A UNIFIED BORN-KIRCHHOFF REPRESENTATION FOR ACOUSTIC MEDIA

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ABSTRACT

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For the modeling of a single-target reflector in a smooth inhomogeneous elastic anisotropic medium, it has been recently shown that the volume Born integral can be transformed into a surface-scattering integral on the reflector. This derived surface integral has been called Born-Kirchhoff as it relates very naturally to the Kirchhoff-Helmholtz integral, thus providing the theoretical link between the two approaches. Here we specify the derivation and main properties of the Born-Kirchhoff integral in the acoustic case, and use a simple synthetic example to provide a comparison between the new integral and its classical counterparts.

KEY WORDS: Born, Helmholtz-Kirchhoff, Born-Kirchhoff, reciprocity.

INTRODUCTION

The Born (volume) and Kirchhoff-Helmholtz (surface) representation integrals are the most widely used descriptions of reflected and transmitted wavefields due to smooth interfaces (see, e.g., Bleistein, 1984; Frazer and Sen, 1985; Lumley and Beydoun, 1993; Wapenaar and Berkhout, 1993; Chapman and Coates, 1994; Druzhinin, 1994; Tygel et al., 1994).

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Although representing basically the same phenomena, the two integrals result from quite independent formulations, and are traditionally kept as completely separate objects. Moreover, besides their fundamental distinction as volume and surface integrals (Wapenaar and Berkhout, 1993), the representations of Born and Kirchhoff-Helmholtz present also other differences, namely (a) Born assumes weak medium perturbations, uses a linearized scattering coefficient and the resulting integral is reciprocal and (b) Kirchhoff-Helmholtz imposes no contrast restrictions for the medium inhomogeneities, approximates the reflected field and its normal derivative on the reflector using the plane-wave reflection coefficient and the incident field, and the resulting integral is nonreciprocal.

Under the application of a generalized form of the divergence theorem as presented in Bleistein (1984), we follow the lines of Spencer et al. (1995), de Hoop and Bleistein (1996) and Ursin and Tygel (1997) to transform the Born volume integral into a corresponding surface-scattering integral. This new integral, called the Born-Kirchhoff integral by Ursin and Tygel (1997) in the context of elastic, anisotropic media, provides the natural theoretical link between the Born and the Kirchhoff-Helmholtz representations.

In this work, we provide a quick derivation of the Born-Kirchhoff integral for the case of acoustic inhomogeneous media, as well as summarizing its main properties. Furthermore, we examine its application to a simple synthetic example to compare the obtained results with those corresponding to its classical counterparts. We also include into the comparison a modified, reciprocal, Kirchhoff-Helmholtz integral introduced by Deregowski and Brown (1983).

FORMULATION OF THE PROBLEM

Referring to Fig. 1, we consider two unbounded, inhomogeneous acoustic media, separated by a smooth interface Σ. We also consider a reference medium characterized by a smooth compression modulus $k(\mathbf{x})$ and smooth mass density $\rho(x)$, where $\mathbf{x} = (x,y,z)$ denotes the location vector in a fixed, global Cartesian coordinate system. The model parameters of the upper medium coincide with those of the reference medium. The lower medium has perturbed parameters $k(\mathbf{x}) + \Delta k(x)$ and $\rho(\mathbf{x}) + \Delta\rho(\mathbf{x})$. The total acoustic pressure due to a point source located at $\mathbf{x}^s = (x^s,y^s,z^s)$ in the upper medium, is denoted in the frequency domain by $P(\mathbf{x},\omega;\mathbf{x}^s)$. It satisfies the acoustic wave equation

$$\nabla\cdot[\{1/\tilde{\rho}(\mathbf{x})\}\nabla P(\mathbf{x},\omega;\mathbf{x}^s)] + \{\omega^2/\tilde{k}(\mathbf{x})\}P(\mathbf{x},\omega;\mathbf{x}^s) = -F(\omega)\delta(\mathbf{x}-\mathbf{x}^s) \quad , \qquad (1)$$

where $F(\omega)$ is the source function and ω is the angular frequency. Moreover, $\tilde{\rho}$ and $\tilde{k}$ denote the acoustic parameters for the actual medium

$$\begin{cases} \tilde{\rho}(\mathbf{x}) = \rho(\mathbf{x}) \,, & \tilde{k}(\mathbf{x}) = k(\mathbf{x}) \,, \quad \text{upper medium} \,, \\ \tilde{\rho}(\mathbf{x}) = \rho(\mathbf{x}) + \Delta\rho(\mathbf{x}) \,, & \tilde{k}(\mathbf{x}) = k(\mathbf{x}) + \Delta k(\mathbf{x}) \,, \quad \text{lower medium} \,. \end{cases} \qquad (2)$$

For observation points in the upper medium, the total pressure field can be decomposed into the superposition

$$P(\mathbf{x},\omega;\mathbf{x}^s) = P^I(\mathbf{x},\omega;\mathbf{x}^s) + P^R(\mathbf{x},\omega;\mathbf{x}^s) \quad , \qquad (3)$$

where P^I is the *incident wavefield* and P^R is the *scattered or reflected wavefield.* Moreover, the incident wavefield is simply

$$P^I(\mathbf{x},\omega;\mathbf{x}^s) = F(\omega)G^s(\mathbf{x},\omega) \quad , \qquad (4)$$

where $G^s(\mathbf{x},\omega)$ is the Green's function in the reference medium (which coincides with the upper medium) for a point source $\mathbf{x}^s$ and observed at $\mathbf{x}$. It satisfies

$$\nabla\cdot[\{1/\rho(\mathbf{x})\}\nabla G^s(\mathbf{x},\omega)] + \{\omega^2/k(\mathbf{x})\}G^s(\mathbf{x},\omega) = -\delta(\mathbf{x}-\mathbf{x}^s) \quad . \qquad (5)$$

It is our aim to discuss various approximate integral representations for the reflected pressure $P^R(\mathbf{x},\omega;\mathbf{x}^s)$ at a given receiver position $\mathbf{x}^r = (x^r,y^r,z^r)$. We first briefly review the two classical ones of Born (volume integral) and Kirchhoff-Helmholtz (surface integral). We next present two alternative representations, both given as surface integrals.

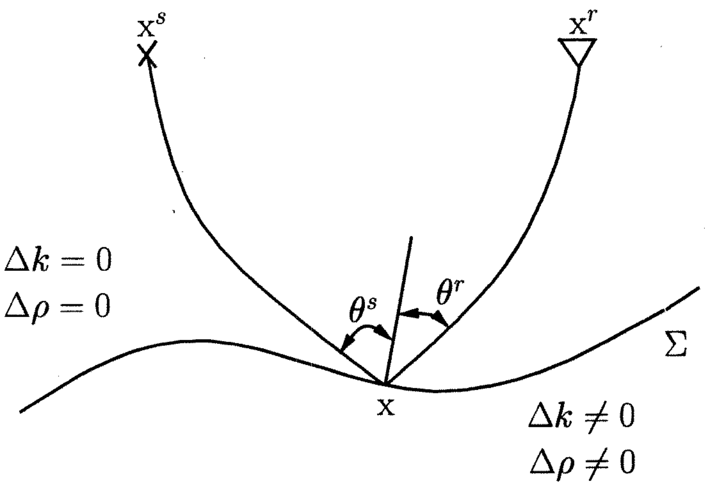

Fig. 1. Two smoothly inhomogeneous, acoustic media separated by a smooth curved interface.

BORN APPROXIMATION

A simple application of Gauss' theorem to the total pressure $P = P(\mathbf{x},\omega;\mathbf{x}^s)$ and Green's function due to a point source placed at the receiver location $G^r = G^r(\mathbf{x},\omega)$, leads to the, at this point exact, representation

$$P^R(\mathbf{x}^r,\omega;\mathbf{x}^s) = F(\omega) \int_\Omega dV[\{\Delta\rho/\rho(\rho+\Delta\rho)\}\nabla P\cdot\nabla G^r - \{\omega^2\Delta k/k(k+\Delta k)\}PG^r] \ . \qquad (6)$$

Here, the domain of integration Ω is the region of nonzero perturbations, namely the lower medium. Making use of the assumptions of low contrasts $|\Delta\rho/\rho| \ll 1$ and $|\Delta k/k| \ll 1$, as well as of weak scattering $|P^R| \ll |P^I|$ and $|\nabla P^R| \ll |\nabla P^I|$, the familiar Born representation (see, e.g., Wapenaar and Berkhout, 1993) can be obtained

$$P^R(\mathbf{x}^r,\omega;\mathbf{x}^s) \approx P^B(\mathbf{x}^r,\omega;\mathbf{x}^s)$$

$$= F(\omega) \int_\Omega dV[(\Delta\rho/\rho^2)\nabla G^s\cdot\nabla G^r - \omega^2(\Delta k/k^2)G^sG^r] \ . \qquad (7)$$

As is easily verified, the resulting approximation P^B is reciprocal.

KIRCHHOFF-HELMHOLTZ APPROXIMATION

To obtain the Kirchhoff-Helmholtz approximation of the scattered field, we restrict ourselves to the upper medium. In this medium, we consider the Kirchhoff-Helmholtz representation of the scattered field as a function of this field and its normal derivative, as well as the corresponding values of the Green's function on the boundary Σ (see, e.g., Wapenaar and Berkhout, 1993), namely

$$P^R(\mathbf{x}^r,\omega;\mathbf{x}^s) = \int_\Sigma dS(1/\rho)[G^r(\partial P^R/\partial\eta) - P^R(\partial G^r/\partial\eta)] \ . \qquad (8)$$

We next substitute the unknown values of the reflected field P^R and its normal derivative $\partial P^R/\partial\eta$ at points $\mathbf{x}$ in Σ, using the celebrated Kirchhoff approximations (see, e.g., Ben-Menahem and Beydoun (1985) for discussions on validity conditions)

$$P^R \approx R(\theta^s)P^I \ , \quad \partial P^R/\partial\eta \approx -R(\theta^s)(\partial P^I/\partial\eta) \ , \qquad (9)$$

where $R(\theta^s)$ is the reflection coefficient of the incidence field (approximated as a plane wave) with respect to the reflector boundary (approximated by its tangent plane) at the reflector point $\mathbf{x}$. Also, θ^s is the angle between these two planes. Substituting relations (9) into equation (8), and also taking into account equation (4), we obtain Kirchhoff-Helmholtz's surface integral approximation

$$P^R(\mathbf{x}^r,\omega;\mathbf{x}^s) \approx P^{KH}(\mathbf{x}^r,\omega;\mathbf{x}^s)$$

$$= F(\omega) \int_\Sigma dS[R(\theta^s)/\rho][G^r(\partial G^s/\partial\eta) + G^s(\partial G^r/\partial\eta)] \ . \qquad (10)$$

Note that $R(\theta^s)$ depends on the reflector geometry and the resulting approximation P^{KH} is nonreciprocal.

HIGH-FREQUENCY RAY APPROXIMATION

The high-frequency, zero-order, ray approximation of the Green's function G and its gradient ∇G can be written (Červený, 1995)

$$G^j(x,\omega) = A^j(x)\exp[i\omega T^j(x)] \ , \quad \nabla G^j(x,\omega) = i\omega\nabla T^j(x)G^j(x,\omega) \ , \qquad (11)$$

where $A^j(x) = A(x;x^j)$ is the amplitude factor and $T^j = T(x,x^j)$ is the traveltime along the ray segments x and x^j, in which j = s (source) or j = r (receiver). In view of the eikonal equation for the traveltime, we can readily find the relationships

$$(\partial G^j/\partial\eta) = (\cos\theta^j/v)G^j \ , \quad \nabla G^s\cdot\nabla G^r = -\omega^2(\cos\theta^{rs}/v^2)G^sG^r \ , \qquad (12)$$

where, once again j = s,r, $v = \sqrt{(k/\rho)}$ is the velocity of the medium at the point x and $\theta^{rs} = \theta^s + \theta^r$. As a consequence of the above relationships, the high-frequency ray approximation corresponding to the Born and Kirchhoff-Helmholtz representation integrals (7) and (10) become, after some simple algebra,

$$P^B(\mathbf{x}^r,\omega;\mathbf{x}^s) = -\omega^2F(\omega)\int_\Omega dV(1/k)[(\Delta k/k) + (\Delta\rho/\rho)\cos\theta^{rs}]G^sG^r \ , \qquad (13)$$

and

$$P^{KH}(\mathbf{x}^r,\omega;\mathbf{x}^s) = i\omega F(\omega)\int_\Sigma dS[R(\theta^s)/\rho][(\cos\theta^s + \cos\theta^r)/v]G^sG^r \ . \qquad (14)$$

The Born integral (13) can be given in a more appealing form upon the introduction of the weak-contrast, linearized reflection acoustic coefficient [see, e.g., Aki and Richards (1980), equation (2.39)]

$$R_L(\theta) = [(\Delta k/k) + (\Delta\rho/\rho)\cos 2\theta][1/4\cos^2\theta] \ . \qquad (15)$$

In fact, setting $\theta = \theta^{rs}/2$, we readily obtain for the Born integral the alternative expression

$$P^B(\mathbf{x}^r,\omega;\mathbf{x}^s) = -\omega^2F(\omega)\int_\Omega dV[R_L(\theta^{rs}/2)/\rho][4\cos^2(\theta^{rs}/2)/v^2]G^sG^r \ . \qquad (16)$$

BORN-KIRCHHOFF INTEGRAL

The above volume Born integral (16) will now be transformed, still within the high-frequency ray approximation, into a surface integral which is very similar to the ray Kirchhoff-Helmholtz integral (14). The procedure is based on the fact that for any sufficiently regular functions f(x) and g(x), we have

$$\nabla\cdot[f e^{i\omega g}\nabla g] \approx i\omega f \|\nabla g\|^2 e^{i\omega g} \ , \quad \nabla[f e^{i\omega g}] \approx i\omega f e^{i\omega g}\nabla g \ , \tag{17}$$

these approximations being valid for the leading order of frequency ω and whenever $\nabla g \neq 0$. Application of the divergence theorem, together with the use of the above approximations and the assumption that the set of zeros of ∇g has zero measure, yields the useful result [Bleistein (1984), equation (2.82)]

$$-\omega^2 \int_V dV\, f \|\nabla g\|^2 e^{i\omega g} \approx i\omega \int_{\partial V} dS\, f(\partial g/\partial \eta) e^{i\omega g} \ . \tag{18}$$

In view of equation (11) we can readily observe that the integral in equation (16) may be recast as

$$P^B(\mathbf{x}^r,\omega;\mathbf{x}^s) = -\omega^2 F(\omega) \int_\Omega dV\, f \|\nabla g\|^2 e^{i\omega g} \ , \tag{19}$$

with

$$f = R_L(\theta^{rs}) A^r A^s \ , \quad \text{and} \quad g = T^s + T^r \ , \tag{20}$$

from which

$$\|\nabla g\|^2 = (4/v^2)\cos^2(\theta^{rs}/2) \ . \tag{21}$$

Applying the high-frequency form of the divergence theorem (18) to the ray Born integral (16), we find what we call the Kirchhoff-Born integral (see Tygel and Ursin, 1997)

$$P^{BK}(\mathbf{x}^r,\omega;\mathbf{x}^s) \approx i\omega F(\omega) \int_\Sigma dS [R_L(\theta^{rs}/2)/\rho][(\cos\theta^s + \cos\theta^r)/v] G^s G^r \ . \tag{22}$$

A simple comparison between the Born-Kirchhoff integral (22) and its classical Kirchhoff-Helmholtz counterpart (14) shows that they only differ by the reflection coefficient and incident angle employed. For each point on the reflector's surface, the Born-Kirchhoff integral utilizes the weak-contrast, linearized, plane-wave reflection coefficient computed for half the angle between the two ray segments that join this point to the fixed source- and -receiver pair. For the same point on the reflector, the Kirchhoff-Helmholtz integral uses the full plane-wave reflection coefficient at the angle between the ray segment that joins this point to the fixed source and the surface normal. It is clear that both angles are equal when the surface point is a specular reflection point. It is to be noted that Deregowski and Brown (1983), concerned with the non-reciprocity

of the classical Kirchhoff-Helmholtz integral, have heuristically proposed the introduction of the half-total source-receiver angle θ^{rs} instead of the incident angle θ^s in the computation of the plane-wave reflection coefficient. We call this approximation Reciprocal Kirchhoff-Helmholtz. Our results show that, at least under the weak-scattering condition, the Reciprocal Kirchhoff-Helmholtz approximation can be well justified.

NUMERICAL EXPERIMENT

To illustrate the four different integral representations discussed in this work, we provide a simple synthetic experiment. The model is shown in Fig.2. It consists of a smooth interface, separating two homogeneous halfspaces with velocities of 3.0 km/s in the upper medium and 3.5 km/s in the lower one. The density in both media is constant and equal to unity. For this model, we simulated a common-shot experiment, with the ray family also shown in Fig. 2.

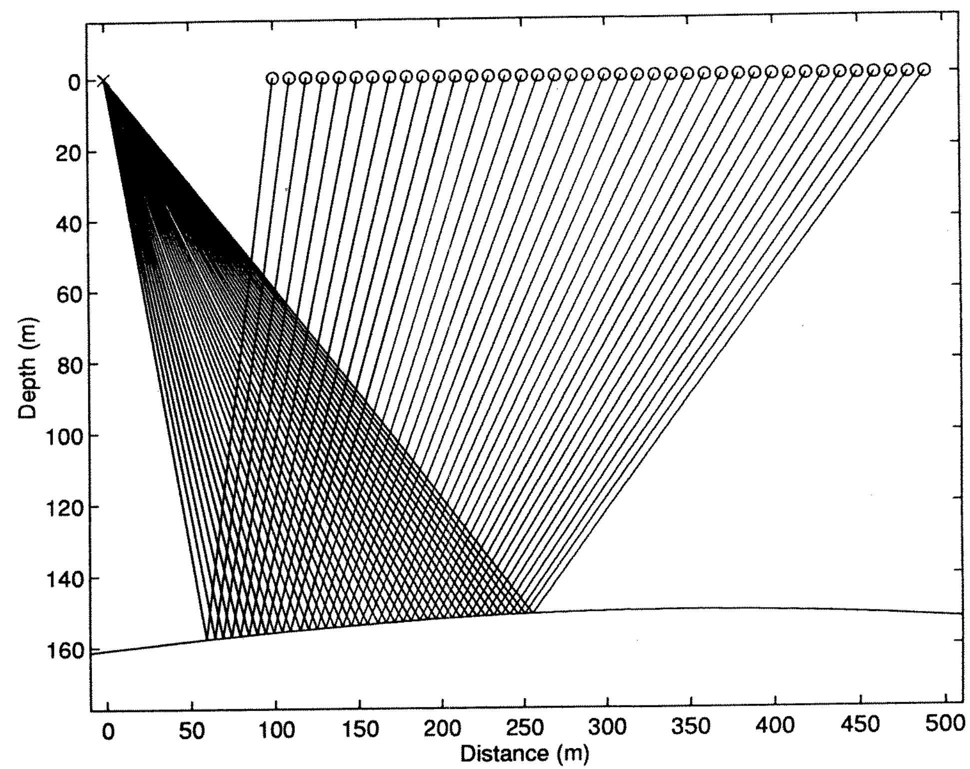

Fig. 2. 2.5-D acoustic model for the experiments.

In Fig. 3, the common-shot section resulting from (a) finite-differences (second-order in time and space) has been compared with the corresponding sections obtained using (b) Kirchhoff-Helmholtz, (c) Reciprocal Kirchhoff-Helmholtz, (d) Born and (e) Born-Kirchhoff representations. To compute the integrals in all cases, we used the trapezoidal rule with a uniform spatial grid $\Delta x = \Delta y = \Delta z = 10$ m. For the point source we have chosen a Küpper wavelet with $\Delta t = 1$ ms.

For a more quantitative analysis, the peak amplitudes along the reflections have been picked for all seismograms. These are shown in Fig. 4. We see that for near offsets the results are quite the same. All approximations systematically overestimate the amplitudes. For farther offsets there are significant differences.

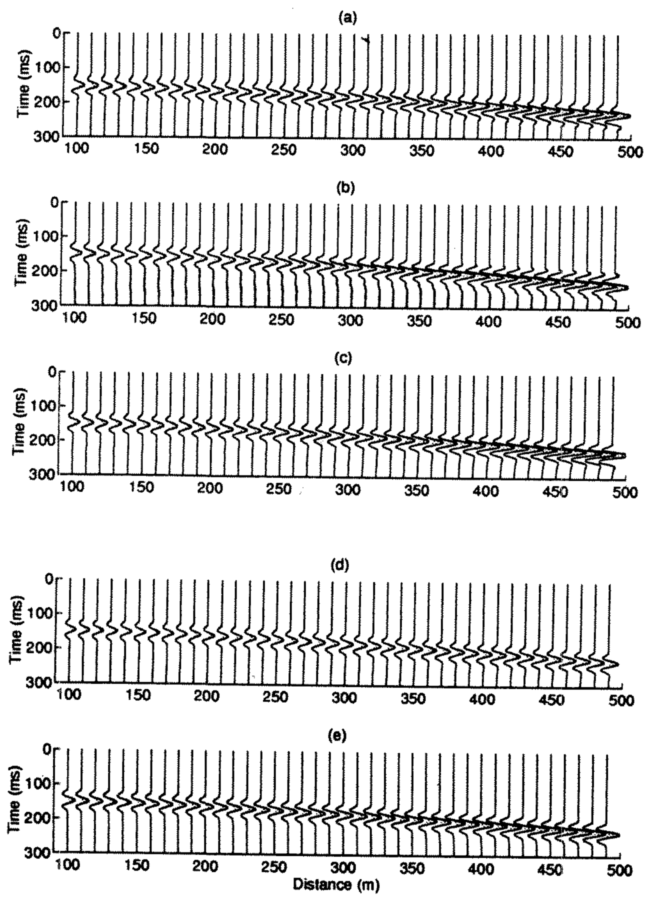

Fig. 3. Synthetic seismograms: (a) Finite-differences, (b) Kirchhoff-Helmholtz, (c) Reciprocal Kirchhoff-Helmholtz, (d) Born and (e) Born-Kirchhoff.

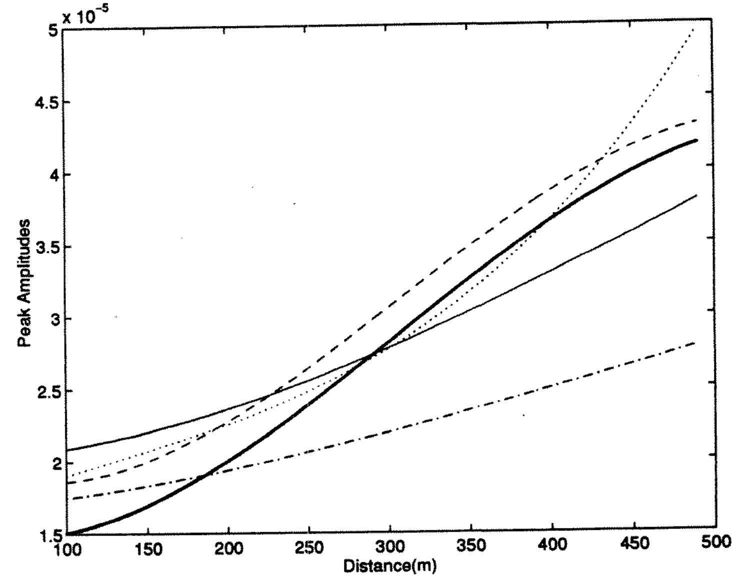

Fig. 4. Peak amplitudes: Finite-differences (bold line), Kirchhoff-Helmholtz (dashed line), Reciprocal-Kirchhoff (dotted line), Born (dash-dotted line) and Born-Kirchhoff (solid line).

These might be due to the use of the trapezoidal rule for the summation, since the integrals considered here are oscillatory and may deserve a better quadrature rule (Frazer, 1988). Nevertheless, it is important to mention that only the Kirchhoff approximation gave reasonable results for the farthest offsets, where the reflections are postcritical. The Reciprocal-Kirchhoff gave a far too high amplitude. We do not have any theoretical or numerical explanation for this behaviour. More systematic testing would be very welcome to fully understand the results obtained upon the use of the various integral representations. This, however, has not been the aim of the present work.

SUMMARY

We have investigated the relationship between the Born volume representation integral and its counterpart Kirchhoff-Helmholtz surface integral for the model of two acoustic inhomogeneous media separated by a curved reflector. Following similar results recently provided by Ursin and Tygel (1997) for elastic, anisotropic media, we have transformed the Born volume integral into a surface Born-Kirchhoff integral of the same form as the classical Kirchhoff-Helmholtz integral. This integral provides the desired link between the two approaches. Still another representation, the Reciprocal-Kirchhoff surface integral has been presented, following a heuristic suggestion of Deregowski and Brown (1983).

For a simple model of a smooth reflector between two homogeneous acoustic media, we have computed the different seismograms corresponding to the described four integral representations. For the same model, we also computed, as a reference, the corresponding seismogram using a central-time and central-space second-order, finite-differences scheme. Some discussion and comments on the obtained results were also provided.

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