SECOND-ORDER APPROXIMATIONS OF THE REFLECTION AND TRANSMISSION COEFFICIENTS BETWEEN TWO VISCO-ELASTIC ISOTROPIC MEDIA

ALEXEY STOVAS and BJØRN URSIN

Norwegian University of Science and Technology (NTNU), Department of Petroleum Engineering and Applied Geophysics, S.P. Andersensvei 15A, N-7491 Trondheim, Norway.

(Received November 3, 2000; revised version accepted February 9, 2001)

ABSTRACT

Stovas, A. and Ursin, B., 2001. Second-order approximations of the reflection and transmission coefficients between two visco-elastic isotropic media. *Journal of Seismic Exploration*, 9: 223-233.

The plane-wave reflection and transmission (R/T) coefficients between two isotropic visco-elastic media where the medium parameters are complex function of frequency, are found from the continuity of stress and displacement. Normalizing the R/T coefficients with respect to vertical energy-flux results in important symmetry properties. By expanding the eigenvector matrices for up- and down-going waves into second-order Taylor series and using the symmetry properties of these matrices we derive simple second-order approximations for the R/T coefficients valid for pre-critical reflections. The standard linear weak-contrast approximations are obtained by neglecting the second-order terms in the relative changes in the medium parameters at the interface.

The two approximations are compared to the exact R/T coefficients in three numerical examples where the attenuation of P- and S-waves are given by the Kolsky-Futterman attenuation relation. In all cases, the new approximations give a better fit to the exact R/T coefficients than the linearized approximations. Inversion of the PP and PS reflection coefficients into the relative changes in the medium parameters shows that the new approximations perform much better than using the linear approximations.

KEY WORDS: plane waves, visco-elasticity, reflection coefficient, transmission coefficient, second-order approximation.

0963-0651/01/$5.00

INTRODUCTION

We consider the plane-wave reflection and transmission (R/T) coefficients between two isotropic visco-elastic media where the medium parameters are complex function of frequency (Ben-Menahem and Singh, 1981). Continuity of stress and displacement velocity gives the exact equations for the R/T coefficients (Aki and Richards, 1980). By normalizing the R/T coefficients with respect to vertical energy-flux results in important symmetry properties (Ursin, 1983; Chapman, 1994). The R/T coefficients are expressed in terms of the eigenvector matrices which define the up- and down-going waves. For pre-critical reflections we expand these matrices in Taylor series up to second order in the relative change in the medium parameters at the interface. Substituting these expressions into the exact equations for the R/T coefficients results in second-order approximations. The linear terms in these approximations give the standard weak-contrast linearized approximation (Aki and Richards, 1980; Ursin and Haugen, 1996; Ursin and Stovas, 2001).

The two approximations are compared to the exact R/T coefficients in three numerical examples where the attenuation of P- and S-waves are given by the Kolsky-Futterman attenuation relation (Futterman, 1962; Aki and Richards, 1980). The inversion of PP and PS reflection data using the second-order approximations for the PP and PS reflection coefficients is compared to inversion using the linear approximation.

THE TRANSMISSION AND REFLECTION COEFFICIENT MATRICES

We consider a plane interface between two homogeneous isotropic visco-elastic media. Continuity of stress and displacement velocity at the interface gives the 2×2 reflection and transmission coefficient matrices for down-going plane-waves (Ursin, 1983, equation (102)):

$$\mathbf{T}_D = \begin{pmatrix} T_D^{PP} & T_D^{PS} \\ T_D^{SP} & T_D^{SS} \end{pmatrix} = 2(\mathbf{C} + \mathbf{D})^{-1}$$

$$\mathbf{R}_D = \begin{pmatrix} R_D^{PP} & R_D^{PS} \\ R_D^{SP} & R_D^{SS} \end{pmatrix} = (\mathbf{C} - \mathbf{D})(\mathbf{C} + \mathbf{D})^{-1} \quad (1)$$

Here T_D^{PS} and R_D^{PS} denote the R/T coefficients for a down-going S-wave and out-going P-wave. $\mathbf{R}_U$ and $\mathbf{T}_U$ can be computed just by interchanging the superscripts 1 and 2 or by using the symmetry relations (Ursin and Stovas, 2001).

$$\mathbf{R}_U = -\mathbf{R}_D \ , \quad \mathbf{T}_U = \mathbf{T}_D^T \ . \quad (2)$$

The matrices **C** and **D** are defined by

$$\mathbf{C} = [\mathbf{L}_2^{(1)}]^T \mathbf{L}_1^{(2)}$$

$$\mathbf{D} = [\mathbf{L}_1^{(1)}]^T \mathbf{L}_2^{(2)} \ , \quad (3)$$

and where superscripts (1) and (2) denote the upper and lower medium, respectively.

The matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ are given by (Ursin, 1983; Ursin and Stovas, 2001)

$$\mathbf{L}_1 = \begin{pmatrix} \sqrt{\frac{q_\alpha}{\rho}} & \frac{p}{\sqrt{\rho q_\beta}} \\ 2\beta^2 p \sqrt{\rho q_\alpha} & (2\beta^2 p^2 - 1)\sqrt{\frac{\rho}{q_\beta}} \end{pmatrix} ,$$

$$\mathbf{L}_2 = \begin{pmatrix} -(2\beta^2 p^2 - 1)\sqrt{\frac{\rho}{q_\alpha}} & 2p\beta^2\sqrt{\rho q_\beta} \\ \frac{p}{\sqrt{\rho q_\alpha}} & -\sqrt{\frac{q_\beta}{\rho}} \end{pmatrix} \quad (4)$$

where $p = k/\omega$ is horizontal slowness, and ρ is density. The P- and S-wave vertical slownesses q_α and q_β are defined by the dispersion relations: $q_\alpha^2 = (1/\alpha^2) - p^2$ and $q_\beta^2 = (1/\beta^2) - p^2$. The squares of the complex P- and S-wave velocities are $\alpha^2 = (\lambda + 2\mu)/\rho$ and $\beta^2 = \mu/\rho$. For visco-elastic media, the Lame parameters λ and μ are complex functions of ω (Ben-Menahem and Singh, 1981). The choice of sign of the square roots for the vertical slownesses and their square roots are discussed in Ursin and Stovas (2001). The matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ are scaled so that the corresponding vertical energy flux is constant. This gives the following properties

$$\mathbf{L}_2^T \mathbf{L}_1 = \mathbf{L}_1^T \mathbf{L}_2 = \mathbf{I}$$

$$\mathbf{L}_1 \mathbf{L}_2^T = \mathbf{L}_2 \mathbf{L}_1^T = \mathbf{I} \ . \quad (5)$$

For pre-critical reflections we can expand the matrices defining **C** and **D** in second-order Taylor series

$$\mathbf{L}_k^{(1)} = \mathbf{L}_k - (1/2)\Delta\mathbf{L}_k + (1/8)\Delta^2\mathbf{L}_k + \ldots$$
$$\mathbf{L}_k^{(2)} = \mathbf{L}_k + (1/2)\Delta\mathbf{L}_k + (1/8)\Delta^2\mathbf{L}_k + \ldots , \quad k = 1,2 \tag{6}$$

where

$$\Delta\mathbf{L}_k = (\partial\mathbf{L}_k/\partial\mathbf{m}^T)\,\Delta\mathbf{m} \tag{7}$$

is the change in $\mathbf{L}_k$ at the interface and the jump in the elastic parameters are $\Delta\mathbf{m} = \mathbf{m}^{(2)} - \mathbf{m}^{(1)}$. The second-order terms are

$$\Delta^2\mathbf{L}_k = \Delta\mathbf{m}^T\,(\partial^2\mathbf{L}_k/\partial\mathbf{m}\partial\mathbf{m}^T)\,\Delta\mathbf{m} \ . \tag{8}$$

In the Taylor series expansion (6) all terms are evaluated for the average of the elastic parameters at the interface, $\mathbf{m} = (\mathbf{m}^{(1)} + \mathbf{m}^{(2)})/2$.

We can also define the discrete forward- and back-scattering matrices **F** and **G** (Ursin and Stovas, 2001)

$$\mathbf{F} = -(1/2)(\mathbf{L}_2^T\Delta\mathbf{L}_1 + \mathbf{L}_1^T\Delta\mathbf{L}_2) = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix} ,$$
$$\mathbf{G} = -(1/2)(\mathbf{L}_2^T\Delta\mathbf{L}_1 - \mathbf{L}_1^T\Delta\mathbf{L}_2) = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \tag{9}$$

with the following symmetries

$$\mathbf{F}^T = -\mathbf{F} \ , \quad \mathbf{G}^T = \mathbf{G} \ . \tag{10}$$

The elements of **F** and **G** are given by

$$\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \\ f \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{1}{\alpha^2 q_\alpha^2} & -8\beta^2p^2 & 1-4\beta^2p^2 \\ 0 & \frac{4p\beta^2}{\sqrt{q_\alpha q_\beta}}\left(q_\alpha q_\beta - p^2\right) & \frac{p}{\sqrt{q_\alpha q_\beta}}\left(1+2\beta^2\left(q_\alpha q_\beta - p^2\right)\right) \\ 0 & -\frac{1}{\beta^2 q_\beta^2}+8\beta^2p^2 & -1+4\beta^2p^2 \\ 0 & -\frac{4p\beta^2}{\sqrt{q_\alpha q_\beta}}\left(q_\alpha q_\beta + p^2\right) & \frac{p}{\sqrt{q_\alpha q_\beta}}\left(1-2\beta^2\left(q_\alpha q_\beta + p^2\right)\right) \end{pmatrix}\begin{pmatrix} \frac{\Delta\alpha}{\alpha} \\ \frac{\Delta\beta}{\beta} \\ \frac{\Delta\rho}{\rho} \end{pmatrix} , \tag{11}$$

In the Appendix the matrices **C** and **D** which define the R/T coefficients are expanded into second-order Taylor series. This gives the second-order approximations

$$\mathbf{T}_D = \left[\mathbf{I} - \mathbf{F} + \frac{\mathbf{G}^2 + \mathbf{F}^2}{2} + \cdots\right]^{-1} \approx \mathbf{I} + \mathbf{F} + \frac{\mathbf{F}^2 - \mathbf{G}^2}{2}$$
$$= \begin{pmatrix} 1-\frac{1}{2}\left(g_{11}^2+g_{12}^2+f^2\right) & f-\frac{1}{2}g_{12}\left(g_{11}+g_{22}\right) \\ -f-\frac{1}{2}g_{12}\left(g_{11}+g_{22}\right) & 1-\frac{1}{2}\left(g_{12}^2+g_{22}^2+f^2\right) \end{pmatrix} = \mathbf{T}_D^{(2)}$$
$$\mathbf{R}_D = -\left[\mathbf{G} - \frac{\mathbf{FG}+\mathbf{GF}}{2} + \cdots\right]\left[\mathbf{I} - \mathbf{F} + \frac{\mathbf{G}^2+\mathbf{F}^2}{2} + \cdots\right]^{-1} \approx -\mathbf{G} + \frac{\mathbf{FG}-\mathbf{GF}}{2} \ .$$
$$= -\begin{pmatrix} g_{11} - g_{12}f & g_{12}+\frac{1}{2}f\left(g_{11}-g_{22}\right) \\ g_{12}+\frac{1}{2}f\left(g_{11}-g_{22}\right) & g_{22}+g_{12}f \end{pmatrix} = \mathbf{R}_D^{(2)} \tag{12}$$

Due to the symmetries of the matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ the second-order derivatives of these matrices with respect to the medium parameters [$\Delta^2\mathbf{L}_k$ in equation (6)] cancel, and they do not appear in the second-order approximations for the R/T coefficients.

The approximations can be interpreted by a multiple weak-scattering effect. It means that each contribution from g_{ij} or f is related to the act of reflection or transmission on artificial interfaces. The signs of g_{ij} or f take into account the symmetries of the down-going and up-going reflection and transmission responses as given in equation (2). All contributions of higher order are related to multiples.

In Fig. 1 there is a scheme for the interpretation of the second-order contributions for $\Delta t_{DPP}^{(2)}$, $\Delta t_{DPS}^{(2)}$, $\Delta r_{DPP}^{(2)}$ and $\Delta r_{DPS}^{(2)}$. For example the second-order contribution for the up-going PS reflection coefficient is given by [see equation (12)]

$$\Delta r_{DPS}^{(2)} = -(1/2)f(g_{11} - g_{22}) = (1/2)[(-g_{11})(f) + (-f)(-g_{22})] \ . \tag{13}$$

The first-order approximations can be obtained from equation (12) by neglecting the second-order terms

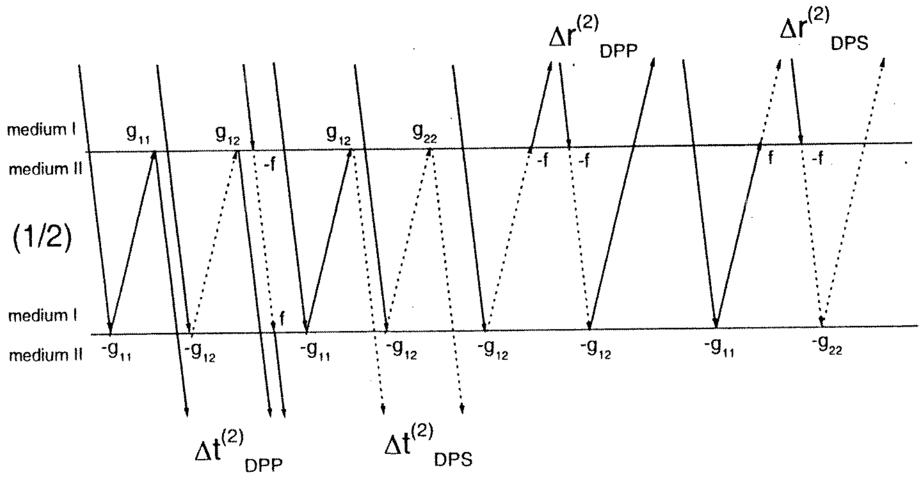

Fig. 1. Contributions from the quadratic terms in the second order approximations for t_{DPP}, t_{DPS}, r_{DPP} and r_{DPS}.

$$\mathbf{T}_D^{(1)} = \mathbf{I} + \mathbf{F} = \begin{pmatrix} 1 & f \\ -f & 1 \end{pmatrix},$$

$$\mathbf{R}_D^{(1)} = -\mathbf{G} = -\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \qquad (14)$$

These are given by Ursin and Stovas (2001), and they are similar to the standard linear approximations in Aki and Richards (1980) except that they are complex and normalized with respect to the vertical energy flux.

NUMERICAL RESULTS

In the numerical calculations we have used the Kolsky-Futterman attenuation law with $Q_\alpha = 150$ and Q_β for all cases, so that there is no change in these parameters across the interface. The approximate R/T coefficients are only valid for pre-critical angles (or horizontal slowness), and then the effect of visco-elasticity is small (Ursin and Stovas, 2001). Consequently, only the real part of the R/T coefficients are shown in the examples.

Table 1. Model parameters ($Q_\alpha = 150$, $Q_\beta = 100$).

	Weak-contrast model			Medium-contrast model			Large-contrast model		
	Upper medium	Lower medium	Con-trast	Upper medium	Lower medium	Con-trast	Upper medium	Lower medium	Con-trast
P-wave velocity (km/s)	3.19	2.92	-0.09	3.37	2.92	-0.14	4.5	2.92	-0.43
S-wave velocity (km/s)	1.53	1.4	-0.09	1.83	1.4	-0.27	2.5	1.4	-0.56
Density (kg/m³)	2350	2180	-0.07	2500	2180	-0.14	2750	2180	-0.23

The reflection and transmission coefficients were computed for three isotropic visco-elastic models with the different contrasts in the medium parameters: a weak-contrast model, a medium-contrast model and a large-contrast model (see Table 1). The exact PP and PS reflection and transmission coefficients as well as approximated ones (equations 12 and 14) are computed for the all models. For the weak-contrast model there is almost no difference between both approximations and the exact solution (Fig. 2). For the medium-contrast model exact PP and PS reflection and transmission coefficients are plotted in Fig. 3. One can see that the second-order approximations give better fit to the exact reflection and transmission coefficients than the standard linear approximation. The results for the large-contrast model shown in Fig. 4 also shows better fit.

The new approximation (12) was also applied to invert the PP and PS reflection data into the relative changes in the medium parameters. For the inversion of non-linear equations for approximated PP and PS reflection coefficients (12) we used an iterative approach. First, the minimum square inversion was done for the linear terms of r_{DPP} and r_{DPS}, and the first value of the vector $(\Delta m_i/m_i)$, $i = 1,2,3$ was obtained, then this value was used to compute f. After that we used f as the known coefficient and applied again the linear inversion for both terms in the expressions for r_{DPP} and r_{DPS}. The results of this iterative non-linear inversion for the large-contrast model are shown in Fig. 5. One can see that the results from the linear inversion (zero iteration) are far from the model data, but after 4-5 iterations they become very close to them. The inversion of the PS data gives worse results than the inversion of the PP data, but nevertheless they are better than ones from the linear inversion. The worst is the estimation of the contrast in P-wave velocity, but this value was not even available from the linear inversion of PS data.

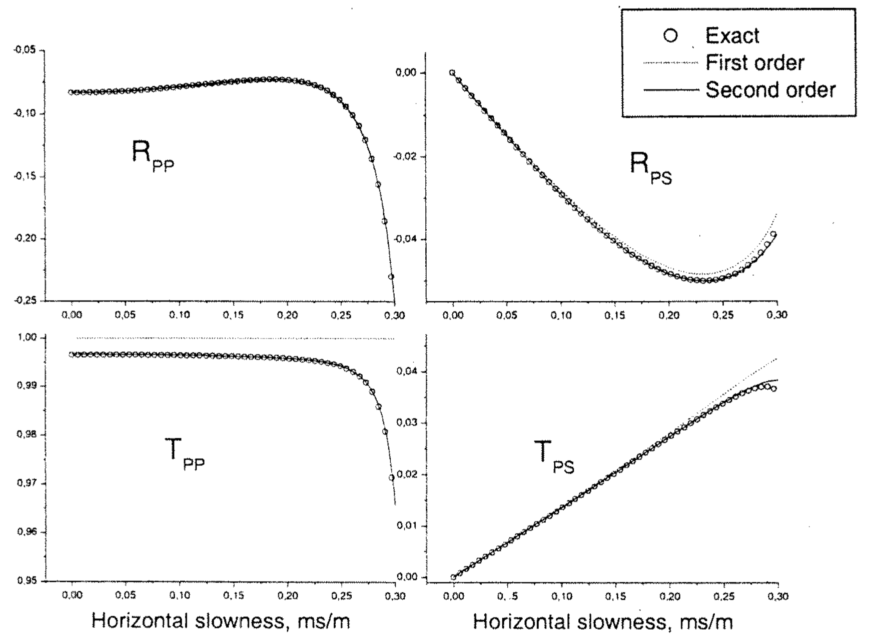

Fig. 2. PP and PS reflection and transmission coefficients for the weak-contrast model.

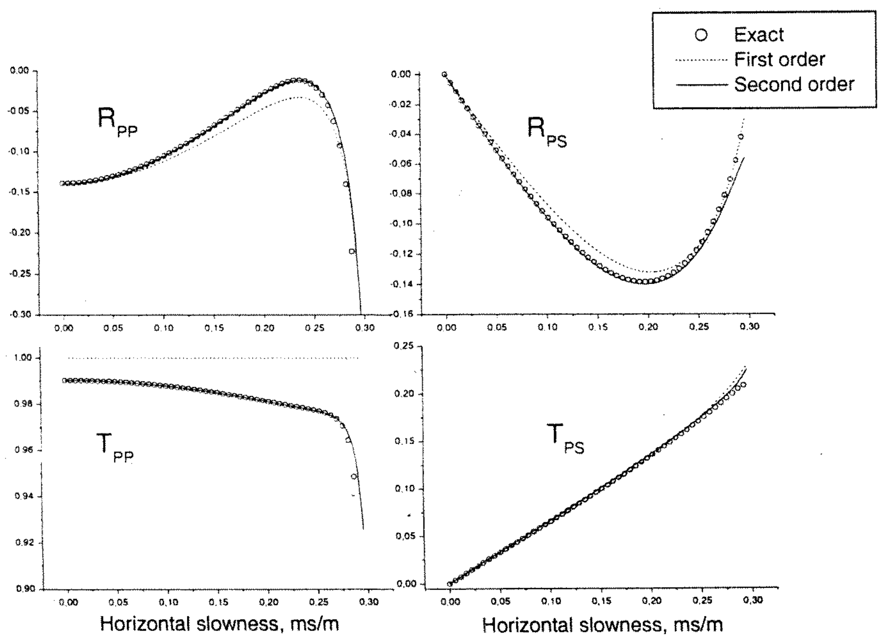

Fig. 3. PP and PS reflection and transmission coefficients for the medium-contrast model.

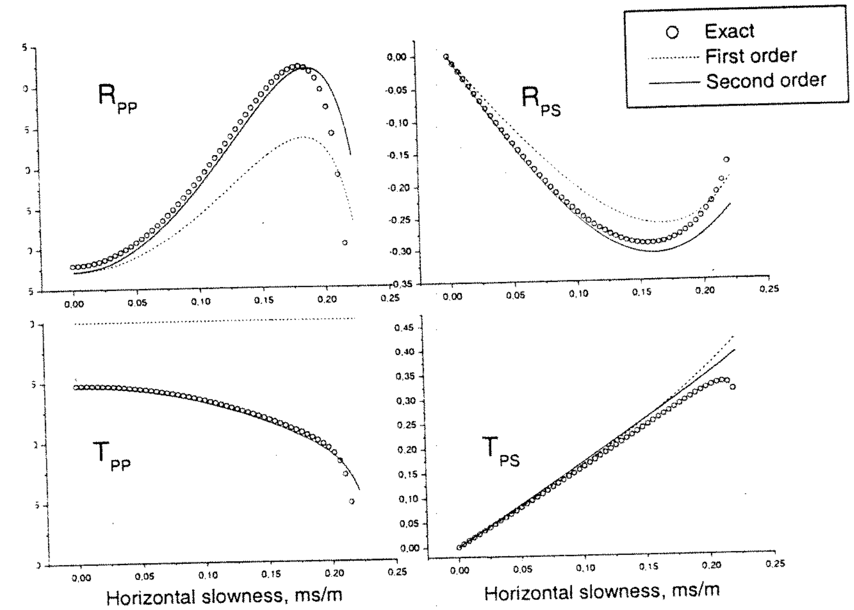

Fig. 4. PP and PS reflection and transmission coefficients for the large-contrast model.

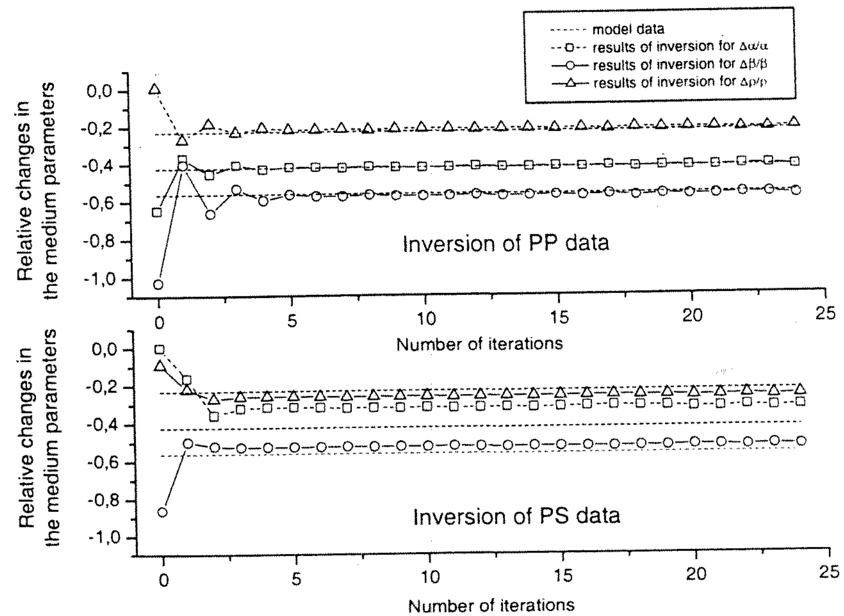

Fig. 5. Inversion of PP and PS reflections for the large-contrast model using the second-order approximations for the PP and PS reflection coefficients.

CONCLUSIONS

We have derived second-order approximations for reflection and transmission coefficients between two isotropic visco-elastic media. They give better fit to the exact reflection and transmission coefficients than the weak-contrast linear approximation, especially when the contrasts in the elastic parameters are not weak. This new approximation can also be useful for the inversion of the reflection data, because all second-order terms are expressed in terms of the first-order derivatives of the eigenvector matrices.

ACKNOWLEDGMENTS

This work has been supported by Norsk Hydro ASA (Bergen) and the European Union under the project "Detection of overpressure zones with seismic and well data".

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APPENDIX

THE SECOND-ORDER TAYLOR SERIES FOR THE MATRICES C AND D

Using the expansions of the matrices $\mathbf{L}_i^{(1)}$ and $\mathbf{L}_i^{(2)}$ from equation (6) we can write the following expression for matrices **C** and **D** [equation (3)]

$$\mathbf{C} = \left[\mathbf{L}_2^{(1)}\right]^T \mathbf{L}_1^{(2)} = \mathbf{L}_2^T\mathbf{L}_1 - \frac{1}{2}\left(\Delta\mathbf{L}_2^T\mathbf{L}_1 - \mathbf{L}_2^T\Delta\mathbf{L}_1\right) + \frac{1}{8}\left(\Delta^2\mathbf{L}_2^T\mathbf{L}_1 - 2\Delta\mathbf{L}_2^T\Delta\mathbf{L}_1 + \mathbf{L}_2^T\Delta^2\mathbf{L}_1\right) + \cdots$$

$$\mathbf{D} = \left[\mathbf{L}_1^{(1)}\right]^T \mathbf{L}_2^{(2)} = \mathbf{L}_1^T\mathbf{L}_2 - \frac{1}{2}\left(\Delta\mathbf{L}_1^T\mathbf{L}_2 - \mathbf{L}_1^T\Delta\mathbf{L}_2\right) + \frac{1}{8}\left(\Delta^2\mathbf{L}_1^T\mathbf{L}_2 - 2\Delta\mathbf{L}_1^T\Delta\mathbf{L}_2 + \mathbf{L}_1^T\Delta^2\mathbf{L}_2\right) + \cdots \quad \text{(A-1)}$$

The similar expressions can be written for the matrix products from equation (5) for the upper medium

$$\left[\mathbf{L}_2^{(1)}\right]^T \mathbf{L}_1^{(1)} = \mathbf{L}_2^T\mathbf{L}_1 - \frac{1}{2}\left(\Delta\mathbf{L}_2^T\mathbf{L}_1 + \mathbf{L}_2^T\Delta\mathbf{L}_1\right) + \frac{1}{8}\left(\Delta^2\mathbf{L}_2^T\mathbf{L}_1 + 2\Delta\mathbf{L}_2^T\Delta\mathbf{L}_1 + \mathbf{L}_2^T\Delta^2\mathbf{L}_1\right) + \cdots = \mathbf{I}$$

$$\left[\mathbf{L}_1^{(1)}\right]^T \mathbf{L}_2^{(1)} = \mathbf{L}_1^T\mathbf{L}_2 - \frac{1}{2}\left(\Delta\mathbf{L}_1^T\mathbf{L}_2 + \mathbf{L}_1^T\Delta\mathbf{L}_2\right) + \frac{1}{8}\left(\Delta^2\mathbf{L}_1^T\mathbf{L}_2 + 2\Delta\mathbf{L}_1^T\Delta\mathbf{L}_2 + \mathbf{L}_1^T\Delta^2\mathbf{L}_2\right) + \cdots = \mathbf{I} \quad \text{(A-2)}$$

From equation (A-2) it follows that, up to second order,

$$\Delta^2\mathbf{L}_2^T\mathbf{L}_1 + \mathbf{L}_2^T\Delta^2\mathbf{L}_1 = -2\Delta\mathbf{L}_2^T\Delta\mathbf{L}_1$$

$$\Delta^2\mathbf{L}_1^T\mathbf{L}_2 + \mathbf{L}_1^T\Delta^2\mathbf{L}_2 = -2\Delta\mathbf{L}_1^T\Delta\mathbf{L}_2 \quad \text{(A-3)}$$

From the definition of matrices **F** and **G** [equation (9)] and equation of symmetries (5) we have the first order (with respect to **F** and **G**) terms

$$\Delta\mathbf{L}_1^T\mathbf{L}_2 = -\mathbf{L}_1^T\Delta\mathbf{L}_2 = \mathbf{F} - \mathbf{G}$$

$$\Delta\mathbf{L}_2^T\mathbf{L}_1 = -\mathbf{L}_2^T\Delta\mathbf{L}_1 = \mathbf{F} + \mathbf{G} \quad \text{(A-4)}$$

and the second order terms

$$\Delta\mathbf{L}_2^T\Delta\mathbf{L}_1 = -(\mathbf{F}+\mathbf{G})(\mathbf{F}+\mathbf{G})$$

$$\Delta\mathbf{L}_1^T\Delta\mathbf{L}_2 = -(\mathbf{F}-\mathbf{G})(\mathbf{F}-\mathbf{G}) \quad \text{(A-5)}$$

Substituting equations (A-3), (A-4) and (A-5) into the equations (A-1), we obtain the second-order expressions for the matrices **C** and **D**

$$\mathbf{C} = \mathbf{I} - (\mathbf{F}+\mathbf{G}) + \frac{(\mathbf{F}+\mathbf{G})(\mathbf{F}+\mathbf{G})}{2} + \cdots$$

$$\mathbf{D} = \mathbf{I} - (\mathbf{F}-\mathbf{G}) + \frac{(\mathbf{F}-\mathbf{G})(\mathbf{F}-\mathbf{G})}{2} + \cdots \quad \text{(A-6)}$$

This gives

$$\mathbf{C}+\mathbf{D} = 2\left[\mathbf{I} - \mathbf{F} + \frac{\mathbf{F}^2+\mathbf{G}^2}{2} + \cdots\right]$$

$$\mathbf{C}-\mathbf{D} = 2\left[-\mathbf{G} + \frac{\mathbf{FG}+\mathbf{GF}}{2} + \cdots\right] \quad \text{(A-7)}$$