ITERATIVE INVERSION/MIGRATION WITH COMPLETE BOUNDARY CONDITIONS FOR THE RESIDUAL MISFIT FIELD

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ABSTRACT

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An integral representation of an elastic inversion algorithm, having a strong formal resemblance to prestack migration schemes, is developed by requiring that the retropropagation of the misfit wavefield should have a complete set of boundary conditions similar to the Kirchhoff integral. This can be achieved by using the error-energy-flux density as a kernel in the misfit function.

In order to have a non-negative objective function for the proposed scheme, the measured data must be preprocessed in order to remove free-surface related multiple reflections.

KEY WORDS: seismic migration, seismic inversion, the Kirchhoff integral, misfit function, boundary conditions.

INTRODUCTION

A physical problem approximated by a differential equation is not solved by solving the differential equation alone, also the boundary conditions to be satisfied must be implemented in order to fix one particular solution. Hyperbolic differential equations with open boundaries (that is, the field can escape to infinity in at least one direction) need Cauchy boundary conditions, implying that the field value and its slope must be specified to give an unique and stable solution. The Kirchhoff integral for migration of acoustic data is a well-known example of such a complete boundary condition.

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Inverse schemes designed to treat generally inhomogeneous media will, for realistically sized problems, have many unknowns and finite difference calculations of Fréchet derivatives are too time consuming. However, the gradient of the misfit function with respect to a physical parameter can be calculated by time correlating forward modeled data with misfit data (Tarantola, 1984). The retropropagation of the misfit data is a boundary value problem. Even if the misfit field is not an observable field, it still requires as many boundary conditions as a real field. In this paper we develop an inversion scheme for anisotropic media which fit in with this requirement. The practical similarity between the first iteration of inversion and prestack migration is well documented (Lailly, 1984). Our scheme will also be formally similar to prestack migration schemes since the retropropagated field is given by a representation theorem equal to that used to initiate the retropropagation of the observed field in prestack migration schemes. It is well known that inverse schemes for generally inhomogeneous media have convergence problems and that true physical parameters are hard to recover. A less ambitious approach is to use the inverse scheme as an iterative prestack migration scheme with a well defined imaging principle. The formalism developed in this paper can be used as an inverse scheme, however, the best approach is probably to use this formalism as an iterative prestack migration scheme. Numerical examples showing how this method can be used as an iterative prestack migration scheme was described in Mittet and Helgesen (1992). An iterative imaging principle, based on the difference between the model at the current iteration and the initial model was also given therein. The scheme has also been used successfully on real offset VSP data giving much improved results as compared to standard processing schemes.

An acoustic field carries two physical properties: energy and momentum. In order to describe the field completely at a boundary surface we need to know both quantities. This can be obtained by a combined measurement of the normal displacement acceleration component (or equivalently the normal derivative of the pressure) and the pressure. The elastic case is slightly more complicated since the field also carries angular momentum. In Appendix A we give the representation theorems for both stress and displacement fields.

In prestack migration schemes (Wapenaar et al., 1987) a sufficient set of boundary conditions is given through the representation theorem (Aki and Richards, 1980). Inverse theory for generally inhomogeneous media was originally developed (Tarantola, 1984; Kolb et al., 1986) with one boundary condition in contrast to the migration schemes which formally uses a complete set of boundary conditions. When recording elastic data on a free surface, it suffices to measure the displacement vector only. The scheme of Mora (1987), using displacement as boundary conditions, is valid for such experimental configurations.

In order to obtain the correct boundary conditions, we have to measure them. The natural way to incorporate the boundary conditions in an inverse scheme, is to include the relevant field components in the misfit function. We propose to let the kernel of the misfit function be the outgoing error-energy-flux density through the recording surface. In this way we obtain representation theorems for the backpropagated misfit fields. These representation theorems are identical to those for the fields themselves, as given in Appendix A.

An important point in the proposed scheme is that using the outgoing error-energy-flux density as kernel in the misfit function requires a preprocessing of the recorded data in the cases where a free surface is parallel to the recording surface. If this is the situation, then free surface related multiples should be removed from the recorded data as proposed by Wapenaar et al. (1990). A complete set of boundary conditions is then known at the receiver surface since their removal, in principle, gives a purely outgoing field at the receiver locations. Displacement velocity can then be calculated from stress (pressure) and vice versa.

MISFIT FUNCTION

We will seek the solution of the seismic inverse problem within an elastic anisotropic 3-D formulation, but give proper reductions to both the isotropic elastic case and the acoustic case.

As kernel of the misfit function we use the error-energy-flux density. Then the misfit function is given as

$$\tilde{\epsilon} = -\sum_{S} \int_{0}^{T} dt \oint dS_g \, [n_i \Delta\tilde{\sigma}_{ij}(\mathbf{x}_g,t \,|\, \mathbf{x}_s) \Delta\tilde{v}_j(\mathbf{x}_g,t \,|\, \mathbf{x}_s)] \quad , \tag{1}$$

which is always positive if the field can escape the volume V enclosed by the recording surface, S_g, within the recording time T. In equation (1) $\Delta\tilde{\sigma}_{ij}(\mathbf{x}_g,t \,|\, \mathbf{x}_s)$ is an observed stress component minus the corresponding stress component generated in the current model, and $\Delta\tilde{v}_j(\mathbf{x}_g,t \,|\, \mathbf{x}_s)$ is an observed displacement velocity component minus the corresponding displacement velocity component generated in the current model. The shot location is denoted $\mathbf{x}_s$, but in the following we suppress the summation over shot locations and the $\mathbf{x}_s$ arguments for simplicity.

It is assumed that the sources outside the volume V under consideration are known. Hence, they do not contribute to $\tilde{\epsilon}$ as error energy flowing into the volume, since this contribution is subtracted by a similar contribution from the forward modeling. All energy inside V is then due to secondary sources and must finally escape V, if we record for a sufficiently long period. However, this norm can be very small if we cannot integrate over a closed surface. Consider a marine-seismic experimental configuration. If an approximately horizontal plane wave is passing a streamer on its way up, it will be reflected at the free surface and pass the streamer on its way down. The time integral may cancel the contributions from the upgoing flux and the equally large downgoing flux. Also, on a free surface, where the upgoing and downgoing fluxes are equal, or alternatively, the stresses are zero, equation (1) is not a suitable misfit function.

The misfit function in equation (1) cannot be used without modifications. The necessary modifications can be implemented by preprocessing the observed data and processing the predicted data in the same way. We follow Wapenaar et al. (1990) and remove free-surface-related multiples from the recorded data if there is a free-surface parallel to the recording surface or the recording surface is on a free surface. If the free-surface-related multiples are removed from data recorded on a free surface, then the equally large up- and down-going fluxes are replaced by a purely up-going flux. This implies that the processed stress components can be calculated from the displacement velocities.

As the modified misfit-functions we propose

$$\begin{aligned} \epsilon &= -\int_0^T dt \int dS_g \, [n_i \Delta\tilde{\sigma}_{ij}(\mathbf{x}_g,t)\Delta\tilde{v}_j(\mathbf{x}_g,t)]^{(\text{free})} \\ &= -\int_0^T dt \int dS_g \, [n_i \Delta\sigma_{ij}(\mathbf{x}_g,t)\Delta v_j(\mathbf{x}_g,t)] \quad , \end{aligned} \tag{2}$$

where $\Delta\sigma_{ij}(\mathbf{x}_g,t)$ is an observed preprocessed stress component minus the correspondingly processed stress component modeled in the current model, and $\Delta v_j(\mathbf{x}_g,t)$ is an observed preprocessed displacement velocity component minus the correspondingly processed displacement velocity component modeled in the current model. The processed predicted data may be obtained simply by removing the relevant free surface from the numerical forward modeling. Equation (2) is a measure of the outgoing error-energy flux at the receiver positions with the effect of the free surface removed. Note that we have replaced the closed surface integral in equation (1) with an integration over a surface which may not necessarily be closed. Data are recorded with limited spatial apertures in a seismic experiment. However, for inverse schemes based on retropropagation, a complete description of the misfit field in a volume V formally requires that the necessary field components are recorded on a surface enclosing this volume. It is also acceptable to let V be the volume between two parallel recording surfaces if these surfaces have infinite area. Most real data are recorded at or near the surface of the earth, and geophone or hydrophone arrays have finite length. Such a reduction of the aperture will inevitably lead to reduced resolution for the gradients.

The choice of misfit function is motivated by the fact that equation (2) leads to boundary conditions for the misfit field which are analogous to those of an observable field and will, in the acoustic case, give a representation theorem similar to the Kirchhoff integral. The objective function will be non-negative with a proper preprocessing of the observed data and proper processing of the predicted data. An inversion scheme based on this misfit function is similar to algorithms like those proposed and discussed by Tarantola (1984), Mora (1987) and Crase et al. (1990), but our scheme also depends on free surface related preprocessing (Wapenaar et al., 1990) of the observed data as the first step.

GRADIENT CALCULATION

The expressions for the gradients with respect to Hooke's tensor and density are derived in Appendix B. We let $\epsilon_{ab}(\mathbf{x},t)$ and $v_n(\mathbf{x},t)$ denote forward propagated strain and displacement velocity fields respectively, generated in the current model. In Appendix B we obtain the backward propagated residual stress in the current model as

$$\tau_{kl}(\mathbf{x},t) = \int dS_g \, [G^0_{klpq}(\mathbf{x},0|\mathbf{x}_g,t) * n_p \Delta a_q(\mathbf{x}_g,t) - \partial^g_j G^0_{kljq}(\mathbf{x},0|\mathbf{x}_g,t) * n_p \Delta\sigma_{pq}(\mathbf{x}_g,t)/\varrho(\mathbf{x}_g)] \quad , \tag{3}$$

which is similar to the surface integral term in equation (A-7), which is the representation theorem for stress. We also define backward propagated strain $E_{mn}(\mathbf{x},t)$ and displacement velocity $w_n(\mathbf{x},t)$ as

$$E_{mn}(\mathbf{x},t) = s_{mnkl}(\mathbf{x})\tau_{kl}(\mathbf{x},t) \quad ; \quad \partial_t w_n(\mathbf{x},t) = \varrho^{-1}(\mathbf{x})\partial_m \tau_{mn}(\mathbf{x},t) \quad . \tag{4}$$

Where $\tau_{kl}(\mathbf{x},t)$ has the same units as stress.

The gradient elements of the elastic stiffness parameters are

$$g_{abmn}(\mathbf{x}) = \int_0^T dt[\partial_t \epsilon_{ab}(\mathbf{x},t)]E_{mn}(\mathbf{x},t) \quad , \tag{5}$$

and the density gradient is

$$g(\mathbf{x}) = -\int_0^T dt[\partial_t v_n(\mathbf{x},t)]w_n(\mathbf{x},t) \quad . \tag{6}$$

In the acoustic case equation (3) reduces to

$$\Delta P(\mathbf{x},t) = \int dS_g \varrho(\mathbf{x}_g)^{-1} n_i [G^0(\mathbf{x},0|\mathbf{x}_g,t) * \partial^g_i \Delta P(\mathbf{x}_g,t) - \partial^g_i G^0(\mathbf{x},0|\mathbf{x}_g,t) * \Delta P(\mathbf{x}_g,t)] \quad , \tag{7}$$

which is the Kirchhoff integral for the residual misfit pressure $\Delta P(\mathbf{x}_g,t)$. $G^0(\mathbf{x},t\,|\,\mathbf{x}',t')$ is here the Green's function for the pressure in the current model. The gradient for the bulk modulus $M(\mathbf{x})$ is obtained from equation (5) as

$$g_M(\mathbf{x}) = [1/M^2(\mathbf{x})] \int_0^T dt[\partial_t P(\mathbf{x},t)]\Delta P(\mathbf{x},t) \quad . \tag{8}$$

As is well known, migration and the first iteration of an inversion are related (Lailly, 1984). Using the misfit function proposed here, this relation is even more prominent: for a smooth initial model or macro model the forward data, $P^D(\mathbf{x},t)$, will be mainly downgoing, and reflections will be negligible. The residual pressure will be approximately equal to the observed pressure $P^{obs}(\mathbf{x}_g,t)$, and equation (7) will be equal to the Kirchhoff integral

$$P^U(\mathbf{x},t) = \int dS_g \varrho(\mathbf{x}_g)^{-1} n_i [G^0(\mathbf{x},0\,|\,\mathbf{x}_g,t) * \partial_i^g P^{obs}(\mathbf{x}_g,t)$$
$$- \partial_i^g G^0(\mathbf{x},0\,|\,\mathbf{x}_g,t) * P^{obs}(\mathbf{x}_g,t)] \quad . \tag{9}$$

Equation (8) can now be written

$$g_M^{U/D}(\mathbf{x}) = [1/M^2(\mathbf{x})] \int_0^T dt[\partial_t P^D(\mathbf{x},t)]P^U(\mathbf{x},t) \quad . \tag{10}$$

which, except for a time derivative on the downgoing pressure and the inverse squared bulk modulus term, is identical to one of the U/D imaging concepts proposed by Claerbout (1971).

UPDATE STRATEGY

In Appendix D we derive the following expressions for the parameter updates

$$\begin{bmatrix} \Delta c_{rstu} \\ \Delta \varrho \end{bmatrix} = - \begin{bmatrix} H_{\alpha\beta\mu\upsilon\, rstu} & H_{\alpha\beta\mu\upsilon} \\ H_{rstu} & H \end{bmatrix}^{-1} \begin{bmatrix} g_{\alpha\beta\mu\upsilon} \\ g \end{bmatrix} , \tag{11}$$

which shows that the update of one parameter class in principle depends on the gradients for all parameter classes. However, a calculation of the Hessian tensors coupling the different parameter classes is numerically very expensive, so these quantities have to be approximated. Under the simplifying approximations given in Appendix D, equation (11) reduces, in the isotropic case, to parameter updates Δm_η ($\eta = \lambda, \mu, \varrho$ and $\phi = \lambda, \mu, \varrho$) for the Lamé parameters and density

$$\Delta m_\eta = -\alpha_{\eta\varphi}\hat{g}_\varphi \quad . \tag{12}$$

Here the preconditioned gradients with respect to the Lamé parameters λ, μ, and density are defined as

$$\hat{g}_\varphi(\mathbf{x}) = B(\mathbf{x}) g_\varphi(\mathbf{x}) \quad , \tag{13}$$

where the preconditioner, $B(\mathbf{x})$ can be approximated by a function proportional to components of the average field energy in position **x** during the forward modeling (Canadas, 1986). The gradients can be obtained from equations (5) and (6) as

$$g_\lambda(\mathbf{x}) = g_{nnmm}(\mathbf{x}) \ ; \quad g_\mu(\mathbf{x}) = 2g_{nmmn}(\mathbf{x}) \ ; \quad g_\rho(\mathbf{x}) = g(\mathbf{x}) \quad . \tag{14}$$

In equation (12) we see that the update of one parameter class depends on all the gradients as in equation (11). The α-coefficients may be determined by a search in the gradient direction. For each α-coefficient we have to perform a separate forward modeling operation. It is possible to perform simplifications in order to reduce the amount of numerical work. If we assume that only the diagonal α-coefficients are non-zero, then we have to perform three additional forward modeling operations per shot and not nine as for the fully coupled case. If the data processing is to be performed over a line with several shots, we may use only a subset of the total number of shots in order to determine these coefficients, which gives a further reduction of forward modeling operations.

CONCLUSIONS

A modified elastic inverse scheme (iterative migration scheme) for generally inhomogeneous media has been proposed. By using the outgoing error-energy-flux density as the kernel in the misfit function, we obtained a representation theorem for the retropropagated misfit field which contained a complete set of boundary conditions.

We further concluded that with the proposed misfit function, a preprocessing of the measured data must be performed in order to remove free-surface-related multiples, if the recording surface is parallel to the free surface.

It was shown, within the acoustic approximation, how the proposed scheme was reduced to a standard prestack migration scheme in the case where a smooth initial model (macro model) was assumed.

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APPENDIX A

Representation theorems

The inversion formalism is simplified utilizing spatial reciprocity relations for the Green's tensors. We must then assume homogeneous boundary conditions on some closed surface. This may not always be a property of the recording surface. One way to circumvent this problem is to introduce the recording surface as an internal artificial surface which is transparent to the Green's tensor and its derivatives (Aki and Richards, 1980) and assume homogeneous boundary conditions or the Sommerfeld radiation condition on some outer surface which, for practical purposes, will be the edges of the numerical model. We have tested and verified these assumptions within our numerical scheme. Hence we assume homogeneous boundary conditions on a closed surface S_0 surrounding the internal recording surface S transparent to the Green's tensors and their partial derivatives. Then we may use spatial reciprocity relations of the Green's tensors in the volume enclosed by S.

The wave equation for the displacement field is

$$\varrho(\mathbf{x})\partial_t^2 u_i(\mathbf{x},t) - \partial_j c_{ijpq}(\mathbf{x})\partial_p u_q(\mathbf{x},t) = f_i(\mathbf{x},t) \quad . \tag{A-1}$$

The Green's tensor for the displacement field is determined by

$$\varrho(\mathbf{x})\partial_t^2 \gamma_{in}(\mathbf{x},t|\mathbf{x}',t') - \partial_j c_{ijpq}(\mathbf{x})\partial_p \gamma_{qn}(\mathbf{x},t|\mathbf{x}',t') = \delta_{in}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \quad . \tag{A-2}$$

From equations (A-1) and (A-2) the representation theorem for the displacement field is obtained as

$$\begin{aligned} u_n(\mathbf{x},t) = & \int_0^{t+} dt' \int d^3x' \gamma_{ni}(\mathbf{x},t-t'|\mathbf{x}',0) f_i(\mathbf{x}',t') \\ & + \int_0^{t+} dt' \oint_S dS(\mathbf{x}') \{\gamma_{ni}(\mathbf{x},t-t'|\mathbf{x}',0) n_j \sigma_{ij}(\mathbf{x}',t') \\ & \qquad - [\partial_j' \gamma_{ni}(\mathbf{x},t-t'|\mathbf{x}',0)] c_{ijpq}(\mathbf{x}') n_p u_q(\mathbf{x}',t')\} \\ & + \int d^3x' \varrho(\mathbf{x}') \{\partial_{t'} \gamma_{ni}(\mathbf{x},t-t'|\mathbf{x}',0) u_i(\mathbf{x}',t') \\ & \qquad - \gamma_{ni}(\mathbf{x},t-t'|\mathbf{x}',0)\partial_{t'} u_i(\mathbf{x}',t')\} \Big|_0^{t+} \quad , \end{aligned} \tag{A-3}$$

where t^+ means $t + \epsilon$, where ϵ is arbitrarily small. This limit prevents the time integration from ending at the peak of a delta function. Assuming zero initial conditions for the field and its time derivative, the last integral in equation (A-3) vanishes.

For a retropropagated displacement component $U_n(\mathbf{x},t)$, with final conditions at time T for the field and its time derivative equal to zero, we obtain in a similar manner

$$
\begin{aligned}
U_n(\mathbf{x},t) = & \int_{t^-}^{T} dt' \int d^3x' \gamma_{ni}(\mathbf{x},0\,|\,\mathbf{x}',t-t') f_i(\mathbf{x}',t') \\
& + \int_{t^-}^{T} dt' \oint_S dS(\mathbf{x}') \{ \gamma_{ni}(\mathbf{x},0\,|\,\mathbf{x}',t-t') n_j \sigma_{ij}(\mathbf{x}',t') \\
& - [\partial_j' \gamma_{ni}(\mathbf{x},0\,|\,\mathbf{x}',t-t')] c_{ijpq}(\mathbf{x}') n_p u_q(\mathbf{x}',t') \} \quad , \qquad \text{(A-4)}
\end{aligned}
$$

where t^- means $t - \epsilon$.

The wave equation for the stress field is

$$
\begin{aligned}
& s_{pqij}(\mathbf{x}) \partial_t^2 \sigma_{ij}(\mathbf{x},t) - (1/2)[\partial_p \varrho^{-1}(\mathbf{x}) \partial_k \sigma_{qk}(\mathbf{x},t) + \partial_q \varrho^{-1}(\mathbf{x}) \partial_k \sigma_{pk}(\mathbf{x},t)] \\
& = (1/2)[\partial_p \varrho^{-1}(\mathbf{x}) f_q(\mathbf{x},t) + \partial_q \varrho^{-1}(\mathbf{x}) f_p(\mathbf{x},t)] \quad . \qquad \text{(A-5)}
\end{aligned}
$$

The Green's tensor for the stress field is given by

$$
\begin{aligned}
& s_{pqij}(\mathbf{x}) \partial_t^2 G_{ijnm}(\mathbf{x},t\,|\,\mathbf{x}',t') \\
& - (1/2)[\partial_p \varrho^{-1}(\mathbf{x}) \partial_k G_{qknm}(\mathbf{x},t\,|\,\mathbf{x}',t') + \partial_q \varrho^{-1}(\mathbf{x}) \partial_k G_{pknm}(\mathbf{x},t\,|\,\mathbf{x}',t')] \\
& = (1/2)[\delta_{pm}\delta_{qn} + \delta_{qm}\delta_{pn}] \delta(\mathbf{x}-\mathbf{x}') \delta(t-t') \quad , \qquad \text{(A-6)}
\end{aligned}
$$

where we have, $s_{ijpq} c_{pqkl} = (1/2)[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$. The representation theorem for the stress field is obtained from equations (A-5) and (A-6) as

$$
\begin{aligned}
\sigma_{mn}(\mathbf{x},t) = & \int_0^{t^+} dt' \int d^3x' G_{mnij}(\mathbf{x},t-t'\,|\,\mathbf{x}',0) \partial_i' \varrho^{-1}(\mathbf{x}') f_j(\mathbf{x}',t) \\
& + \int_0^{t^+} dt' \oint_S dS(\mathbf{x}') \{ G_{mnij}(\mathbf{x},t-t'\,|\,\mathbf{x}',0) n_i a_j(\mathbf{x}',t') \\
& - [\partial_i' G_{mnij}(\mathbf{x},t-t'\,|\,\mathbf{x}',0)][n_p \sigma_{jp}(\mathbf{x}',t')/\varrho(\mathbf{x}')] \} \\
& + \int d^3x' \{ [\partial_{t'} G_{mnij}(\mathbf{x},t-t'\,|\,\mathbf{x}',0)] \epsilon_{ij}(\mathbf{x}',t') \\
& - G_{mnij}(\mathbf{x},t-t'\,|\,\mathbf{x}',0) \partial_{t'} \epsilon_{ij}(\mathbf{x}',t') \} \,\Big|_0^{t^+} \quad . \qquad \text{(A-7)}
\end{aligned}
$$

Assuming zero initial conditions for the field and its time derivative, the last integral in equation (A-7) vanishes.

For a retropropagated stress component $\Sigma_{mn}(\mathbf{x},t)$, with final conditions at time T equal to zero, we obtain

$$\Sigma_{mn}(\mathbf{x},t) = \int_{t-}^{T} dt' \int d^3x' G_{mnij}(\mathbf{x},0|\mathbf{x}',t-t')\partial'_i \varrho^{-1}(\mathbf{x}')f_j(\mathbf{x}',t)$$
$$+ \int_{t-}^{T} dt' \oint_S dS(\mathbf{x}')\{G_{mnij}(\mathbf{x},0|\mathbf{x}',t-t')n_i a_j(\mathbf{x}',t')$$
$$- [\partial'_i G_{mnij}(\mathbf{x},0|\mathbf{x}',t-t')][n_p \sigma_{jp}(\mathbf{x}',t')/\varrho(\mathbf{x}')]\} \quad . \qquad \text{(A-8)}$$

Displacement and stress are not independent fields for the elastic problem, they are linked through the equation of motion and the constitutive relation. Hence, the Green's tensors for displacement and stress are not independent. This can be expressed in several ways, we give two expressions here

$$\varrho^{-1}(\mathbf{x})\partial_j G_{ijmn}(\mathbf{x},t|\mathbf{x}',t') = -c_{ljmn}(\mathbf{x}')\partial'_j \gamma_{il}(\mathbf{x},t|\mathbf{x}',t')$$
$$\partial_m \gamma_{ni}(\mathbf{x},t|\mathbf{x}',t') = -\varrho^{-1}(\mathbf{x}')s_{mnkl}(\mathbf{x})\partial'_\mu G_{kli\mu}(\mathbf{x},t|\mathbf{x}',t') \quad . \qquad \text{(A-9)}$$

APPENDIX B

Gradient calculation

Following Kolb et al. (1986), we assume that the gradient of the misfit function with respect to the Hooke's tensor and density is determined by a small variation of the misfit function of the general form

$$\delta\epsilon = \int d^3x[g_{abmn}(\mathbf{x})\delta c_{abmn}(\mathbf{x}) + g(\mathbf{x})\delta\varrho(\mathbf{x})] \quad , \qquad \text{(B-1)}$$

where $g_{abmn}(\mathbf{x})$ is the gradient tensor with respect to the Hooke's tensor $c_{abmn}(\mathbf{x})$ and $g(\mathbf{x})$ is the gradient with respect to the density $\varrho(\mathbf{x})$. We obtain these gradients by first calculating the change in misfit function due to a change in $c_{ijkl}(\mathbf{x})$ and $\varrho(\mathbf{x})$. Let $L_{pq}[c^0_{ijkl}(\mathbf{x}),\varrho^0(\mathbf{x})](\mathbf{x}_g,t)$ be the modeling operator determining the stress tensor at the receivers in the current medium with a given source function, and let $L_q[c^0_{ijkl}(\mathbf{x}),\varrho^0(\mathbf{x})](\mathbf{x}_g,t)$ be the modeling operator determining the displacement velocity in the current medium with the same source. A change in the medium parameters will cause the following change in the error norm,

$$\Delta\epsilon = -\int_0^T dt \int dS_g n_p\{(L_{pq}[c^0_{ijkl}+\Delta c_{ijkl}, \varrho^0+\Delta\varrho] - L_{pq}[c^0_{ijkl}, \varrho^0])\Delta v_q(\mathbf{x}_g,t)$$
$$+ (L_q[c^0_{ijkl} + \Delta c_{ijkl}, \varrho^0+\Delta\varrho] - L_q[c^0_{ijkl}, \varrho^0])\Delta\sigma_{pq}(\mathbf{x}_g,t)\} \quad , \qquad \text{(B-2)}$$

when we retain terms linear in the error fields. The difference in the modeled stress and displacement velocity fields is obtained by use of the Lippmann-Schwinger equations, given in Appendix C, as follows: we move the single Green's tensor, generated in the unperturbed model to the left side of the equations (C-1) and (C-2), then we convolve these expressions with the proper source terms. With $f_i(\mathbf{x},t)$ equal to a component of the force density we use

$$T_{ij}(\mathbf{x},t) = (1/2)[\partial_j \varrho^{-1}(\mathbf{x}) f_i(\mathbf{x},t) + \partial_i \varrho^{-1}(\mathbf{x}) f_j(\mathbf{x},t)] \quad ,$$

$$\dot{f}_i(\mathbf{x},t) = \partial_t f_i(\mathbf{x},t) \quad .$$

Hence, we obtain

$$\begin{aligned}
\Delta\epsilon = & \int dt \int dS_g \int d^3x' \int d^3x'' G^0_{pqmn}(\mathbf{x}_g,t|\mathbf{x}',0) \, * \\
& [\Delta s_{mnrs}(\mathbf{x}')\partial_t^2 G_{rsij}(\mathbf{x}',t|\mathbf{x}'',0) - \partial'_m \Delta b(\mathbf{x}')\partial'_k G_{nkij}(\mathbf{x}',t|\mathbf{x}'',0)] \, * \\
& T_{ij}(\mathbf{x}'',t) n_p \Delta v_q(\mathbf{x}_g,t) \\
& + \int dt \int dS_g \int d^3x' \int d^3x'' \gamma^0_{qn}(\mathbf{x}_g,t|\mathbf{x}',0) \, * \\
& [\Delta\varrho(\mathbf{x}')\partial_t^2 \gamma_{ni}(\mathbf{x}',t|\mathbf{x}'',0) - \partial'_j \Delta c_{njab}(\mathbf{x}')\partial'_a \gamma_{bi}(\mathbf{x}',t|\mathbf{x}'',0)] \, * \\
& \dot{f}_i(\mathbf{x}'',t) n_p \Delta\sigma_{pq}(\mathbf{x}_g,t) \quad . \qquad \text{(B-3)}
\end{aligned}$$

We convolve the unperturbed Green's tensors G^0_{pqmn} and γ^0_{qn} with the difference fields Δv_q and $\Delta\sigma_{pq}$ to obtain the back propagated fields, and using equation (A-9) we express all back propagated fields with the Green's tensor for stress. At this step we also assume that the physical parameters can be measured on the recording surface. There will be additional terms giving corrections to the gradients on the recording surface only, if the physical parameters are not known in these positions.

In order to find a gradient expression for ϵ with respect to the physical parameters, we assume small perturbations and keep terms to first order in the variation of the physical parameters. Hence, we replace G_{ijkl} and γ_{in} with G^0_{ijkl} and γ^0_{in} respectively, which are the Green's tensors calculated in the known medium (macro model). We let $\Delta\epsilon \rightarrow \delta\epsilon$, $\Delta\varrho \rightarrow \delta\varrho$, $\Delta b \rightarrow -\delta\varrho/\varrho^2$, $\Delta c_{abmn} \rightarrow \delta c_{abmn}$, $\Delta s_{abmn} \rightarrow \delta s_{abmn}$, and for small perturbations we have to first order $c_{klpq}\delta s_{pqij} = -s_{ijpq}\delta c_{pqkl}$.

Equation (B-3) can now be divided into the following four terms

$$\delta\epsilon_I = \int d^3x \int_0^T dt[\partial_t\epsilon_{ab}(\mathbf{x},t)]$$

$$s_{mnkl}(\mathbf{x})[\int dS_g G^0_{klpq}(\mathbf{x},0|\mathbf{x}_g,t) * n_p\Delta a_q(\mathbf{x}_g,t)]\delta c_{abmn}(\mathbf{x}) \quad , \qquad \text{(B-4)}$$

$$\delta\epsilon_{II} = -\int d^3x \int_0^T dt[\partial_t\epsilon_{ab}(\mathbf{x},t)]$$

$$s_{mnkl}(\mathbf{x})[\int dS_g\{\partial^g_j G^0_{kljq}(\mathbf{x},0|\mathbf{x}_g,t)\} * n_p\Delta\sigma_{pq}(\mathbf{x}_g,t)/\varrho(\mathbf{x}_g)]\delta c_{abmn}(\mathbf{x}) \quad , \qquad \text{(B-5)}$$

$$\delta\epsilon_{III} = \int d^3x \int_0^T dt v_n(\mathbf{x},t)$$

$$\varrho^{-1}(\mathbf{x})[\partial_m \int dS_g G^0_{mnpq}(\mathbf{x},0|\mathbf{x}_g,t) * n_p\Delta a_q(\mathbf{x}_g,t)]\delta\varrho(\mathbf{x}) \quad , \qquad \text{(B-6)}$$

$$\delta\epsilon_{IV} = -\int d^3x \int_0^T dt v_n(\mathbf{x},t)$$

$$\varrho^{-1}(\mathbf{x})[\partial_m \int dS_g\{\partial^g_j G^0_{mnjq}(\mathbf{x},0|\mathbf{x}_g,t)\} * n_p\Delta\sigma_{pq}(\mathbf{x}_g,t)/\varrho(\mathbf{x}_g)]\delta\varrho(\mathbf{x}) \quad , \qquad \text{(B-7)}$$

where $\epsilon_{ab}(\mathbf{x},t)$ and $v_n(\mathbf{x},t)$ are forward propagated strain and displacement velocity fields respectively, generated in the current model. We define the backward propagated stress in the current model as

$$\tau_{kl}(\mathbf{x},t) = \int dS_g[G^0_{klpq}(\mathbf{x},0|\mathbf{x}_g,t) * n_p\Delta a_q(\mathbf{x}_g,t)$$

$$-\{\partial^g_j G^0_{kljq}(\mathbf{x},0|\mathbf{x}_g,t)\} * n_p\Delta\sigma_{pq}(\mathbf{x}_g,t)/\varrho(\mathbf{x}_g)] \quad , \qquad \text{(B-8)}$$

which is similar to the surface integral term in equation (A-8), which is the representation theorem for stress. We also define backward propagated strain $E_{mn}(\mathbf{x},t)$ and displacement velocity $w_n(\mathbf{x},t)$ as

$$E_{mn}(\mathbf{x},t) = s_{mnkl}(\mathbf{x})\tau_{kl}(\mathbf{x},t) \quad ,$$

$$\partial_t w_n(\mathbf{x},t) = \varrho^{-1}(\mathbf{x})\partial_m\tau_{mn}(\mathbf{x},t) \quad . \qquad \text{(B-9)}$$

Note that $\tau_{kl}(\mathbf{x},t)$ is not a 'stress-like' field but has the same units as stress.

Comparing equations (B-4) to (B-7) with equation (B-1), we obtain for the gradient of elastic stiffness parameters

$$g_{abmn}(\mathbf{x}) = \int_0^T dt[\partial_t\epsilon_{ab}(\mathbf{x},t)]E_{mn}(\mathbf{x},t) \quad , \qquad \text{(B-10)}$$

and for the density gradient

$$g(\mathbf{x}) = -\int_0^T dt[\partial_t v_n(\mathbf{x},t)]w_n(\mathbf{x},t) \quad . \tag{B-11}$$

The gradients (B-10) and (B-11) have the same form as given by Tarantola (1987), except for a time derivative on the backpropagated fields. This difference can be explained by the fact that Tarantola (1987) uses a different misfit function. Actually, our boundary condition for the backpropagated field is spatially more high frequent than in Tarantola (1987), this will compensate for the mentioned time derivative such that gradients generated with both methods will have the same spatial frequency content.

APPENDIX C

The Lippmann-Schwinger equations

The Lippmann-Schwinger equations for the displacement Green's tensors are obtained in a similar way as the representation theorems. These equations require that the Green's tensors are known in a given model and describe how the Green's tensors change if the medium changes. The change in medium parameters need not be small for these equations to be valid. The relations between the physical parameters in the given model and the modified model are $\varrho(\mathbf{x}) = \varrho^0(\mathbf{x}) + \Delta\varrho(\mathbf{x})$, $b(\mathbf{x}) = b^0(\mathbf{x}) + \Delta b(\mathbf{x})$, $c_{ijkl}(\mathbf{x}) = c^0_{ijkl}(\mathbf{x}) + \Delta c_{ijkl}(\mathbf{x})$ and $s_{ijkl}(\mathbf{x}) = s^0_{ijkl}(\mathbf{x}) + \Delta s_{ijkl}(\mathbf{x})$, where $b(\mathbf{x})$ is inverse density. The Lippmann-Schwinger equation for the displacement Green's tensor is

$$\gamma_{mi}(\mathbf{x},t|\mathbf{x}'',t'') = \gamma^0_{mi}(\mathbf{x},t|\mathbf{x}'',t'') + \int d^3x' \int_0^{t+} dt' \gamma^0_{mn}(\mathbf{x},t|\mathbf{x}',t')$$

$$[-\{\Delta\varrho(\mathbf{x}')\partial^2_{t'}\gamma_{ni}(\mathbf{x}',t'|\mathbf{x}'',t'') - \partial'_j\Delta c_{njpq}(\mathbf{x}')\partial'_p\gamma_{qi}(\mathbf{x}',t'|\mathbf{x}'',t'')\}] \quad . \tag{C-1}$$

The Lippmann-Schwinger equation for the stress Green's tensor is

$$G_{pqij}(\mathbf{x},t|\mathbf{x}'',t'') = G^0_{pqij}(\mathbf{x},t|\mathbf{x}'',t'') + \int d^3x' \int_0^{t+} dt' G^0_{pqmn}(\mathbf{x},t|\mathbf{x}',t')$$

$$[-\{\Delta s_{mnrs}(\mathbf{x}')\partial^2_{t'}G_{rsij}(\mathbf{x}',t'|\mathbf{x}'',t'') - \partial'_m\Delta b(\mathbf{x}')\partial'_k G_{nkij}(\mathbf{x}',t'|\mathbf{x}'',t'')\}] \quad . \tag{C-2}$$

Here $\gamma_{mi}(\mathbf{x},t|\mathbf{x}'',t'')$ and $G_{pqij}(\mathbf{x},t|\mathbf{x}'',t'')$ are Green's tensors for the modified or perturbed medium. $\gamma^0_{mi}(\mathbf{x},t|\mathbf{x}'',t'')$ and $G^0_{pqij}(\mathbf{x},t|\mathbf{x}'',t'')$ are the assumed known Green's tensor calculated in the unperturbed medium. The fields can be obtained by convolving these expressions with the proper source terms.

APPENDIX D

Parameter update with a fully coupled subspace method

Let $\Gamma_Q(\mathbf{x}_g,t|\mathbf{x})$ and $\Psi_Q^{\dagger}(\mathbf{x}|\mathbf{x}_g,t)$ be shorthand notations for integral operators with kernels $K_Q(\mathbf{x}_g,t|\mathbf{x})$ and $H_Q^{\dagger}(\mathbf{x}|\mathbf{x}_g,t)$ respectively, where Q may be a set of indices and the dagger ($\dagger$) denotes a Hermitian conjugate. As an example,

$$\Delta v_i(\mathbf{x}_g,t) = \Gamma_i(\mathbf{x}_g,t|\mathbf{x})\Delta m(\mathbf{x}) \equiv \int d^3x K_i(\mathbf{x}_g,t|\mathbf{x})\Delta m(\mathbf{x}) \quad .$$

The Lippmann-Schwinger equations (C-1) and (C-2) describe an integral operator transforming a change in medium parameters into a change in the resulting wavefield,

$$\begin{bmatrix} \Delta v_i(\mathbf{x}_g,t) \\ \Delta\sigma_{ij}(\mathbf{x}_g,t) \end{bmatrix} = \begin{bmatrix} \Gamma_{iknpq}(\mathbf{x}_g,t|\mathbf{x}) & \Gamma_i(\mathbf{x}_g,t|\mathbf{x}) \\ \Gamma_{ijknpq}(\mathbf{x}_g,t|\mathbf{x}) & \Gamma_{ij}(\mathbf{x}_g,t|\mathbf{x}) \end{bmatrix} \begin{bmatrix} \Delta c_{knpq}(\mathbf{x}) \\ \Delta\varrho(\mathbf{x}) \end{bmatrix} . \qquad \text{(D-1)}$$

On the other hand, the gradients are obtained from an integral operator on the difference data,

$$\begin{bmatrix} g_{\alpha\beta\mu\nu}(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = - \begin{bmatrix} \Psi^{\dagger}_{\alpha\beta\mu\nu i}(\mathbf{x}|\mathbf{x}_g,t) & \Psi^{\dagger}_{\alpha\beta\mu\nu ij}(\mathbf{x}|\mathbf{x}_g,t) \\ \Psi^{\dagger}_i(\mathbf{x}|\mathbf{x}_g,t) & \Psi^{\dagger}_{ij}(\mathbf{x}|\mathbf{x}_g,t) \end{bmatrix} \begin{bmatrix} \Delta v_i(\mathbf{x}_g,t) \\ \Delta\sigma_{ij}(\mathbf{x}_g,t) \end{bmatrix} . \qquad \text{(D-2)}$$

In the linear case $\Psi_Q^{\dagger}(\mathbf{x}|\mathbf{x}_g,t)$ can be shown to be equal to $\Gamma_Q^{\dagger}(\mathbf{x}|\mathbf{x}_g,t)$. Both $\Psi_Q^{\dagger}(\mathbf{x}|\mathbf{x}_g,t)$ and $\Gamma_Q^{\dagger}(\mathbf{x}|\mathbf{x}_g,t)$ are then obtained by using the first Born approximation to the Lippmann-Schwinger equations.

In the following we drop spatial and temporal arguments for simplicity. Substituting (D-1) into (D-2) we obtain the relation

$$\begin{bmatrix} g_{\alpha\beta\mu\nu} \\ g \end{bmatrix} = - \begin{bmatrix} H_{\alpha\beta\mu\nu\, rstu} & H_{\alpha\beta\mu\nu} \\ H_{rstu} & H \end{bmatrix} \begin{bmatrix} \Delta c_{rstu} \\ \Delta\varrho \end{bmatrix} , \qquad \text{(D-3)}$$

with the inverse relation

$$\begin{bmatrix} \Delta c_{rstu} \\ \Delta\varrho \end{bmatrix} = - \begin{bmatrix} H_{\alpha\beta\mu\nu\, rstu} & H_{\alpha\beta\mu\nu} \\ H_{rstu} & H \end{bmatrix}^{-1} \begin{bmatrix} g_{\alpha\beta\mu\nu} \\ g \end{bmatrix} . \qquad \text{(D-4)}$$

Equation (D-4) has now a well-known form for the parameter update at iteration n:

$$m_Q^{n+1} = m_Q^n + \Delta m_Q^n \quad ; \qquad \Delta m_Q^n = -H_{QP}^{-1} g_P^n \quad , \qquad \text{(D-5)}$$

where also P may be a set of indices. Here m_Q denotes elastic parameters or density.

For the isotropic case we assume that (D-4) can be written

$$\begin{bmatrix} \Delta c_{rstu} \\ \Delta\varrho \end{bmatrix} = - \begin{bmatrix} R_{rstu\alpha\beta\mu\nu} & R_{rstu} \\ R_{\alpha\beta\mu\nu} & R \end{bmatrix} \begin{bmatrix} g_{\alpha\beta\mu\nu} \\ g \end{bmatrix} \quad . \tag{D-4}$$

and further that the R_Q operators consist of a common spatial term multiplied with a term with constant coefficients resembling the behavior of c_{ijkl}. As an example,

$$R_{rstu\alpha\beta\mu\nu}(\mathbf{x}) = \int d^3x' R(\mathbf{x}|\mathbf{x}')[a\delta_{rs}\delta_{tu} + b(\delta_{rt}\delta_{su} + \delta_{ru}\delta_{st})]$$
$$[c\delta_{\alpha\beta}\,\delta_{\mu\nu} + d(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\beta\mu}\delta_{\alpha\nu})] \quad . \tag{D-7}$$

We let $B(\mathbf{x})$ represent the diagonal of $R(\mathbf{x}|\mathbf{x}')$, that is, $B(\mathbf{x}) = R(\mathbf{x}|\mathbf{x})$. It may be shown that $B(\mathbf{x})$ can be approximated by a function proportional to components of the average field energy in position $\mathbf{x}$ during the forward modeling.

We also observe that for the isotropic case, equation (B-1) reduces to

$$\delta\epsilon = \int d^3x[g_{abmn}(\mathbf{x})\delta c_{abmn}(\mathbf{x}) + g(\mathbf{x})\delta\varrho(\mathbf{x})]$$
$$= \int d^3x[g_{aamm}(\mathbf{x})\delta\lambda(\mathbf{x}) + 2g_{amma}(\mathbf{x})\delta\mu(\mathbf{x}) + g(\mathbf{x})\delta\varrho(\mathbf{x})] \quad . \tag{D-8}$$

The gradients with respect to the Lamé parameters λ and μ, and density are given by equation (14). We define the preconditioned gradients as

$$\hat{g}_\lambda(\mathbf{x}) = B(\mathbf{x})g_\lambda(\mathbf{x}) \ ; \ \hat{g}_\mu(\mathbf{x}) = B(\mathbf{x})g_\mu(\mathbf{x}) \ ; \ \hat{g}_\varrho(\mathbf{x}) = B(\mathbf{x})g_\varrho(\mathbf{x}) \ . \tag{D-9}$$

In the following we let Roman indices be related to spatial directions whereas Greek indices are related to the physical parameters λ, μ and ϱ. Using (D-6), (D-7) and (D-9) we find for the parameter updates

$$\Delta m_\eta = -\alpha_{\eta\varphi}\hat{g}_\varphi \quad . \tag{D-10}$$

Note that the update of one parameter class is in principle dependent on all the gradients.

The processed error norm given in equation (2) is to be minimized with respect to the steplengths $\alpha_{\eta\varphi}$. Defining

$$\Delta W_{j\eta\varphi}(\mathbf{x}_g,t) = [L_j(m_\eta + \epsilon\hat{g}_\varphi) - L_j(m_\eta)]/\epsilon \quad , \tag{D-11}$$

and

$$\Delta\Sigma_{ij\eta\varphi}(\mathbf{x}_g,t) = [L_{ij}(m_\eta + \epsilon\hat{g}_\varphi) - L_{ij}(m_\eta)]/\epsilon \quad , \tag{D-12}$$

we obtain for the parameter update

$$H_{\beta\gamma\eta\varphi}\alpha_{\eta\varphi} = \Theta_{\beta\gamma} \quad , \tag{D-13}$$

where

$$\Theta_{\beta\gamma} = \int d^2x\,\hat{g}^\dagger_\beta(\mathbf{x})g_\gamma(\mathbf{x}) \quad , \tag{D-14}$$

and

$$H_{\beta\gamma\eta\varphi} = -\int_0^T dt \int dS_g[\Delta\Sigma^\dagger_{\beta\gamma ij}(\mathbf{x}_g,t)n_i\Delta W_{j\eta\varphi}(\mathbf{x}_g,t)$$

$$+ \Delta W^\dagger_{\beta\gamma j}(\mathbf{x}_g,t)n_i\Delta\Sigma_{ij\eta\varphi}(\mathbf{x}_g,t)]^{free} \quad , \tag{D-15}$$

where again the effect of free surfaces must be removed. $\alpha_{\eta\phi}$ is given by a 9×9 system of linear equations. We have the possibility to reduce this system assuming less coupling between the physical parameters. A further reduction in numerical costs can be obtained by using a subset of all shots when calculating these steplengths.

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