Vertical propagation of low-frequency waves in finely layered media

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ABSTRACT

Multiple scattering in finely layered sediments is important for interpreting stratigraphic data, matching well-log data with seismic data, and seismic modeling. Two methods have been used to treat this problem in seismic applications: the O'Doherty-Anstey approximation and Backus averaging. The O'Doherty-Anstey approximation describes the stratigraphic-filtering effects, while Backus averaging defines the elastic properties for an effective medium from the stack of the layers. It is very important to know when the layered medium can be considered as an effective medium. In this paper, we only investigate vertical propagation. Therefore, no anisotropy effect is taken into consideration. Using the matrix-propagator method, we derive equations for transmission and reflection responses from the stack of horizontal layers. From the transmission response, we compute the phase velocity and compare the zero-frequency limit with the effective-medium velocity from Backus averaging. We also investigate how the transition from time-average medium to effective medium depends on contrast; i.e., strength of the reflection-coefficient series. Using numerical examples, we show that a transition zone exists between the effective medium (low-frequency limit) and the time-average medium (high-frequency limit), and that the width of this zone depends on the strength of the reflection-coefficient series.

INTRODUCTION

In seismic interpretation, matching well-log data with seismic data and seismic modeling requires the relating of wave-velocity measurements at a scale of tens of meters to velocity measurements at a scale of centimeters. Borehole logs show earth layering on scales down to a few centimeters. Wave propagation through a finely layered medium is dispersed and attenuated (O'Doherty and Anstey, 1971; Burridge and Chang, 1989).

Shapiro et al. (1996) and Shapiro and Treitel (1997) provided generalized O'Doherty-Anstey formulas for randomly multilayered 1D media. This problem was first studied in the classical paper by O'Doherty and Anstey (1971), but the original study was limited to single and double scattering.

In the infinite-wavelength limit, finely layered media can be regarded as an effective homogeneous medium (Bruggerman, 1937; Backus, 1962). Folstad and Schoenberg (1992) investigated models with different layer thicknesses and concluded that fine layering of the order of one-tenth of the smallest wavelength effectively could be regarded as a homogeneous medium. Shapiro and Treitel (1997) showed that the classical O'Doherty-Anstey approximation can be derived in a purely deterministic way from the reflection-coefficient series. The theoretical estimate of the error in making this approximation is given by Berlyand and Burridge (1995). In Shapiro et al. (1996) and Shapiro and Hubral (1999), the O'Doherty-Anstey approximation was extended to calculate both amplitudes and phase factors in random media.

The wave-propagation velocity strongly depends on the ratio of the dominant wavelength to the typical layer thickness λ/d. When the wavelength is large compared to the layer thickness, the wave velocity is given by an average of the properties of individual layers (Backus, 1962), and waves behave as if propagating in an effective-anisotropic homogeneous medium (Helbig, 1984). In contrast, when the wavelength is small compared to the layer thickness, waves can be described as rays with wave velocities larger than that of the effective medium, although a velocity in an individual layer may exceed the effective velocity. For intermediate values λ/d waves are generally dispersive and velocities change rapidly with frequency (Rio et al., 1996).

Several attempts have been made to establish the minimum value of λ/d for which effective-medium theory is still valid. For periodically layered media, Helbig (1984) concludes that the minimum value of λ/d is larger than 3 for SH-wave propagation, while Melia and Carlson (1984) found from laboratory experiments on periodically layered media that the minimum value of λ/d lies be-

tween 10 and 100, depending on material properties and layer thicknesses. Marion et al. (1994) and Rio et al. (1995) performed laboratory experiments and found that the minimum value of λ/d lies in the range between 8 and 15. Carcione et al. (1991) found from numerical experiments that for periodically layered media, the minimum value of λ/d depends on the reflection coefficients of the medium. Folstad and Schoenberg (1992) concluded from numerical experiments that the minimum value of λ/d was approximately 10 in a randomly layered medium. Hovem (1995) suggested that for periodically layered media, the minimum value of λ/d strongly depends on the impedance of the layers. Thus, the region of validity of the effective-medium theory still is not defined clearly.

We use the propagator-matrix method (Hovem, 1995) to derive transmission and reflection responses for vertical-wave propagation through the stack of layers. The result is similar to that obtained by Shapiro and Treitel (1997), but we do not use the Goupillaud model in our derivation. We also obtain an equation for phase velocity in the weak-contrast and zero-frequency limit to compare with the effective velocity from Backus averaging. We show that the zero-frequency-limit phase velocity computed from the matrix-propagator method is different from the vertical velocity computed from Backus averaging. We show that an O'Doherty-Anstey type approach also can be used to approximate phase velocity.

In this paper, we study vertical-wave propagation in a plane-layered medium and show that the region of validity of effective-medium theory depends on the values of reflection coefficients. The minimum value of λ/d, for which effective-medium theory is still valid, tends to increase with increasing reflection coefficients.

In the following section, we derive simple approximate expressions for transmission amplitude, phase velocity, and attenuation for waves propagating in finely layered media. The expression for transmission amplitude is similar to the well known O'Doherty and Anstey (1971) expression, but we derived ours in a purely deterministic way with no statistical assumption. In this respect, our result is similar to that derived in Shapiro and Treitel (1997) but without the limitations of the Goupillaud model.

In the section on numerical results, we show that the phase velocity given by the O'Doherty and Anstey formula approximates surprisingly well the exact numerical calculation. In this section, we also compute the minimum ratio of dominant wavelength to typical layer thickness as a function of reflectivity contrast and give the region of validity for effective-medium theory and time-average theory.

THE TRANSMISSION RESPONSE FROM A STACK OF LAYERS

To compute transmission and reflection responses from a stack of plane layers, we use the propagator-matrix method (Haskell, 1953; Kennett, 1983). The propagator matrix $\mathbf{Q}$ for N layers (with layer thickness d_j, velocity v_j, and density ρ_j for the jth layer) is the product of N elementary matrices (Appendix A), as shown by Hovem (1995) with

$$\mathbf{Q}_N = \begin{pmatrix} A_N & B_N \\ B_N^* & A_N^* \end{pmatrix} = \prod_{j=1}^{N} \mathbf{Q}_j, \qquad (1)$$

where the asterisk denotes complex conjugate. The elements are given by

$$A_N = \frac{e^{i\vartheta_N}}{\prod_{j=1}^{N}(1-r_j)}\left[1 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j e^{-2i(\vartheta_j-\vartheta_k)} + \ldots\right] \qquad (2)$$

and

$$B_N = \frac{e^{i\vartheta_N}(\sum_{j=1}^{N} r_j e^{-2i(\vartheta_N-\vartheta_j)} + \ldots)}{\prod_{j=1}^{N}(1-r_j)}, \qquad (3)$$

where $(+ \ldots)$. remains for higher-order multiple terms. The cumulated phase functions are

$$\vartheta_k = \sum_{j=1}^{k} \theta_j, \qquad (4)$$

and r_j is the reflection coefficient at the bottom of layer j, and $\theta_j = \omega d_j/v_j$.

The determinant of the total propagator matrix is given by

$$\det \mathbf{Q}_N = |A_N|^2 - |B_N|^2 = \prod_{j=1}^{N} \det \mathbf{Q}_j = \prod_{j=1}^{N} \frac{1+r_j}{1-r_j}. \quad (5)$$

Note that for a periodic medium, $\det \mathbf{Q}_N = 1$; therefore, $\det \mathbf{Q}_N$ can be used as the characteristic of periodicity.

The transmission response (Appendix A) is given by

$$\begin{aligned} t_D^{(N)} &= \frac{e^{i\vartheta_N}\prod_{k=1}^{N}(1-r_k)}{[1 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j e^{2i(\vartheta_j-\vartheta_k)} + \ldots]} \\ &= \frac{e^{i\vartheta_N}\prod_{k=1}^{N}(1-r_k)}{[1 + \Phi]} \\ &= |t_D^{(N)}|e^{i\varphi_N}. \end{aligned} \qquad (6)$$

The transmission amplitude

$$|t_D^{(N)}| = \frac{\prod_{k=1}^{N}(1-r_k)}{|1 + \Phi|} \qquad (7)$$

consists of two terms: $\prod_{k=1}^{N}(1-r_k)$, which is responsible for attenuation because of transmission, and $|1 + \Phi|^{-1}$, which accounts for attenuation due to scattering.

The transmission phase

$$\varphi_N = \vartheta_N - \tan^{-1}\left(\frac{\operatorname{Im}\Phi}{1 + \operatorname{Re}\Phi}\right) \qquad (8)$$

also consists of two terms: the time-average term ϑ_N and the scattering term Φ. The function Φ, which is responsible for scattering, may be considered the correlation function for the reflection-coefficient series $r_j, j = 1, N$, and is given by

$$\Phi = \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j e^{2i(\vartheta_j - \vartheta_k)} + \; \ldots \; . \tag{9}$$

The reflection response (Appendix A) is given by

$$r_D^{(N)} = \frac{e^{2i\vartheta_N}(\sum_{j=1}^{N} r_j e^{-2i(\vartheta_N - \vartheta_j)} + \; \ldots \;)}{[1 + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j e^{2i(\vartheta_j - \vartheta_k)} + \; \ldots \;]}$$

$$= \frac{e^{2i\vartheta_N}(\sum_{j=1}^{N} r_j e^{-2i(\vartheta_N - \vartheta_j)} + \; \ldots \;)}{[1 + \Phi]}. \tag{10}$$

The phase velocity associated with transmission through the layers can be computed from equation 8 and is given by

$$\frac{1}{V(\omega)} = \frac{1}{\omega D} \tan^{-1}\left(\frac{\mathrm{Im}\, t_D}{\mathrm{Re}\, t_D}\right)$$

$$= \frac{1}{V_{TA}} - \frac{1}{\omega D} \tan^{-1}\left(\frac{\mathrm{Im}\, \Phi}{1 + \mathrm{Re}\, \Phi}\right)$$

$$= \frac{1}{V_{TA}} - \frac{1}{\omega D} \tan^{-1}\left(\frac{\sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j \sin 2(\vartheta_j - \vartheta_k) + \ldots}{1 + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j \cos 2(\vartheta_j - \vartheta_k) + \ldots}\right), \tag{11}$$

where $V_{TA} = \omega D / \vartheta_N$ is the time-average velocity, and D is the total thickness of the stack. The zero-frequency limit of equation 11 is given by

$$\frac{1}{V_0} = \lim_{\omega \to 0} \frac{1}{V(\omega)} = \frac{1}{V_{TA}}$$

$$- \frac{2}{D} \frac{\sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j (\tau_j - \tau_k) + \ldots}{1 + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j + \ldots}, \tag{12}$$

where $\tau_j = \vartheta_j / \omega$ is one-way propagation times. From equation 12, we can see that the phase velocity's zero-frequency limit can be either larger or smaller than the time-average velocity (high-frequency limit).

THE WEAK-CONTRAST APPROXIMATION VERSUS THE O'DOHERTY-ANSTEY APPROXIMATION

The scattering function Φ contains an infinite number of even-power correlation functions related to internal-multiple legs. The reflection term in the denominator of equation 10 can be interpreted similarly but with an odd number of sums in each term. In the weak-contrast approximation, we assume that the reflection coefficients are very small ($|r_j| \ll 1$); therefore, we can neglect higher-order terms both in the scattering function and in the reflection term. In practice, this means that in all aforementioned equations, we skip the terms hidden under the $+\ldots$.

To a certain degree, the O'Doherty-Anstey (ODA) type approximation ($1 + \Phi \approx e^{\Phi}$) reconstructs the neglected terms because the exponential function has an infinite number of terms in its Taylor

series. The transmission amplitude of equation 7 reduces to the well-known O'Doherty-Anstey formula (Appendix B) with

$$\left|t_D^{(N)}\right| = e^{-\sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j \cos 2(\vartheta_j - \vartheta_k)} \prod_{k=1}^{N} (1 - r_k), \tag{13}$$

and the zero-frequency limit of equation 12 reduces to

$$V_0 = V_{TA} e^{2/\tau_N \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j (\tau_j - \tau_k)}. \tag{14}$$

BACKUS EQUATION

Another approach for describing an effective medium was proposed by Bruggeman (1937) and Backus (1962). We derive the velocity from Backus averaging in terms of reflection coefficients. From the original Backus definition, we obtain

$$\frac{1}{V_{EF}^2} = \frac{1}{D^2}\left(\sum_{j=1}^{N} d_j \rho_j\right)\left(\sum_{j=1}^{N} \frac{d_j}{\rho_j v_j^2}\right)$$

$$= \frac{1}{V_{TA}^2} + \frac{1}{D^2}\left[\sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \frac{d_j d_k}{v_j v_k}\left(\frac{\rho_j v_j}{\rho_k v_k} + \frac{\rho_k v_k}{\rho_j v_j} - 2\right)\right]$$

$$= \frac{1}{V_{TA}^2} + \frac{4}{D^2}\left[\sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \frac{d_j d_k}{v_j v_k} \frac{r_{jk}^2}{(1 - r_{jk}^2)}\right], \tag{15}$$

where r_{jk} is the reflection coefficient computed from interfaces between layers j and k. The reflection coefficient r_{jk} can be obtained from the reflection-coefficient series r_i by using the determinant of the propagator matrix (equation 5) that is computed for the stack of layers between layer j and layer $k-1$:

$$r_{jk} = \frac{\det \mathbf{Q}_j^{k-1} - 1}{\det \mathbf{Q}_j^{k-1} + 1}, \quad \det \mathbf{Q}_m^n = \prod_{i=m}^{n} \frac{1 + r_i}{1 - r_i}. \tag{16}$$

Note that because the term in the square brackets in equation 15 is always positive, the velocity in the Backus limit is less than the time-average velocity $V_{EF} < V_{TA}$.

The zero-frequency limit V_0 from equation 12 generally is different from the Backus velocity V_{EF} given in equation 15. This can be explained by the different averaging techniques used. Let us introduce the transmission response from equation 6 as $t_D^{(N)} = e^{i\Psi_N}$, with Ψ_N as the phase function (Shapiro and Hubral, 1999). Backus averaging is applied to the total wave field $\langle e^{i\Psi_N}\rangle$, while the zero-frequency limit is computed from the phase only $e^{i\langle\Psi_N\rangle}$. The system of differential equations for vertical propagation of only the vertical component is given by

$$\frac{d}{dz} U_z = c_{33}^{-1} S_z$$

$$\frac{d}{dz} S_z = -\rho\omega^2 U_z, \tag{17}$$

where U_z and S_z are Fourier-Hankel transformed vertical components of displacement and stress. Backus averaging leads to aver-

aging equation 17 coefficients $\langle c_{33}^{-1} \rangle$ and $\langle \rho \rangle$. Therefore, the Backus velocity (equation 15) is defined for slowness squared, while the zero-frequency limit (equation 12) is defined for slowness. Note that in our comparison, we are limited by vertical propagation, while Backus averaging is valid also for nonvertical propagation.

For a binary medium, series r_{jk} reduces to only one coefficient r, and equation 15 reduces to the Floquet solution (Floquet, 1883) for effective medium velocity (Hovem, 1995) that can be given in terms of the reflection coefficient as

$$\frac{1}{V_{EF}^2} = \frac{1}{V_{TA}^2} + \frac{4}{D^2} \frac{d_1 d_2}{v_1 v_2} \frac{r^2}{(1 - r^2)}. \qquad (18)$$

Despite the fact that zero-frequency limit and effective-velocity limit are generally different, both reduce to the same expression for a binary medium (Schoenberg, 1983).

Applying the weak-contrast approximation in equation 16 results in

$$r_{jk} = \frac{\prod_{i=j}^{k-1} (1 + r_i) - \prod_{i=j}^{k-1} (1 - r_i)}{\prod_{i=j}^{k-1} (1 + r_i) + \prod_{i=j}^{k-1} (1 - r_i)}$$

$$\approx \frac{\sum_{i=j}^{k-1} r_i}{1 + \sum_{m=j}^{k-2} \sum_{n=m+1}^{k-1} r_m r_n}. \qquad (19)$$

Substituting equation 19 into equation 15 and neglecting high-order terms in reflection-coefficient products, we obtain

$$\frac{1}{V_{EF}^2} = \frac{1}{V_{TA}^2} + \frac{4}{D^2} \left[\sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \frac{d_j d_k}{v_j v_k} \frac{(\sum_{m=j}^{k-1} r_m)^2}{(1 - \sum_{m=j}^{k-1} r_m^2)} \right]. \qquad (20)$$

Rewriting equation 12 in a similar way, we obtain

$$\frac{1}{V_0^2} = \frac{1}{V_{TA}^2} - \frac{4}{D^2} \frac{\left[\sum_{i=1}^{N} \frac{d_i}{v_i} \right] \left[\sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j \sum_{i=j}^{k-1} \frac{d_i}{v_i} \right]}{1 + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} r_k r_j}. \qquad (21)$$

By comparing equations 20 and 21, we conclude that the Backus-velocity limit and zero-frequency limit account for internal multiples in completely different ways. Backus averaging always guaranties the inequality $V_{EF} < V_{TA}$. However, the zero-frequency limit in the weak-contrast approximation (equation 21) does not guarantee that.

NUMERICAL RESULTS

To investigate low-frequency wave propagation through a stack of fine layers, we use models with different layer thicknesses and variable contrast. To examine the influence of layer thickness, we use real data from one well log (sampled at 0.125 m) as model M1, and constructed from it the set of models M2, M4, and M8, by di-

viding the layer spacing by a factor of 2, 4, and 8, respectively (Figure 1). The resulting models M2, M4, and M8 are then duplicated 2, 4, and 8 times, preserving the total-depth interval of 500 m. Such repeated lithologic sequences can be found in turbidite systems, for example.

To change the reflectivity contrast γ in the stack, while keeping the velocity profile unchanged, we introduce the following transformation to the density profile $\rho_i = Z_i/V_i$, where acoustic impedances are recursively transformed by the formulas $Z_i = Z_{i-1}(1 + \gamma r_{i-1}/1 - \gamma r_{i-1})$, $i = 2,3,\ldots$ and $Z_1 = \rho_1 V_1$. Therefore, the new reflection-coefficient series is defined as $r_i^{(\gamma)} = \gamma r_i^{(1)}$. To compute the reflection and transmission response, we use the matrix-propagator method (equations 6 and 10).

In Figure 2, we compare for model M1 the Backus limit, zero-frequency limit of equation 12, and zero-frequency limit from the ODA approximation versus the strength of the reflection-coefficient series. Note that even though all these limits are differ-
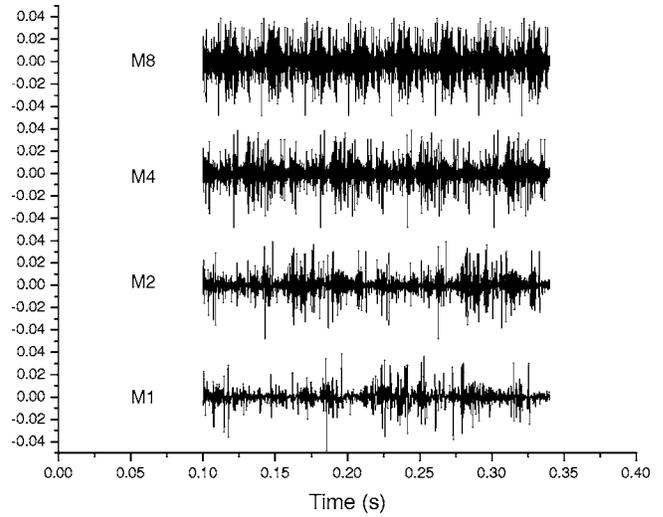
Figure 1. Reflection-coefficient time series for models M1, M2, M4, and M8.
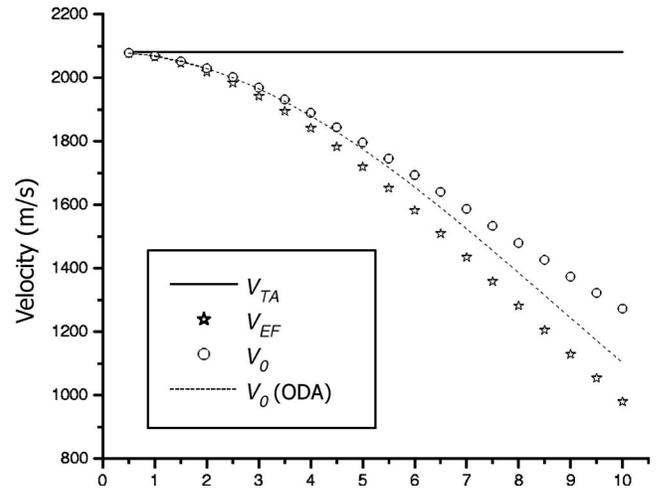
Figure 2. Time-average velocity V_{TA}, Backus velocity limit V_{EF}, zero-frequency limit V_0, and zero-frequency limit from the O'Doherty-Anstey approximation versus reflectivity contrast computed for model M1.
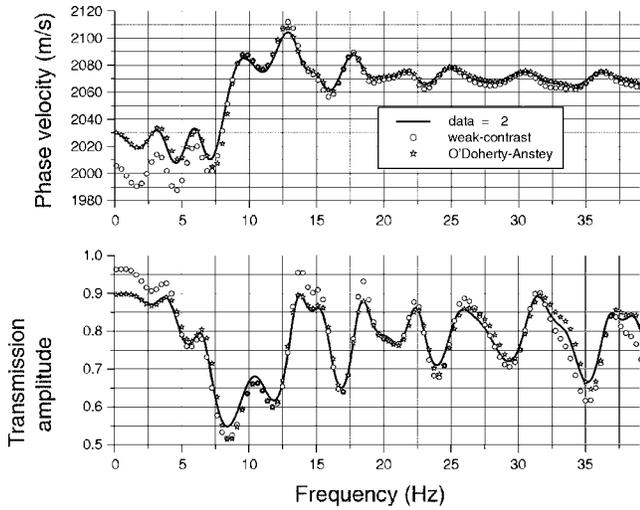
Figure 3. Phase velocity and transmission amplitude versus frequency exactly computed for model M1, using the weak-contrast and O'Doherty-Anstey approximations with reflectivity contrast $\gamma = 2$.

ent, the limit values are very similar for relatively small reflection coefficients. When $\gamma \to 0$, all velocity limits converge to the time-average limit. For large values of γ, we use the Backus limit because only this limit has physical meaning from effective-medium theory point of view. Note that all velocity limits decrease with increasing γ; hence, low-frequency waves propagate slowly in a medium with high-contrast impedance.

Figure 3 compares the exact-values phase velocity and transmission amplitude with those obtained from weak-contrast approximation and from ODA. The weak-contrast approximation uses equation 7 for transmission amplitude and equation 11 for phase velocity but neglects the higher-order terms in the correlation function Φ. For the ODA approximation, the function Φ is defined by equation B-2. Note that for both transmission amplitude and phase velocity, the weak-contrast approximation is pure at some frequencies, and the ODA approximation is much more accurate, especially at low frequencies.

In Figure 4, transmission and reflection responses are shown for a Gaussian wavelet with 15-Hz peak frequency for models M1, M2,..., 32 and $\gamma = 1$ and 4. This figure shows a transition zone be-
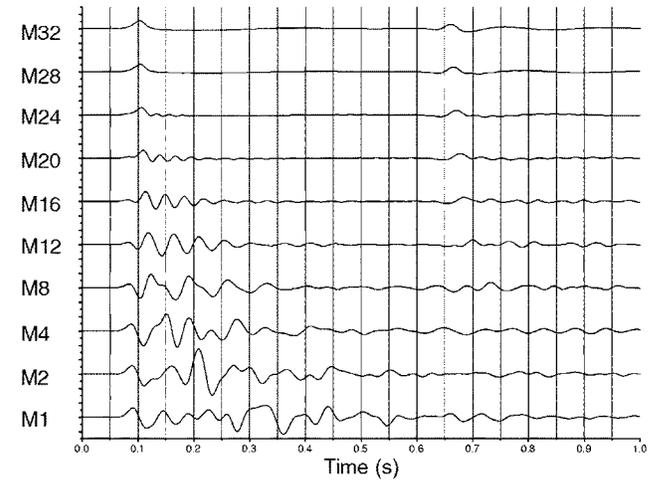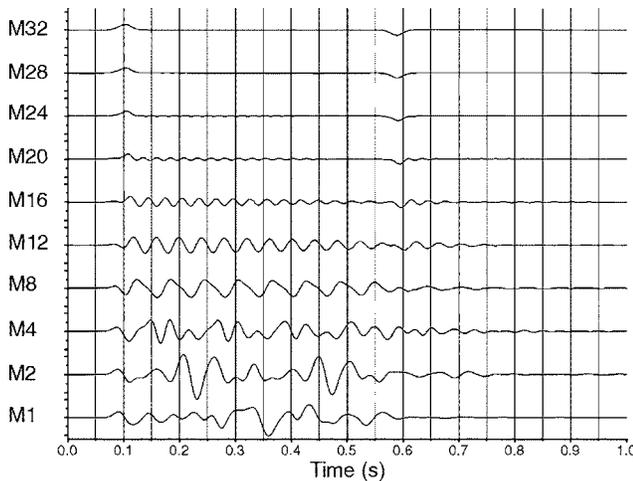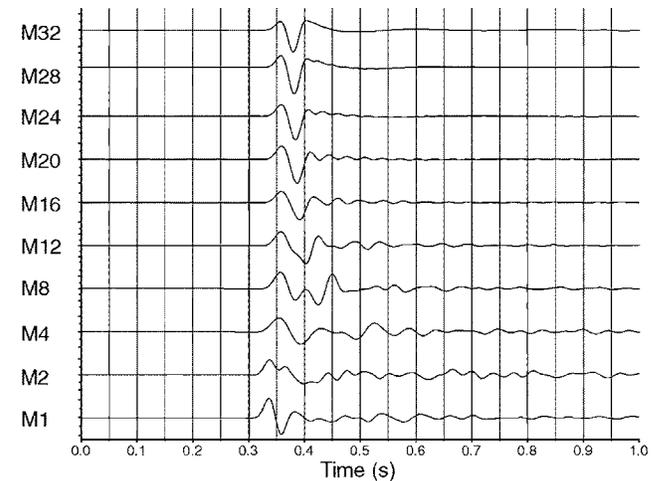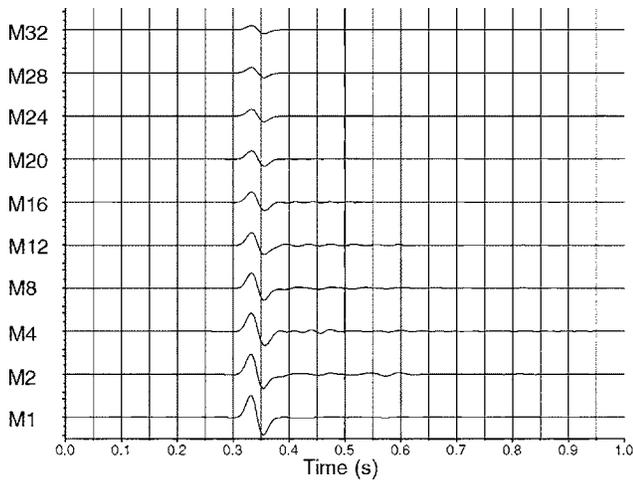
Figure 4. Transmission (top) and reflection (bottom) responses for the set of models M1, M2,…, M32, computed for reflectivity contrast $\gamma = 1$ (left) and $\gamma = 4$ (right).
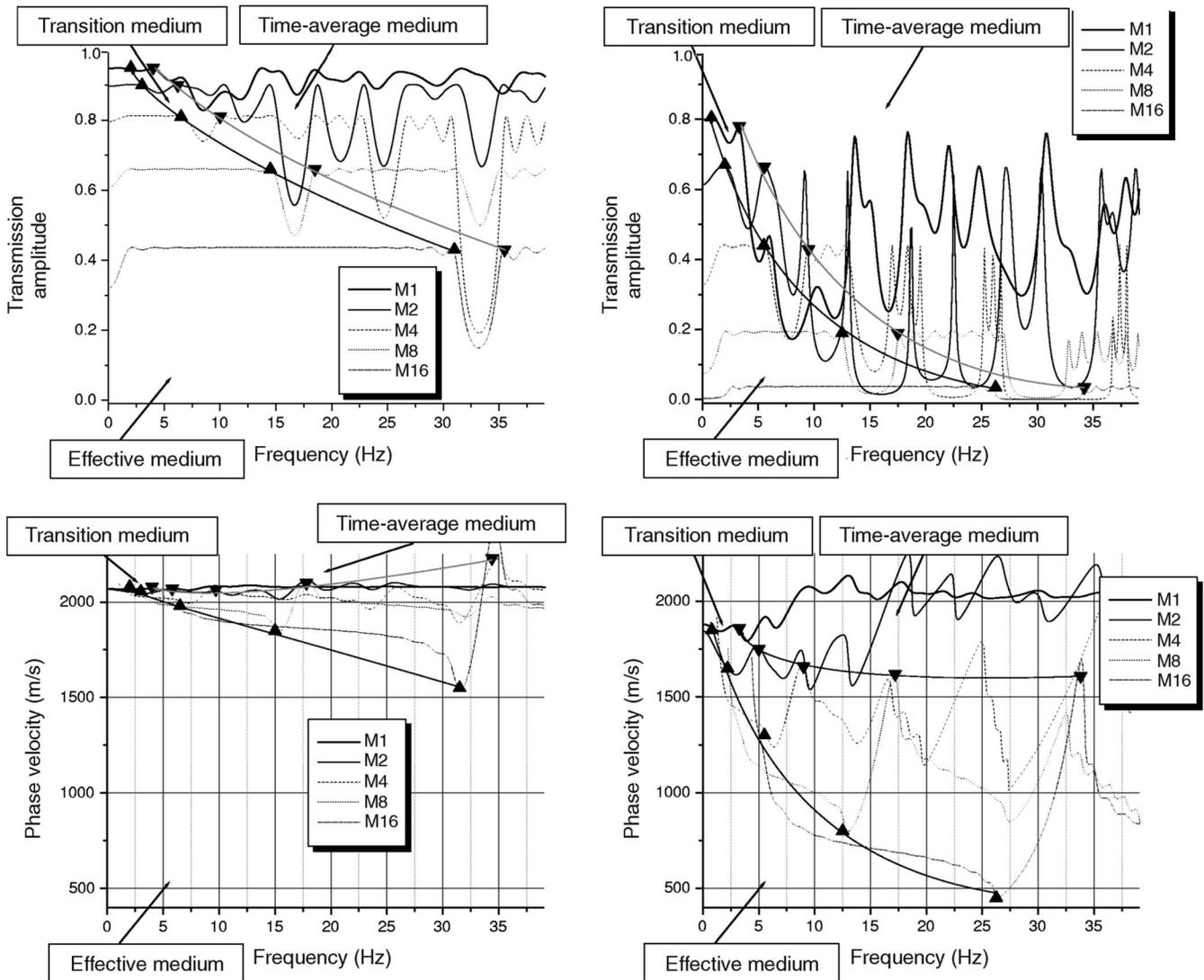
Figure 5. Transmission amplitude (top) and phase velocity (bottom) versus frequency for the set of models M1, M2,…, M16, computed for reflectivity contrast $\gamma = 1$ (left) and $\gamma = 4$ (right). The critical frequencies for each curve are shown by triangles.
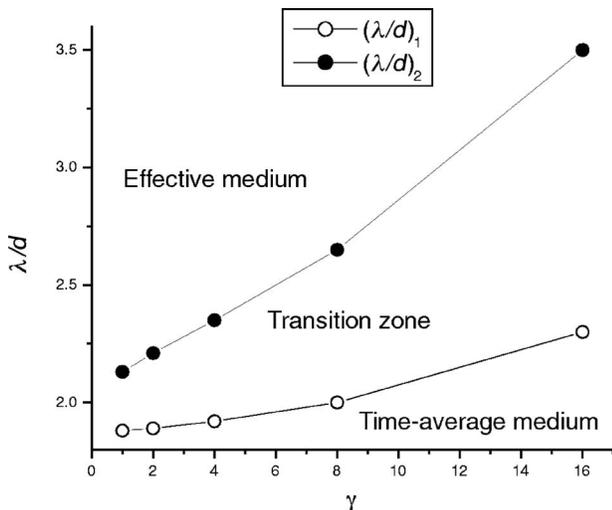
Figure 6. The critical λ/d ratio versus reflectivity.

tween the effective medium and the time-average medium, where the position of the zone depends on the strength of the reflection-coefficient series defined by parameter γ. Effective-medium parameters also depend on γ (the reflections from the bottom of the effective medium have different polarities for $\gamma = 1$ and $\gamma = 4$).

Figure 5 shows the phase velocity and transmission amplitude for models M1, M2,…, M16 and contrasts $\gamma = 1$ and 4. The transition zone on the transmission-amplitude curve is the first local minimum following the pedestal. The phase-velocity curve reveals the abrupt increase in velocity from the low-frequency limit to the time-average limit. The position of the transition zone on each curve is marked by triangles.

In Figure 6, the critical λ/d ratio is plotted against the strength of the reflection-coefficient series γ. With increasing reflectivity, the transition zone becomes larger. This means that the transition between the effective medium and time-average medium is defined by the constant λ/d ratio, and that it is also strongly reflectivity dependent.

CONCLUSIONS

Based on the matrix-propagator method, we derived equations for transmission and reflection responses from a stack of horizontal layers. We also derived expressions for the phase velocity and its zero-frequency limit that are different from the effective velocity derived by Backus averaging. The result of applying different averaging techniques is that difference increases with increasing strength of the reflection-coefficient series. Because only Backus averaging has physical meaning, it should be used regardless of the strength of the reflection-coefficient series.

By ignoring high-order terms in the scattering function, we obtain weak-contrast approximations for transmission amplitude and phase velocity. Using an O'Doherty-Anstey type approximation improves the weak-contrast approximation for both transmission amplitude and phase velocity.

A transition zone exists between the effective medium and the time-average medium. Transition frequencies are dependent on the strength of reflection coefficients, with larger reflection coefficients producing a wider transition zone.

ACKNOWLEDGMENTS

APPENDIX A

TRANSMISSION AND REFLECTION RESPONSES

The single-layer, 1D-propagator matrix is given by Hovem (1995) as

$$\mathbf{Q}_j = \frac{1}{t_j}\begin{pmatrix} e^{i\theta_j} & r_j e^{i\theta_j} \\ r_j e^{-i\theta_j} & e^{-i\theta_j} \end{pmatrix}, \qquad \text{(A-1)}$$

where $t_j = 1 - r_j$ is the transmission coefficient at the jth interface. For an elastic medium, $\mathbf{Q}_j = \mathbf{Q}_j^H$. Therefore, the product of N matrices $\mathbf{Q}_N = \prod_{j=1}^{N}\mathbf{Q}_j$ is a matrix with the same type of symmetry. The elements of the total-propagator matrix can be written

$$\text{as } Q_N(1,1) = \frac{e^{i\vartheta_N}}{\prod_{j=1}^{N}(1 - r_j)}$$
$$\times \left[1 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j e^{-2i(\vartheta_j - \vartheta_k)} + \dots \right] \qquad \text{(A-2)}$$

and

$$Q_N(1,2) = \frac{e^{i\vartheta_N}\left(\sum_{j=1}^{N} r_j e^{-2i(\vartheta_N - \vartheta_j)} + \dots \right)}{\prod_{j=1}^{N}(1 - r_j)}, \qquad \text{(A-3)}$$

where the cumulative phase function is

$$\vartheta_k = \sum_{j=1}^{k} \theta_j. \qquad \text{(A-4)}$$

The down-going transmission and reflection responses can be defined from Ursin (1983) as

$$t_D^{(N)} = Q_N^{-1}(2,2) = [Q_N^*(1,1)]^{-1}$$
$$= \frac{e^{i\vartheta_N}\prod_{k=1}^{N}(1 - r_k)}{\left[1 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j e^{2i(\vartheta_j - \vartheta_k)} + \dots \right]} \qquad \text{(A-5)}$$

and

$$r_D^{(N)} = Q_N(1,2)Q_N^{-1}(2,2) = Q_N(1,2)[Q_N^*(1,1)]^{-1}$$
$$= \frac{e^{2i\vartheta_N}\left(\sum_{j=1}^{N} r_j e^{-2i(\vartheta_N - \vartheta_j)} + \dots \right)}{\left[1 + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j e^{2i(\vartheta_j - \vartheta_k)} + \dots \right]}. \qquad \text{(A-6)}$$

APPENDIX B

O'DOHERTY-ANSTEY APPROXIMATION

The O'Doherty-Anstey type approximation can be given symbolically by

$$1 + y = e^y. \qquad \text{(B-1)}$$

This approximation contains an infinite number of terms, meaning that we can add a polynomial type of term $y^2/2! + y^3/3! + \dots$. They are not quite the same terms as those neglected in the weak-contrast approximation (that are, in fact, convolutional-type terms $y^*y + y^*y^*y + \dots$). However, the O'Doherty-Anstey type approximation reconstructs the exact solution.

If we approximate function $1 + \Phi$ as

$$1 + \Phi = e^{\sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j[\cos 2(\vartheta_j - \vartheta_k) + i \sin 2(\vartheta_j - \vartheta_k)]}, \qquad \text{(B-2)}$$

then the transmission response (equation 6) reduces to

$$t_D^{(N)} = e^{i[\vartheta_N - \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j \sin 2(\vartheta_j - \vartheta_k)]}$$
$$\times e^{-\sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j \cos 2(\vartheta_j - \vartheta_k)}\prod_{k=1}^{N}(1 - r_k), \qquad \text{(B-3)}$$

with the phase function

$$\varphi_N = \vartheta_N - \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j \sin 2(\vartheta_j - \vartheta_k). \qquad \text{(B-4)}$$

If we apply both weak-contrast and O'Doherty-Anstey approximations to the pure transmission term $\Pi_{k=1}^{N}(1-r_k)$, we can simplify equation B-3 to

$$t_D^{(N)} = e^{i[\vartheta_N - \Sigma_{k=1}^{N-1}\Sigma_{j=k+1}^{N}r_k r_j \sin 2(\vartheta_j - \vartheta_k)]}$$

$$\times e^{-\Sigma_{k=1}^{N}r_k + 2\Sigma_{k=1}^{N-1}\Sigma_{j=k+1}^{N}r_k r_j \sin^2(\vartheta_j - \vartheta_k)}. \qquad \text{(B-5)}$$

Note for a binary medium, the first term in the amplitude exponent disappears: $e^{-\Sigma_{k=1}^{N}r_k} = 1$.

The reflection response in ODA is given by

$$r_D^{(N)} = e^{i[2\vartheta_N - \Sigma_{k=1}^{N-1}\Sigma_{j=k+1}^{N}r_k r_j \sin 2(\vartheta_j - \vartheta_k)]} e^{-\Sigma_{k=1}^{N-1}\Sigma_{j=k+1}^{N}r_k r_j \cos 2(\vartheta_j - \vartheta_k)}$$

$$\times \left(\sum_{j=1}^{N} r_j e^{-2i(\vartheta_N - \vartheta_j)} \right). \qquad \text{(B-6)}$$

The phase velocity in ODA reduces to

$$\frac{1}{V(\omega)} = \frac{1}{V_{TA}} - \frac{1}{\omega D}\sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j \sin 2(\vartheta_j - \vartheta_k),$$

$$\text{(B-7)}$$

with the zero-frequency limit defined by

$$\frac{1}{V_0} = \frac{1}{V_{TA}} - \frac{2}{D}\sum_{k=1}^{N-1}\sum_{j=k+1}^{N} r_k r_j (\tau_j - \tau_k). \qquad \text{(B-8)}$$

Applying the ODA for equation B-8, we derive the very convenient equation

$$V_0 = V_{TA} e^{2\Sigma_{k=1}^{N-1}\Sigma_{j=k+1}^{N}r_k r_j(\tau_j - \tau_k)/\tau_N}. \qquad \text{(B-9)}$$

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