N}) \end{bmatrix}. \tag{11.53}$$

The composition of two S matrices is given by Redheffer's star product (Redheffer 1961). This composition rule was also derived by Kennett (1974).

The star product is defined by (Ursin 1983, Equation 93):

$$\begin{bmatrix} T_D(x_{30}, x_{3N}), & R_U(x_{30}, x_{3N}) \\ R_D(x_{30}, x_{3N}), & T_U(x_{30}, x_{3N}) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{T_D(x_{30}, x_3)T_D(x_3, x_{3N})}{1 - R_U(x_{30}, x_3)R_D(x_3, x_{3N})}, & R_U(x_3, x_{3N}) + \dfrac{T_D(x_3, x_{3N})R_U(x_{30}, x_3)T_U(x_3, x_{3N})}{1 - R_U(x_{30}, x_3)R_D(x_3, x_{3N})} \\ R_D(x_{30}, x_3) + \dfrac{T_U(x_{30}, x_3)R_D(x_3, x_{3N})T_D(x_{30}, x_3)}{1 - R_U(x_{30}, x_3)R_D(x_3, x_{3N})}, & \dfrac{T_U(x_{30}, x_3)T_U(x_3, x_{3N})}{1 - R_U(x_{30}, x_3)R_D(x_3, x_{3N})} \end{bmatrix} \tag{11.54}$$

and denoted by

$$S(x_{30}, x_{3N}) = S(x_{30}, x_3) * S(x_3, x_{3N}). \tag{11.55}$$

THE RICCATI EQUATION

Synthetic seismograms may be computed by computing the reflection and transmission response of a layered medium. This is done by solving a set of differential equations in the inhomogeneous layers and applying interface conditions (derived from the boundary conditions of the acoustic equations) at the interfaces between the layers. The general differential equations are given in Ursin (1983, Equation 105), and by replacing

$s\Lambda$, F, and G with $-\mathrm{i}k_3$, γ and $-\gamma$ this gives

$$\begin{aligned}\frac{\partial T_U}{\partial x_3}&=(\mathrm{i}k_3-\gamma)\,T_U+\gamma R_U T_U\\ \frac{\partial R_U}{\partial x_3}&=\mathrm{i}2k_3R_U+\gamma(R_U^2-1)\\ \frac{\partial T_D}{\partial x_3}&=(\mathrm{i}k_3+\gamma)\,T_D+\gamma R_U T_D\\ \frac{\partial R_D}{\partial x_3}&=\gamma T_U T_D\end{aligned} \tag{11.56}$$

where $T_U=T_U(x_{30},x_3)$ and so on. The second equation is a scalar Riccati equation (also derived by Schelkunoff 1951) which can be solved first. The other equations can then be solved directly (assuming there are no interfaces between x_{30} and x_3):

$$\begin{aligned}T_U(x_{30},x_3)&=T_{U0}\exp\left[\int_{x_{30}}^{x_3}[\mathrm{i}k_3(z)+\gamma(z)\,(R_U(x_{30},z)-1)]\,\mathrm{d}z\right]\\ T_D(x_{30},x_3)&=T_{D0}\exp\left[\int_{x_{30}}^{x_3}[\mathrm{i}k_3(z)+\gamma(z)\,(R_U(x_{30},z)+1)]\,\mathrm{d}z\right]\\ R_D(x_{30},x_3)&=R_{D0}+\int_{x_{30}}^{x_3}\gamma(z)\,T_D(x_{30},z)\,T_U(x_{30},z)\,\mathrm{d}z.\end{aligned} \tag{11.57}$$

In the case that there is a homogeneous half-space above x_{30} the initial conditions are $T_{U0}=T_{D0}=1$ and $R_{U0}=R_{D0}=0$.

In the case that $|R_U|^2 \ll 1$ the Riccati equation in (11.56) can be approximated by

$$\frac{\partial R_U}{\partial x_3}=\mathrm{i}2k_3R_U-\gamma \tag{11.58}$$

with solution

$$R_U(x_{30},x_3)=R_{U0}\exp\left[2\mathrm{i}\int_{x_{30}}^{x_3}k_3(z)\,\mathrm{d}z\right]-\int_{x_{30}}^{x_3}\gamma(z)\exp\left[2\mathrm{i}\int_{z}^{x_3}k_3(\xi)\,\mathrm{d}\xi\right]\mathrm{d}z. \tag{11.59}$$

The four equations in (11.56) are coupled at the interfaces between the inhomogeneous layers, and the interface conditions are given by the star product. Let the S matrix be S_- above the interface at x_{3k}, S_+ below the interface and S_k for the interface itself. Using Equation (11.55) for $S_+=S_-*S_k$ gives

$$\begin{aligned}S_+&=\begin{bmatrix}T_{D+}, & R_{U+}\\ R_{D+}, & T_{U+}\end{bmatrix}\\ &=\begin{bmatrix}\dfrac{T_D-T_{Dk}}{1-R_{U-}R_{Dk}}, & R_{Uk}+\dfrac{T_{Dk}R_{U-}T_{Uk}}{1-R_{U-}R_{Dk}}\\ R_{D-}+\dfrac{T_{U-}R_{Dk}T_{D-}}{1-R_{U-}R_{Dk}}, & \dfrac{T_{U-}T_{Uk}}{1-R_{U-}R_{Dk}}\end{bmatrix}.\end{aligned} \tag{11.60}$$

If $|R_{U-}R_{Dk}| \ll 1$ we can use the approximation

$$S_+ \approx \begin{bmatrix} T_{D-}T_{Dk}, & R_{Uk}+R_{U-}T_{Dk}T_{Uk} \\ R_{D-}+R_{Dk}T_{D-}T_{U-}, & T_{U-}T_{Uk} \end{bmatrix} \tag{11.61}$$

in which multiple reflections involving interface number k have been neglected.

The approximate interface conditions (11.61) and the solutions in Equation (11.57) give the approximate solution

$$T_D(x_{30}, x_3) = \prod_{x_{30} \leq x_{3k} < x_3} T_{Dk} \exp\left[\int_{x_{30}}^{x_3} (\mathrm{i}k_3(z) + \gamma(z)(R_U(x_{30}, z) + 1))\,\mathrm{d}z\right]$$

$$T_U(x_{30}, x_3) = \prod_{x_{30} \leq x_{3k} < x_3} T_{Uk} \exp\left[\int_{x_{30}}^{x_3} (\mathrm{i}k_3(z) + \gamma(z)(R_U(x_{30}, z) - 1))\,\mathrm{d}z\right] \tag{11.62}$$

$$R_D(x_{30}, x_3) = \int_{x_{30}}^{x_3} \left[\gamma(z) + \sum_{x_{30} \leq x_{3k} < z} R_{Dk}\,\delta(z - x_{3k})\right]$$

$$\prod_{x_{30} \leq x_{3j} < z} (T_{Dj}T_{Uj}) \exp\left(2\int_{x_{30}}^{z} (\mathrm{i}k_3(\xi) + R_U(\xi))\,\mathrm{d}\xi\right)\mathrm{d}z$$

The approximate interface conditions (11.61) and the approximate solution in Equation (11.59) give the approximation

$$R_U(x_{30}, x_3) = \int_{x_{30}}^{x_3} \left[\sum_{x_{30} \leq x_{3k} < x_3} R_{Uk}\,\delta(z - x_{3k}) - \gamma(z)\right]$$

$$\prod_{z < x_{3j} \leq x_3} (T_{Dj}T_{Uj}) \exp\left(2\mathrm{i}\int_{z}^{x_3} k_3(\xi)\,\mathrm{d}\xi\right)\mathrm{d}z. \tag{11.63}$$

In the appendix it is shown that for plane waves at normal incidence, the first equation in (11.62) can be simplified to an approximation derived by Resnick, Lerche and Shuey (1985). By assuming homogeneous layers of equal travel time this can be further simplified to the approximation derived by O'Doherty and Anstey (1971).

In inverse problems it is of interest to remove the effect of the top layers (layer stripping). This can be done by computing $S(x_3, x_{3N})$ for increasing x_3 and removing the effect of the interfaces. From Ursin (1983, Equations 112 and 114) it follows that

$$\frac{\partial T_D}{\partial x_3} = (-\mathrm{i}k_3 - \gamma + \gamma R_D)\,T_D$$

$$\frac{\partial T_U}{\partial x_3} = (-\mathrm{i}k_3 + \gamma + \gamma R_D)\,T_U \tag{11.64}$$

$$\frac{\partial R_D}{\partial x_3} = -2\mathrm{i}k_3 R_D + \gamma(R_D^2 - 1)$$

$$\frac{\partial R_U}{\partial x_3} = \gamma T_D T_U$$

where $T_D = T_D(x_3, x_{3N})$ and so on. Using the same notation as before the star product $S_- = S_k * S_+$ is

$$\begin{bmatrix} T_{D-}, & R_{U-} \\ R_{D-}, & T_{U-} \end{bmatrix} = \begin{bmatrix} \dfrac{T_{D+}T_{Dk}}{1-R_{Uk}R_{D+}}, & R_{U+} + \dfrac{T_{D+}R_{Uk}T_{U+}}{1-R_{Uk}R_{D+}} \\ R_{Dk} + \dfrac{T_{Uk}R_{D+}T_{Dk}}{1-R_{Uk}R_{D+}}, & \dfrac{T_{U+}T_{Uk}}{1-R_{Uk}R_{D+}} \end{bmatrix} \tag{11.65}$$

which, together with Equation (11.15), give

$$S_+ = \begin{bmatrix} \dfrac{T_{D-}T_{Uk}}{1+R_{D-}R_{Uk}}, & R_{U-} - R_{Uk}\dfrac{T_{D-}T_{U-}}{1+R_{D-}R_{Uk}} \\ -R_{Dk} + R_{D-}\dfrac{T_{Dk}T_{Uk}}{1+R_{D-}R_{Uk}}, & \dfrac{T_{U-}T_{Dk}}{1+R_{D-}R_{Uk}} \end{bmatrix}. \tag{11.66}$$

In the case that $|R_{Uk}R_{D+}| \ll 1$ Equation (11.65) gives the approximate interface conditions:

$$S_+ \approx \begin{bmatrix} T_{D-}T_{Dk}^{-1}, & R_{U-} - R_{Uk}\dfrac{R_{D-}T_{U-}}{T_{Dk}T_{Uk}} \\ \dfrac{R_{D-} - R_{Dk}}{T_{Uk}T_{Dk}}, & T_{U-}T_{Uk}^{-1} \end{bmatrix}. \tag{11.67}$$

Neglecting the R_D^2 term in Equation (11.64) and using the approximate interface conditions result in approximate formulae for layer stripping.

Conclusion

The equations of linear acoustics in a stack of vertically inhomogeneous layers were transformed into a set of equations for up- and downgoing waves for which WKB approximations were derived.

Two propagation invariant forms provided several useful relationships between the reflection and transmission responses. These include a general proof of the critical reflection theorem. The reflection and transmission responses can be computed by finite integrals.

Synthetic seismograms may be computed from the propagator matrix by solving a set of linear differential equations for each frequency and wavenumber or by solving a set of differential equations including a Riccati equation for the reflection and transmission responses. The interface conditions for the propagator matrix are given by an ordinary matrix product, and for the reflection and transmission responses by the star product.

Dropping the quadratic term in the Riccati equation and simplifying the interface conditions resulted in explicit approximate solutions for the reflection and transmission responses.

Layer stripping can be performed by solving a set of equations including a Riccati equation and applying the inverse star product at the interfaces between the inhomogeneous layer.

Appendix

The O'Doherty–Anstey Approximation

O'Doherty and Anstey (1971) derived an approximation for the downward transmission response of a stack of homogeneous layers for plane waves at normal incidence. For normal incidence $k_3 = \omega/c$ and Equation (11.62) gives

$$T_D(x_{30}, x_{3N}) = \prod_{k=1}^{N} T_{Dk} \left[\frac{(\rho c)(x_{3N})}{(\rho c)(x_{30})}\right]^{\frac{1}{2}}$$
$$\exp\left[\mathrm{i}\omega \int_{x_{30}}^{x_{3N}} \frac{\mathrm{d}z}{c(z)}\right] \exp\left[\int_{x_{30}}^{x_{3N}} \gamma(z)\, R_U(x_{30}, z)\, \mathrm{d}z\right]. \tag{A11.1}$$

The first factors represent transmission losses, and the first exponential factor is a delay. The last exponential factor becomes

$$\exp\left[-\int_{x_{30}}^{x_{3N}} \gamma(z) \int_{x_{30}}^{z} \gamma(\xi) \exp\left(2\mathrm{i}\omega \int_{\xi}^{z} \frac{\mathrm{d}\eta}{c(\eta)}\right) \mathrm{d}\xi\, \mathrm{d}z\right] \tag{A11.2}$$

when the approximation

$$R_U(x_{30}, x_3) = -\int_{x_{30}}^{x_3} \gamma(z) \exp\left(2\mathrm{i}\omega \int_{z}^{x_3} \frac{\mathrm{d}\xi}{c(\xi)}\right) \mathrm{d}z \tag{A11.3}$$

is used. The expression in Equation (A11.2) is the approximation derived by Resnick, Lerche and Shuey (1985).

For a stack of homogeneous layers, the approximation

$$\gamma(z) \approx \sum_{k=0}^{N} R_{dk} \delta(z - x_{3k}) \tag{A11.4}$$

is used in Equation (A11.2). Then the exponential can be written

$$\exp\left[-\sum_{k=0}^{N} R_{Dk} \sum_{j=0}^{k} R_{Dj} \exp\left(\mathrm{i}\omega \sum_{m=j+1}^{k} \Delta\tau_m\right)\right]$$
$$= \exp\left[-\sum_{k=0}^{N} R_{Dk} \exp(\mathrm{i}\omega\tau_k) \sum_{j=0}^{k} R_{Dj} \exp(-\mathrm{i}\omega\tau_j)\right] \tag{A11.5}$$

where $\Delta\tau_m = 2(x_{3m} - x_{3m-1})/c_m$ is the two-way travel time in the mth layer, and

$$\tau_k = \sum_{m=1}^{k} \Delta\tau_m \tag{A11.6}$$

is the two-way travel time to the interface at x_{3k} (with $\tau_0 = 0$). With equal travel time in each layer ($\Delta\tau_m = \Delta\tau$) Equation (A11.5) gives the same result as Equation (9) in O'Doherty and Anstey (1971).

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