GEOPHYSICS, VOL. 65, NO. 5 (SEPTEMBER-OCTOBER 2000); P. 1604–1612, 11 FIGS., 1 TABLE.

A new insight into the reciprocity principle

Børge Arntsen* and José M. Carcione‡

ABSTRACT

Reciprocity is usually applied to wavefields associated with concentrated point forces and point receivers, but, reciprocity has a much wider application potential. In many cases, however, it is not used at its full potential because (1) a variety of source and receiver types are not considered or (2) its implementation is not well understood.

We obtain reciprocity relations for inhomogeneous, anisotropic, viscoelastic solids and for distributed sources and receivers, and test these relations with a full-wave numerical modeling algorithm. The theory and the numerical experiments show that, in addition to the usual relations involving directional forces, (1) the diagonal components of the strain tensor are reciprocal for dipole sources (single couple without moment), (2) the off-diagonal components of the stress tensor are reciprocal for double couples with moments, (3) the dilatation due to a directional force is reciprocal to the particle velocity due to a dilatational source, and (4) some combinations of the off-diagonal strains are reciprocal for single couples with moments.

INTRODUCTION

The reciprocity principle relates two wavefields in a medium where the sources and the field receivers are interchanged. The reciprocity principle for static displacements is credited to Betti (1872). Rayleigh (1873) extended the principle to elastodynamic fields and included the action of dissipative forces. Lamb (1888) showed how the reciprocal theorems of von Helmholtz (in the theory of least action in acoustics and optics) and of Lord Rayleigh (in acoustics) can be derived from a remarkable formula established by Lagrange in his 1909 *Méchanique Analytique* (Fung, 1965). In the twentieth century, the work of Graffi (1939, 1954, 1963) is notable. Graffi derived the first convolutional reciprocity theorem for an isotropic, homogeneous, perfectly elastic solid with its extension to inhomogeneous elastic anisotropic media achieved by Knopoff and Gangi (1959). Gangi (1970) developed a volume integral, time-convolution formulation of the reciprocity principle for inhomogeneous, anisotropic linearly elastic media. This formulation allows the use of distributed sources as well as multicomponent sources (i.e., couples with and without moment). He also derived a representation of particle displacement in terms of Green's theorem.

De Hoop (1966) generalized the principle to the viscoelastic anisotropic case in the time domain. A direct numerical test of the principle in the inhomogeneous elastic anisotropic case was performed by Carcione and Gangi (1998). A notable contribution is the work of Boharski (1983), who distinguished between convolution-type and correlation-type reciprocity relations. Recently, De Hoop and Stam (1988) derived a general reciprocity theorem valid for solids with relaxation, including reciprocity for stress, as well as for particle velocity. Useful applications of the reciprocity principle can be found in Fokkema and van den Berg (1993).

Reciprocity holds for the very general case of an inhomogeneous anisotropic viscoelastic solid in the presence of boundary surfaces satisfying Dirichlet and/or Neumman boundary conditions (e.g., Lamb's problem) (Fung, 1965). However, it is not clear how the principle is applied when the sources are couples (Fenati and Rocca, 1984). For instance, Mittet and Hokstad (1995) used reciprocity to transform walkaway VSP data into reverse VSP data, for offshore acquisition. Nyitrai et al. (1996) claim that the analytical solution to Lamb's problem (expressed in terms of particle displacement) for a dilatational point source does not exhibit reciprocity when the source and receiver locations are interchanged. Hence, the following question arises: what, if any, source-receiver configuration is reciprocal in this particular situation? In order to answer this question, we applied the reciprocity principle to the case of sources of couples and demonstrate that for any particular source, there is a corresponding receiver configuration which makes the source-receiver pair reciprocal.

We shall consider a viscoelastic transversely isotropic (VTI) constitutive equation, where anelasticity is described by the standard-linear-solid rheology, and perform numerical

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*Statoil Research Center, Postuttak, N-7005 Trondheim, Norway. E-mail: barn@statoil.no.
‡Osservatorio Geofisico Sperimentale, Borgo Gota Gigante 42c, 34010 Sgonico, Trieste, Italy. E-mail: jcarcione@ogs.trieste.it.

experiments in an inhomogeneous model. The 2-D time-domain equations for propagation in a heterogeneous VTI medium can be found in Carcione (1995, 1997). They are given in the particle velocity-stress formulation, and are solved numerically by using the Fourier pseudospectral method to compute the spatial derivatives. A fourth-order Runge-Kutta technique is used to compute the wavefield recursively in time. Lamb's problem is solved with a similar algorithm, except that the modified Chebysehev differential operator is used along the direction perpendicular to the free surface (Carcione, 1992).

RECIPROCITY PRINCIPLE FOR AN ANISOTROPIC VISCOELASTIC MEDIUM

Let us consider a volume V, enclosed by a surface S, in a viscoelastic solid of density $\rho(\mathbf{x})$ and relaxation tensor $\psi_{ijkl}(\mathbf{x},t)$, where $\mathbf{x}=(x,y,z)=(x_1,x_2,x_3)$ denotes the position and t denotes the time. The equation of motion and the stress-strain relation take the following form:

$$\rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t) = \partial_j\sigma_{ij}(\mathbf{x},t) + f_i(\mathbf{x},t), \tag{1}$$

$$\sigma_{ij}(\mathbf{x},t) = \psi_{ijkl}(\mathbf{x},t) * e_{kl}(\mathbf{x},t), \tag{2}$$

where the indices i and j refer to x, y, and z; u_i are the displacement components; σ_{ij} are the stress components; f_i are the body-force components; and e_{ij} are the strain components given by

$$e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i). \tag{3}$$

The notation ∂_j is used to indicate a spatial partial derivative in the x, y, or z direction. The operator $*$ denotes the time convolution, defined by

$$a(t) * b(t) = \int_0^t d\tau\, a(t-\tau)b(\tau), \tag{4}$$

where a and b are arbitrary time functions. Double dots above a variable indicate a second-order time derivative, and Einstein's summation convention is used.

A reciprocity theorem valid for a general viscoelastic anisotropic medium can be derived from the equation of motion (1) and the constitutive equation (2) (Knopoff and Gangi, 1959; De Hoop, 1966) and written in the form of a volume integral:

$$\int d\mathbf{x}[u_i(\mathbf{x},t) * f_i'(\mathbf{x},t) - f_i(\mathbf{x},t) * u_i'(\mathbf{x},t)] = 0. \tag{5}$$

Here u_i is the ith component of the displacement due to the source $\mathbf{f}$, whereas u_i' is the ith component of the displacement due to the source $\mathbf{f}'$. The derivation of equation (5) assumes that the stresses are zero on the boundary S. Zero initial conditions for the displacements are also assumed. Equation (5) is well known and can conveniently be used for deriving representations of the displacement in terms of Green's tensor (Gangi, 1970).

In the following, it is assumed that the time dependence of the displacements and sources are such that the Fouriér transforms over time exists. Then, equation (5) can be Fouriér transformed into the frequency domain:

$$\int d\mathbf{x}[U_i(\mathbf{x},\omega)F_i'(\mathbf{x},\omega) - F_i(\mathbf{x},\omega)U_i'(\mathbf{x},\omega)] = 0, \tag{6}$$

where U_i, F_i', F_i, and U_i' are the Fouriér transforms of u_i, f_i', f_i, and u_i', respectively, and ω is the angular frequency.

Equation (6) can also be expressed in terms of the particle velocity $V_i(\mathbf{x},\omega) = i\omega U_i(\mathbf{x},\omega)$ by multiplying both sides with $i\omega$:

$$\int d\mathbf{x}[V_i(\mathbf{x},\omega)F_i'(\mathbf{x},\omega) - F_i(\mathbf{x},\omega)V_i'(\mathbf{x},\omega)] = 0. \tag{7}$$

In the time domain, equation (7) reads

$$\int d\mathbf{x}[v_i(\mathbf{x},t) * f_i'(\mathbf{x},t) - f_i(\mathbf{x},t) * v_i'(\mathbf{x},t)] = 0, \tag{8}$$

where v_i is the particle velocity. In the special case that the sources f_i and f_i' have the same time dependence and can be written as a product of a spatial part $g(\mathbf{x})$ and a temporal part $h(t)$,

$$\begin{aligned} f_i(\mathbf{x},t) &= h(t)g_i(\mathbf{x}), \\ f_i'(\mathbf{x},t) &= h(t)g_i'(\mathbf{x}), \end{aligned} \tag{9}$$

equation (7) reads

$$\int d\mathbf{x}[V_i(\mathbf{x},\omega)g_i'(\mathbf{x})H(\omega) - g_i(\mathbf{x})H(\omega)V_i'(\mathbf{x},\omega)] = 0, \tag{10}$$

where $H(\omega)$ is the Fouriér transform of $h(t)$. Equation (10) is equivalent to

$$\int d\mathbf{x}[V_i(\mathbf{x},\omega)g_i'(\mathbf{x}) - g_i(\mathbf{x})V_i'(\mathbf{x},\omega)] = 0, \tag{11}$$

because V_i and V_i' is zero whenever H is zero.

In the time domain, equation (11) reads

$$\int d\mathbf{x}[v_i(\mathbf{x},t)g_i'(\mathbf{x}) - g_i(\mathbf{x})v_i'(\mathbf{x},t)] = 0. \tag{12}$$

RECIPROCITY OF PARTICLE VELOCITY

In the following, the indices m and p indicate either x, y, or z. The spatial part g_i of the body force f_i will simply be referred to as the body force.

To indicate the direction of the body force, a superscript is used so that the ith component g_i^m of a body force acting at $\mathbf{x}=\mathbf{x}_0$ in the m-direction is specified by

$$g_i^m(\mathbf{x};\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0)\delta_i^m. \tag{13}$$

Here, $\delta(\mathbf{x})$ is Dirac's delta function, and the symbol δ_i^m is defined by

$$\delta_i^m = \begin{cases} 1 & m = i \\ 0 & m \neq i \end{cases}. \tag{14}$$

The ith component g_i^p of a body force acting at $\mathbf{x}=\mathbf{x}_0'$ in the p-direction is similarly given by

$$g_i^p(\mathbf{x};\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0')\delta_i^p. \tag{15}$$

We will occasionally refer to body forces of the type given by equations (13) and (15) as monopoles. In the following, we use a superscript on the particle velocity to indicate the direction of the corresponding body force; v_i^m then indicates the ith component of the particle velocity due to a body force acting in the

m-direction, v_i^p indicates the ith component of the particle velocity due to a body force acting in the p-direction. In addition, we also indicate the position of the source in the argument of the particle velocity such that a complete specification of the ith component of the particle velocity due to a body force acting at position $\mathbf{x}_0$ in the m-direction is written as $v_i^m(\mathbf{x}, t; \mathbf{x}_0)$. For the primed system, we similarly have $v_i^p(\mathbf{x}, t; \mathbf{x}'_0)$. The only difference between the primed and unprimed systems is that in the latter the source is acting in the p-direction and is located at position $\mathbf{x}'_0$.

In the Appendix [see equation (A-3)], the well-known result (Knopoff and Gangi, 1959)

$$v_p^m(\mathbf{x}'_0, t; \mathbf{x}_0) = v_m^p(\mathbf{x}_0, t; \mathbf{x}'_0), \tag{16}$$

is derived using equation (12).

This equation reveals a fundamental symmetry of the wavefield in a viscoelastic medium: in any given experiment, the source and receiver positions may be interchanged provided that the particle velocity component indices and the force component indices are interchanged. Note that this equation only applies to the situation where the source consists of a simple body force. In order to illustrate the interpretation of equation (16), Figure 1 shows three possible 2-D reciprocity experiments.

RECIPROCITY OF STRAIN

For sources more complex than a body force directed along one of the coordinate axes, the reciprocity relation will differ from equation (16). Equation (12) is, however, valid for an arbitrary spatially distributed source and will be used to derive reciprocity relations for couples of forces. Reviews of the use of couples for modeling earthquake sources can be found in Aki and Richards (1980) and Pilant (1979). We investigate the use of force couples of different types in the following subsection.

Couples

We shall be concerned with sources consisting of force couples where the ith component of the body force takes the particular form

$$g_i^{mn}(\mathbf{x}; \mathbf{x}_0) = \partial_j \delta(\mathbf{x} - \mathbf{x}_0)\delta_i^m \delta_j^n. \tag{17}$$

Here, the double superscript mn is used to indicate that the force couple depends on the m and n directions. In the primed system the source are similarly specified by

$$g_i^{pq}(\mathbf{x}; \mathbf{x}'_0) = \partial_j \delta(\mathbf{x} - \mathbf{x}'_0)\delta_i^p \delta_j^q. \tag{18}$$

Following Aki and Richards (1980), the forces in equations (17) and (18) may be thought of as composed of a simple (point) force in the positive m-direction and another force of equal magnitude in the negative m-direction separated by a small distance in the n-direction. The magnitude of the forces must be chosen such that the product of the distance between the forces and the magnitude is unity. This is illustrated by the examples in Figures 2 and 3. Although these examples are two-dimensional, we can think of the third (y-) axis as normal to the plane defined by the x- and z-axes. The source in the top left experiment in Figure 2 can be obtained from equation (17) by putting $\mathbf{x}_0$ at the origin with $m = n = x$. Then, $g_y^{xx} = g_z^{xx} = 0$ and

$$g_x^{xx}(\mathbf{x}; 0) = \partial_x \delta(\mathbf{x}). \tag{19}$$

Consider now the source in the top left experiment in Figure 3. Using equation (17) and assuming $m = x$ and $n = z$, then $g_y^{xz} = g_z^{xz} = 0$ and

$$g_x^{xz}(\mathbf{x}; 0) = \partial_z \delta(\mathbf{x}). \tag{20}$$

This body force possesses a moment around the y-axis, in contrast to the source considered in Figure 2, which has zero moment around the y-axis. Whenever $m = n$, the body force is referred to as a couple without moment, whereas when $m \neq n$

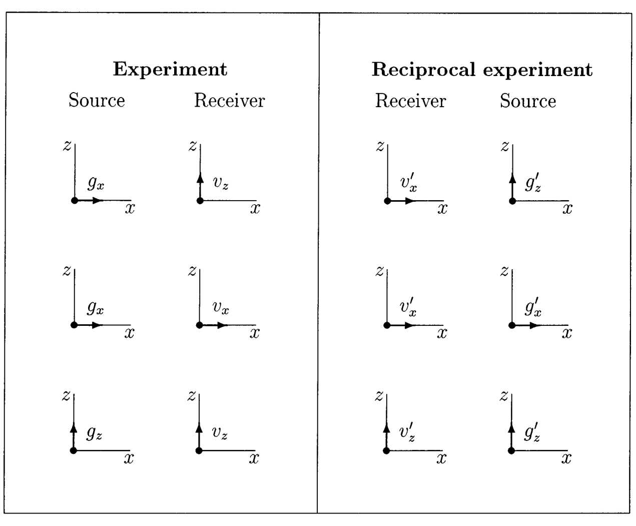

FIG. 1. Some 2-D reciprocal experiments for single forces. The body forces are positioned at $\mathbf{x}_0$ and $\mathbf{x}'_0$. Also $g_x = g_z = \delta(\mathbf{x} - \mathbf{x}_0)$ while $g'_x = g'_z = \delta(\mathbf{x} - \mathbf{x}'_0)$, where δ is Dirac's delta function.

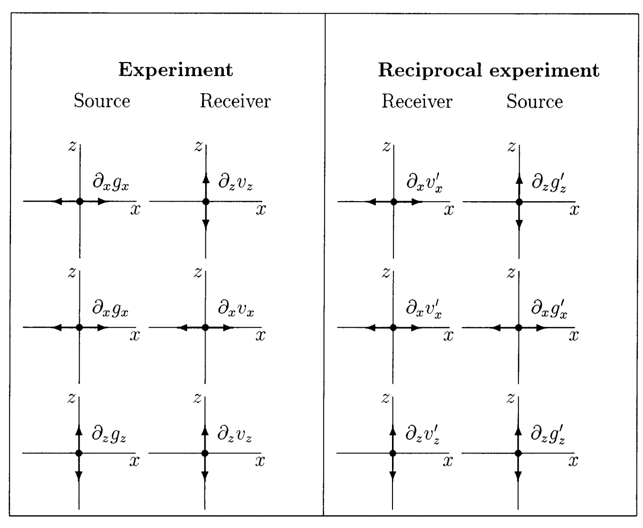

FIG. 2. Some 2-D reciprocal experiments for couples without moment. The body forces are positioned at $\mathbf{x}_0$ and $\mathbf{x}'_0$. Also $g_x = g_z = \delta(\mathbf{x} - \mathbf{x}_0)$ while $g'_x = g'_z = \delta(\mathbf{x} - \mathbf{x}'_0)$, where δ is a Dirac delta function.

the corresponding body force is referred to as a couple with moment. Figures 2 and 3 also illustrate other examples of couples with and without moment.

The particle velocity corresponding to the body force given by equation (17) is written as $v_i^{mn}(\mathbf{x}, t; \mathbf{x}_0)$, using a double superscript. The particle velocity in the primed system is correspondingly written as $v_i^{pq}(\mathbf{x}, t; \mathbf{x}_0')$.

Substituting equations (17) and (18) into equation (12) and using properties of Dirac's delta function, yields the reciprocity relation for couple forces as derived in the Appendix [equation (A-12)]:

$$\partial_q\left[v_p^{mn}(\mathbf{x}_0', t; \mathbf{x}_0)\right] = \partial_n\left[v_m^{pq}(\mathbf{x}_0, t; \mathbf{x}_0')\right]. \tag{21}$$

The interpretation of equation (21) is similar to that of equation (16), except that the derivatives of the particle velocity are reciprocal instead of the particle velocity itself.

Single couples without moment.—When $m = n$ and $p = q$ in equation (21), the derivatives are calculated along the force directions. The resulting couples have orientations depending on the force directions. This is illustrated in Figure 2 for three different experiments.

Single couples with moment.—This situation corresponds to the case $m \neq n$ and $p \neq q$ in equation (21). The resulting couples have moments, with three possible experiments illustrated in Figure 3.

Double couples.—Two perpendicular couples without moments constitute a dilatational source.

This source can be expressed as

$$g_i(\mathbf{x}; \mathbf{x}_0) = \partial_i \delta(\mathbf{x} - \mathbf{x}_0). \tag{22}$$

The ith component of the corresponding particle velocity is denoted by $v_i(\mathbf{x}, t; \mathbf{x}_0)$.

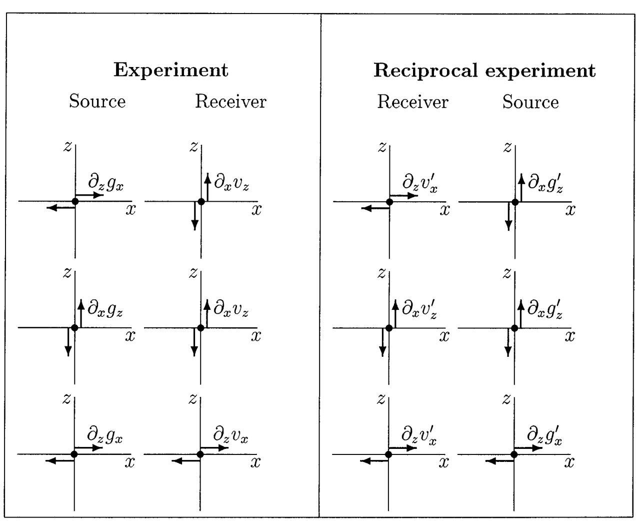

FIG. 3. Some 2-D reciprocal experiments for single couples with moment. The body forces are positioned at $\mathbf{x}_0$ and $\mathbf{x}_0'$. Also $g_x = g_z = \delta(\mathbf{x} - \mathbf{x}_0)$ while $g_x' = g_z' = \delta(\mathbf{x} - \mathbf{x}_0')$, where δ is a Dirac delta function.

The above dilatational force leads to a reciprocity relation of the type [see equation (A-17)]

$$e(\mathbf{x}_0', t; \mathbf{x}_0) = e(\mathbf{x}_0, t; \mathbf{x}_0'), \tag{23}$$

where

$$e = \partial_x v_x + \partial_y v_y + \partial_z v_z \tag{24}$$

is the time derivative of the dilatation. Equation (24) indicates that for a dilatational point source (explosion), the trace of the time derivative of the strain tensor (dilatation) is unchanged when the source and receiver are interchanged.

Consider the case of a double couple without moment in the unprimed system [equation (22)], with corresponding particle-velocity components denoted by $v_i(\mathbf{x}, t; \mathbf{x}_0)$, and in the primed system there is a monopole [equation (15)] with corresponding particle-velocity components given by $v_i^m(\mathbf{x}, t, \mathbf{x}_0')$. Here m indicates the direction of the monopole body force. The reciprocity relation for this case is given in the Appendix [equation (A-22)]:

$$v_m(\mathbf{x}_0', t; \mathbf{x}_0) = e^m(\mathbf{x}_0, t; \mathbf{x}_0'), \tag{25}$$

where $e^m = \partial_i v_i^m = \partial_x v_x^m + \partial_y v_y^m + \partial_z v_z^m$ is the trace of the time derivative of the strain tensor.

Equation (25) indicates that the particle velocity must be substituted for the trace of the time derivative of the strain tensor when the source and receiver are interchanged. The case $v^z = e^z$ is illustrated in Figure 4 (top).

Next, we consider the case when in the unprimed system there is a double couple without moment [equation (22)], with corresponding particle-velocity components given by $v_i(\mathbf{x}, t; \mathbf{x}_0)$, and in the primed system there is a single couple [equation (21)] with particle-velocity components given by $v_i^{mn}(\mathbf{x}, t; \mathbf{x}_0')$. The corresponding reciprocity relation is derived in the Appendix [equation (A-27)]:

$$\partial_n v_m(\mathbf{x}_0', t; \mathbf{x}_0) = e^{mn}(\mathbf{x}_0, t; \mathbf{x}_0'), \tag{26}$$

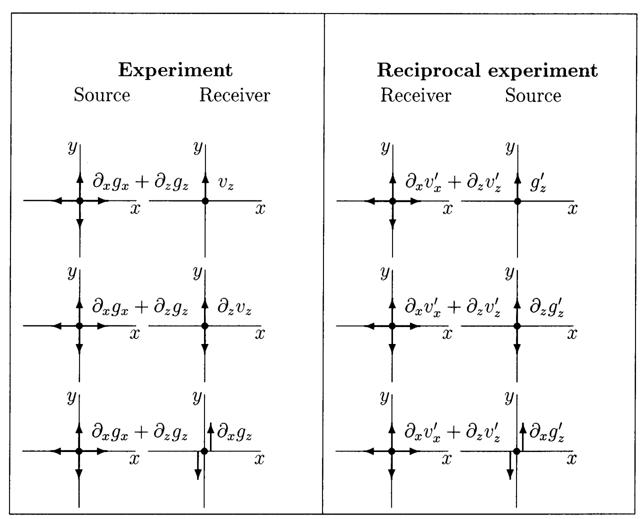

FIG. 4. Some 2-D reciprocal experiments for double couples without moment and single couples. The body forces are positioned at $\mathbf{x}_0$ and $\mathbf{x}_0'$. Also $g_x = g_z = \delta(\mathbf{x} - \mathbf{x}_0)$ while $g_x' = g_z' = \delta(\mathbf{x} - \mathbf{x}_0')$, where δ is a Dirac delta function.

where $e^{mn} = \partial_i v_i^{mn} = \partial_x v_x^{mn} + \partial_y v_y^{mn} + \partial_z v_z^{mn}$. Here, the trace of the time derivative of the strain must be substituted for the derivatives of the particle velocity when the source and receiver are interchanged. Two examples are illustrated in Figure 4 (middle and bottom).

RECIPROCITY OF STRESS

Although equation (8) only involves the particle velocity, a proper choice of the body forces f_i and f_i' leads to reciprocity relations for stress. This occurs for the following body-force components:

$$f_i^{mn}(\mathbf{x}, t; \mathbf{x}_0) = \left[\psi_{ijkl}(\mathbf{x}_0, t)\partial_j \delta(\mathbf{x} - \mathbf{x}_0)\delta_k^m \delta_l^n\right] * h(t), \tag{27}$$

and

$$f_i'^{pq}(\mathbf{x}, t; \mathbf{x}_0') = \left[\psi_{ijkl}(\mathbf{x}_0', t)\partial_j \delta(\mathbf{x} - \mathbf{x}_0')\delta_k^p \delta_l^q\right] * h(t). \tag{28}$$

They consist of couples with direction of forces and moments defined by superscripts m, n, p, and q and the components of the relaxation tensor at the source point. The components of the stress tensor corresponding to those forces are denoted by $\sigma_{ij}^{mn}(\mathbf{x}, t; \mathbf{x}_0)$ and $\sigma_{ij}^{pq}(\mathbf{x}, t; \mathbf{x}_0')$ respectively.

The reciprocity relation for stress is derived in the Appendix [equation (A-41)] and given as

$$\dot{\sigma}_{pq}^{mn}(\mathbf{x}_0', t; \mathbf{x}_0) = \dot{\sigma}_{mn}^{pq}(\mathbf{x}_0, t; \mathbf{x}_0'). \tag{29}$$

The interpretation of equation (29) is the following. The pq stress component at position $\mathbf{x}_0'$ due to a body force with ith component given by f_i^{mn} at position $\mathbf{x}_0$ equals the mn stress component at position $\mathbf{x}_0$ due to a body force with ith component given by f_i^{pq} located at position $\mathbf{x}_0'$. Figure 5 illustrates the source and receiver configuration for an experiment corresponding to reciprocity of stress.

NUMERICAL EXPERIMENTS

A set of numerical experiments verify the reciprocity relations, based on the source-receiver configuration and model illustrated in Figure 6. Table 1 gives the material properties of the different TIV media, where $V_P = \sqrt{c_{33}/\rho}$ and $V_S = \sqrt{c_{55}/\rho}$ denote the unrelaxed vertical velocities, with c_{IJ} the unrelaxed stiffnesses in abbreviated notation, and ρ the material density; and

$$c_{11} = (2\epsilon + 1)c_{33},$$
$$c_{13} = \left[2c_{33}(c_{33} - c_{55})\delta - (c_{33} - c_{55})^2\right]^{1/2} - c_{55},$$

where ϵ and δ are the (unrelaxed) anisotropy coefficients defined by Thomsen (1986). Finally, Q_P and Q_S are the quality factors, due to single relaxation mechanisms related to dilatation and shear deformations, respectively (see Carcione, 1997).

The numerical mesh has 231×231 points with a grid spacing $D_X = D_Z = 10$ m. The source is a Ricker-type wavelet and has a central frequency of 25 Hz. The peak frequencies of the relaxation mechanisms were also chosen to be 25 Hz. In grid points, the source (with receiver in the reciprocal experiment) location is (10, 150) and the receiver (with source in the reciprocal experiment) location is (75, 110). The verification of the reciprocity relations is shown in Figure 7 [(a) first experiment in Figure 1 and (b) first experiment in Figure 2], Figure 8 [(a) first experiment in Figure 3 and (b) first experiment in Figure 4], and Figure 9 (experiment in Figure 5), where the dots corresponds to the reciprocal experiments. The sources of the

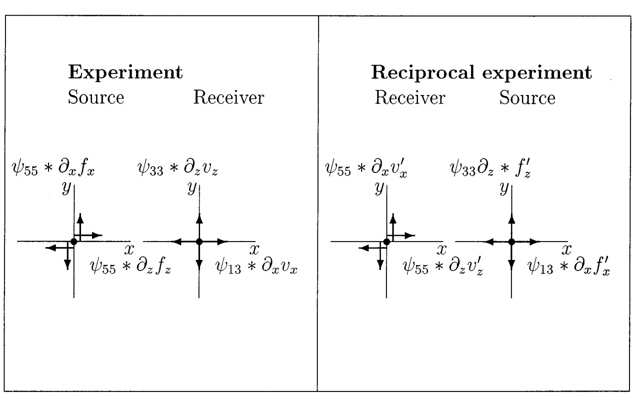

FIG. 5. Source and receiver configuration for reciprocal stress experiments. The body forces are positioned at $\mathbf{x}_0$ and $\mathbf{x}_0'$. Also $f_x = f_z = \delta(\mathbf{x} - \mathbf{x}_0)$ while $f_x' = f_z' = \delta(\mathbf{x} - \mathbf{x}_0')$, where δ is a Dirac delta function.

Table 1. Material properties.

Medium	V_P (km/s)	V_S (km/s)	ϵ	δ	Q_P	Q_S	ρ (g/cm^3)
1	2	1	0.195	0.175	50	40	2
2	2.2	1.3	0.11	0.09	100	80	2.2
3	2.6	1.45	0.015	0.06	150	100	2.3
4	3	1.7	0.2	−0.05	40	20	2.4
5	3.8	2.1	0	0	100	75	2.2
6	3.6	1.7	0.065	0.059	200	150	2.6

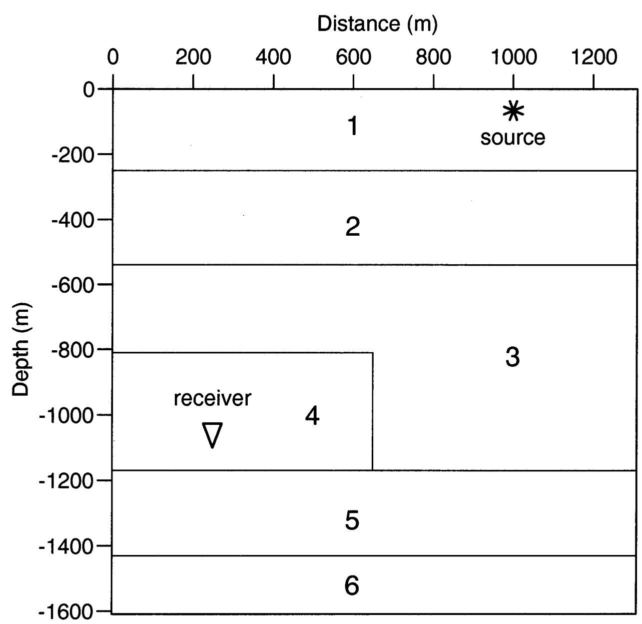

FIG. 6. Model and source-receiver configuration for testing the reciprocity relations due to different sources. The material properties are given in Table 1.

experiments in Figure 9 are

$$f_x^{xz}(\mathbf{x}, t; \mathbf{x}_0) = \psi_{55} * h(t)\partial_z \delta(\mathbf{x} - \mathbf{x}_0),$$
$$f_z^{xz}(\mathbf{x}, t; \mathbf{x}_0) = \psi_{55} * h(t)\partial_x \delta(\mathbf{x} - \mathbf{x}_0)$$

and

$$f_x^{zz}(\mathbf{x}, t; \mathbf{x}'_0) = \psi_{13} * h(t)\partial_x \delta(\mathbf{x} - \mathbf{x}'_0),$$
$$f_z^{zz}(\mathbf{x}, t; \mathbf{x}'_0) = \psi_{33} * h(t)\partial_z \delta(\mathbf{x} - \mathbf{x}'_0),$$

where ψ_{IJ} are relaxation components (Carcione, 1995) (in the elastic case, $\psi_{IJ}(t) = c_{IJ} H(t)$, where H is the Heaviside function). In this experiment, σ_{zz}^{xz} is equal to σ_{xz}^{zz} when the source and receiver positions are interchanged.

Figure 10 represents the model and source-receiver configuration to test the reciprocity relation for an elastic half-space with stress-free boundary conditions. The calculation uses a numerical mesh with $N_x = 135$ and $N_z = 81$, a horizontal grid spacing of 20 m, and a maximum vertical grid spacing of 20 m. The source emits a pulse of peak frequency 30 Hz and is located at 4 m from the free surface, while the receiver is 1200 m apart from the source and at the same depth. Figure 11a compares the dilatations due to dilatational sources (the superposition of the first and second experiments in Figure 2), and Figure 11b compares the particle velocity v_z due to a dilatational source and the dilatation due to a vertical force (dots, first experiment of Figure 4). The first pulse is the compressional wave and the second pulse is a superposition of the Rayleigh and shear body waves.

CONCLUSIONS

We obtained reciprocity relations for different sources in a linear, inhomogeneous, anisotropic and viscoelastic medium, and verified these relations by means of numerical

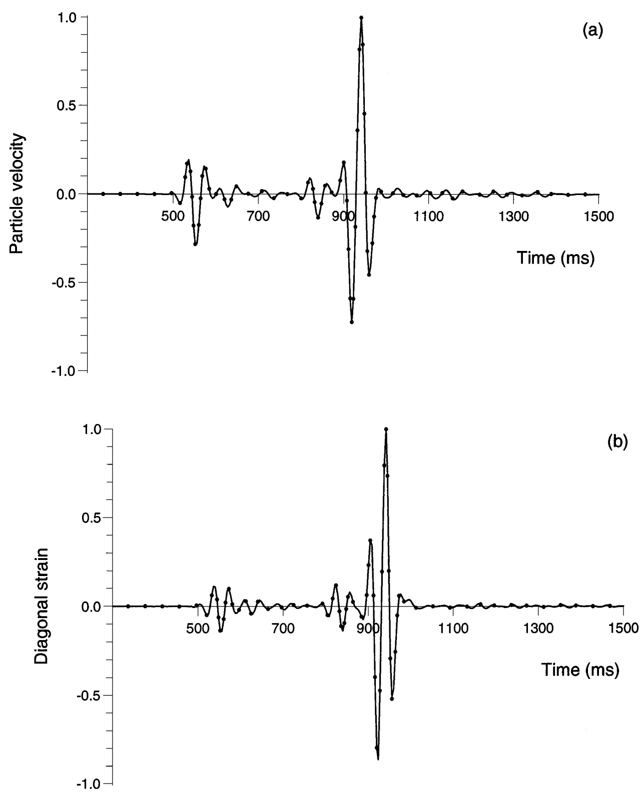

FIG. 7. Verification of the reciprocity relations for (a) single body forces, and (b) single couples without moment. They refer to the first experiments in Figures 1 and 2, respectively (the dots corresponds to the reciprocal experiments).

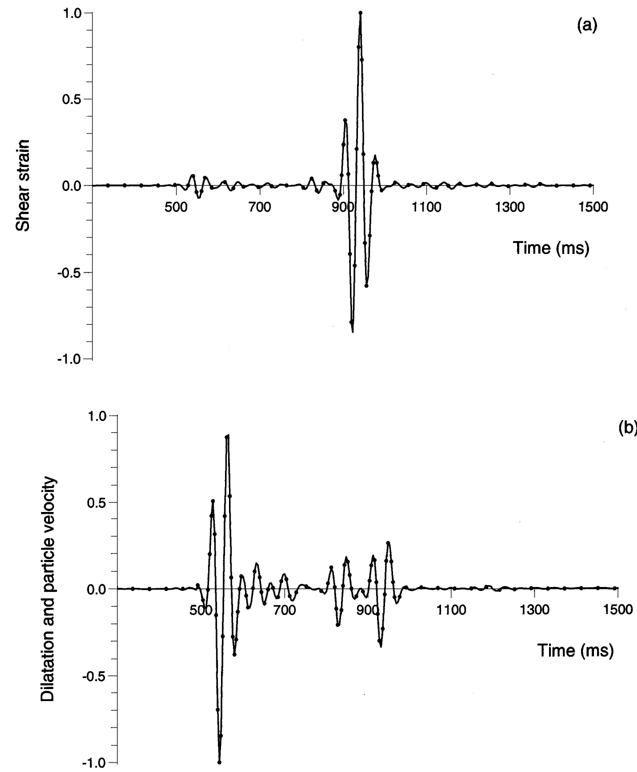

FIG. 8. Verification of the reciprocity relations for (a) single couples with moment and (b) comparison of the particle velocity v_z due to a dilatational source and the dilatation due to a vertical force. They refer to the first experiments in Figures 3 and 4, respectively (the dots corresponds to the reciprocal experiments).

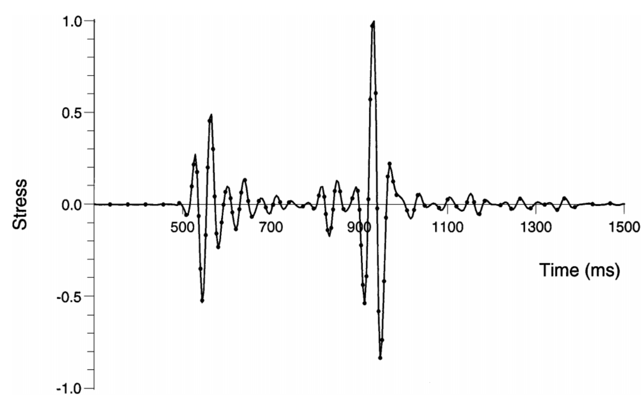

FIG. 9. Verification of the reciprocity relations for stress fields (the dots corresponds to the reciprocal experiments). The source/receiver configuration is shown in Figure 5.

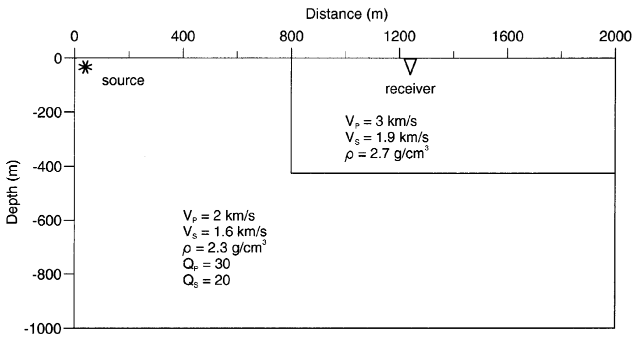

FIG. 10. Model and source-receiver configuration to test the reciprocity relation for an elastic half-space with stress-free boundary conditions.

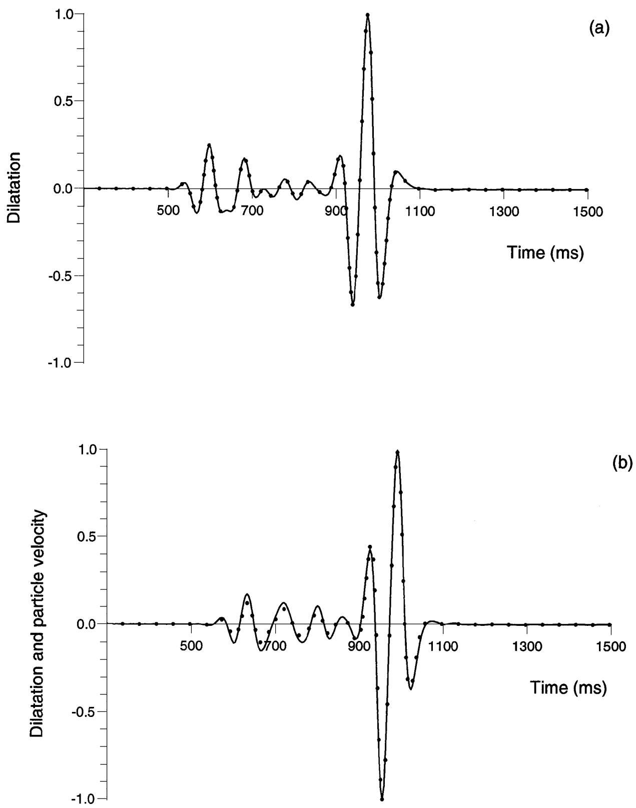

FIG. 11. Reciprocity experiments for the model configuration illustrated in Figure 10: (a) dilatations due to dilatational sources and (b) particle velocity v_z due to a dilatational source and dilatation due to a vertical force. They refer to the first experiment in Figure 4 and an experiment obtained by superposition of the first and second experiments in Figure 2 (the dots corresponds to the reciprocal experiments).

experiments. For many types of sources [e.g., dipoles or explosions (dilatations)], there is a field which satisfies the reciprocity principle. An application of the reciprocity relations can be found, for instance, in off-shore seismic experiments, where the sources are of dilatational type and the hydrophones records the pressure field (i.e., the dilatation multiplied by the water bulk modulus). In land acquisition, the reciprocity relations can be useful in borehole seismic experiments, where couples and pressure sources and receivers are employed.

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APPENDIX

RECIPROCITY RELATIONS

Consider the monopole body forces defined by

$$g_i^m(\mathbf{x};\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0)\delta_i^m, \tag{A-1}$$

with corresponding particle velocity $v_i^m(\mathbf{x},t;\mathbf{x}_0)$, and

$$g_i^p(\mathbf{x};\mathbf{x}_0') = \delta(\mathbf{x}-\mathbf{x}_0')\delta_i^p, \tag{A-2}$$

with particle velocity $v_i^p(\mathbf{x},t;\mathbf{x}_0')$. Using this notation, the reciprocity relation given in the main text by equation (12) can be written as

$$\int d\mathbf{x}\big[v_i^m(\mathbf{x},t;\mathbf{x}_0)g_i^p(\mathbf{x};\mathbf{x}_0) - g_i^m(\mathbf{x};\mathbf{x}_0)v_i^p(\mathbf{x},t;\mathbf{x}_0')\big] = 0. \tag{A-3}$$

In the following, we develop the principle for different sources and receiver types.

Reciprocity for particle velocity

Substituting equations (A-1) and (A-2) into equation (A-3) yields

$$\int d\mathbf{x}\big[v_i^m(\mathbf{x},t;\mathbf{x}_0)\delta(\mathbf{x}-\mathbf{x}_0')\delta_i^p - \delta(\mathbf{x}-\mathbf{x}_0)\delta_i^m v_i^p(\mathbf{x},t;\mathbf{x}_0')\big] = 0. \tag{A-4}$$

Using the property of Dirac's delta function for an arbitrary function $s(\mathbf{x})$,

$$a(\mathbf{x}_0) = \int d\mathbf{x}\, a(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0), \tag{A-5}$$

and performing the summation over the index i gives

$$v_p^m(\mathbf{x}_0',t;\mathbf{x}_0) = v_m^p(\mathbf{x}_0,t;\mathbf{x}_0'). \tag{A-6}$$

Reciprocity for strain

Consider sources consisting of force couples of the type

$$g_i^{mn}(\mathbf{x};\mathbf{x}_0) = \partial_j\delta(\mathbf{x}-\mathbf{x}_0)\delta_i^m\delta_j^n, \tag{A-7}$$

and

$$g_i^{pq}(\mathbf{x};\mathbf{x}_0') = \partial_j\delta(\mathbf{x}-\mathbf{x}_0')\delta_i^p\delta_j^q. \tag{A-8}$$

The ith component of the particle velocity generated by the source in equation (A-7) is denoted by $v_i^{mn}(\mathbf{x},t;\mathbf{x}_0)$, while the ith component of the particle velocity generated by the source in equation (A-8) is denoted by $v_i^{pq}(\mathbf{x},t;\mathbf{x}_0')$.

The reciprocity relation (12) takes the form

$$\int d\mathbf{x}\big[v_i^{mn}(\mathbf{x},t;\mathbf{x}_0)g_i^{pq}(\mathbf{x};\mathbf{x}_0') - g_i^{mn}(\mathbf{x};\mathbf{x}_0)v_i^{pq}(\mathbf{x},t;\mathbf{x}_0')\big] = 0. \tag{A-9}$$

Inserting equations (A-7) and (A-8) into equation (A-9), one gets

$$\int d\mathbf{x}\big[v_i^{mn}(\mathbf{x},t,\mathbf{x}_0)\partial_j\delta(\mathbf{x}-\mathbf{x}_0')\delta_i^p\delta_j^q - \partial_j\delta(\mathbf{x}-\mathbf{x}_0)\delta_i^m\delta_j^n v_i^{pq}(\mathbf{x},t;\mathbf{x}_0')\big] = 0. \tag{A-10}$$

The delta function has the property

$$\partial_i a(\mathbf{x}_0) = \int d\mathbf{x}\, a(\mathbf{x})\partial_i\delta(\mathbf{x}-\mathbf{x}_0), \tag{A-11}$$

where $a(\mathbf{x})$ is an arbitrary function. Carrying out the spatial integral using this property and performing the implied summation over the i and j indices yields

$$\partial_q v_p^{mn}(\mathbf{x}_0',t;\mathbf{x}_0) = \partial_n v_m^{pq}(\mathbf{x}_0,t;\mathbf{x}_0'). \tag{A-12}$$

Single couple with and without moment.—The reciprocity relation given (A-12) applies directly to the case of a single couple with moment if $m \neq n$ and $p \neq p$.

If $m = n$ and $q = p$, equation (A-12) reduces to a reciprocity relation for single couple without moment.

Double couple without moment (dilatation).—Consider force couples of the type

$$g_i(\mathbf{x};\mathbf{x}_0) = \partial_i\delta(\mathbf{x}-\mathbf{x}_0), \tag{A-13}$$

and

$$g_i(\mathbf{x};\mathbf{x}_0') = \partial_i\delta(\mathbf{x}-\mathbf{x}_0'). \tag{A-14}$$

The ith component of the particle velocity corresponding to the source (A-13) is written as $v_i(\mathbf{x},t;\mathbf{x}_0)$, while the ith component of the particle velocity due to the source (A-14) is denoted by $v_i(\mathbf{x},t;\mathbf{x}_0')$.

Substituting equations (A-13) and (A-14) into equation (12) of the main text gives

$$\int d\mathbf{x}[v_i(\mathbf{x},t;\mathbf{x}_0)\partial_i\delta(\mathbf{x}-\mathbf{x}_0') - \partial_i\delta(\mathbf{x}-\mathbf{x}_0)v_i(\mathbf{x},t;\mathbf{x}_0')] = 0. \tag{A-15}$$

Using equation (A-11), the integration can be performed and we get

$$\partial_i v_i(\mathbf{x}_0',t;\mathbf{x}_0) - \partial_i v_i(\mathbf{x}_0,t;\mathbf{x}_0') = 0, \tag{A-16}$$

and defining $e = \partial_i v_i = \partial_x v_x + \partial_y v_y + \partial_z v_z$, equation (A-16) can be written as

$$e(\mathbf{x}_0',t;\mathbf{x}_0) = e(\mathbf{x}_0,t;\mathbf{x}_0'). \tag{A-17}$$

Double couple without moment and monopole force.—Consider force couples of the type

$$g_i(\mathbf{x};\mathbf{x}_0) = \partial_i\delta(\mathbf{x}-\mathbf{x}_0), \tag{A-18}$$

and

$$g_i^m(\mathbf{x};\mathbf{x}_0') = \delta(\mathbf{x}-\mathbf{x}_0')\delta_i^m. \tag{A-19}$$

The ith component of the particle velocity due to the source (A-18) is denoted by $v_i(\mathbf{x},t;\mathbf{x}_0)$, while the ith component of the particle velocity due to the source (A-19) is denoted by

$v_i^m(\mathbf{x}, t; \mathbf{x}_0')$. Inserting equation (A-18) and (A-19) into equation (12) gives

$$\int d\mathbf{x}\big[v_i(\mathbf{x}, t; \mathbf{x}_0)\delta(\mathbf{x}-\mathbf{x}_0')\delta_i^m - \partial_i\delta(\mathbf{x}-\mathbf{x}_0)v_i^m(\mathbf{x}, t; \mathbf{x}_0')\big] = 0. \tag{A-20}$$

Using equations (A-5) and (A-11), the integration can be performed and we get

$$\big[v_i(\mathbf{x}_0', t; \mathbf{x}_0)\delta_i^m - \partial_i v_i^m(\mathbf{x}_0, t; \mathbf{x}_0')\big] = 0. \tag{A-21}$$

Performing the summation over i and defining $e^m = \partial_i v_i^m = \partial_x v_x^m + \partial_y v_y^m + \partial_z v_z^m$ yields

$$v_m(\mathbf{x}_0', t; \mathbf{x}_0) = e^m(\mathbf{x}_0, t; \mathbf{x}_0'). \tag{A-22}$$

Double couple without moment and single couple.—Consider force couples of the type

$$g_i(\mathbf{x}; \mathbf{x}_0) = \partial_i\delta(\mathbf{x} - \mathbf{x}_0), \tag{A-23}$$

and

$$g_i^{mn}(\mathbf{x}; \mathbf{x}_0) = \partial_j\delta(\mathbf{x} - \mathbf{x}_0')\delta_i^m\delta_j^n. \tag{A-24}$$

The ith component of the particle velocity due to the source (A-23) is denoted by $v_i(\mathbf{x}, t; \mathbf{x}_0)$, while the ith component of the particle velocity due to the source (A-24) is denoted by $v_i^{mn}(\mathbf{x}, t; \mathbf{x}_0')$

Substituting equations (A-23) and (A-24) into equation (12) of the main text gives

$$\int d\mathbf{x}\big[v_i(\mathbf{x}, t; \mathbf{x}_0)\partial_j\delta(\mathbf{x} - \mathbf{x}_0')\delta_i^m\delta_j^n - \partial_i\delta(\mathbf{x} - \mathbf{x}_0)v_i^{mn}(\mathbf{x}, t; \mathbf{x}_0')\big] = 0. \tag{A-25}$$

Using equation (A-11), the integration can be performed and we obtain

$$\big[\partial_j v_i(\mathbf{x}_0', t; \mathbf{x}_0)\delta_i^m\delta_j^n - \partial_i v_i^{mn}(\mathbf{x}_0, t; \mathbf{x}_0')\big] = 0. \tag{A-26}$$

Performing the summation over i and j and defining $e^{mn} = \partial_i v_i^{mn} = \partial_x v_x^{mn} + \partial_y v_y^{mn} + \partial_z v_z^{mn}$ yields

$$\partial_n v_m(\mathbf{x}_0', t; \mathbf{x}_0) = e^{mn}(\mathbf{x}_0, t; \mathbf{x}_0'). \tag{A-27}$$

Reciprocity of stress

Consider the following source:

$$f_i^{mn}(\mathbf{x}, t; \mathbf{x}_0) = \big[\psi_{ijkl}(\mathbf{x}_0, t)\partial_j\delta(\mathbf{x} - \mathbf{x}_0)\delta_k^m\delta_l^n\big] * h(t), \tag{A-28}$$

and

$$f_i^{pq}(\mathbf{x}, t; \mathbf{x}_0') = \big[\psi_{ijkl}(\mathbf{x}_0', t)\partial_j\delta(\mathbf{x} - \mathbf{x}_0')\delta_k^p\delta_l^q\big] * h(t). \tag{A-29}$$

The ith component of the particle velocity due to the source (A-28) is indicated by $v_i(\mathbf{x}, t; \mathbf{x}_0)$, while the ith component of the particle velocity due to the source (A-29) is indicated by $v_i(\mathbf{x}, t; \mathbf{x}_0')$, and we omit, for simplicity, the superscripts. The corresponding components of the stress tensor are denoted by $\sigma_{ij}^{mn}(\mathbf{x}, t; \mathbf{x}_0)$ and $\sigma_{ij}^{pq}(\mathbf{x}, t; \mathbf{x}_0')$. The mn and pq superscripts of the stress tensor components correspond to the mn and pq superscripts of the force components. Inserting equations (A-28) and (A-29) into equation (8) gives

$$\int d\mathbf{x}\big\{v_i(\mathbf{x}, t; \mathbf{x}_0) * \big[\psi_{ijkl}(\mathbf{x}_0', t)\partial_j\delta(\mathbf{x} - \mathbf{x}_0')\delta_k^p\delta_l^q\big] * h(t) \tag{A-30}$$

$$- v_i(\mathbf{x}, t; \mathbf{x}_0') * \big[\psi_{ijkl}(\mathbf{x}_0, t)\partial_j\delta(\mathbf{x} - \mathbf{x}_0)\delta_k^m\delta_l^n\big] * h(t)\big\} = 0. \tag{A-31}$$

Using equation (A-11), the integration can be performed and we get

$$\big[\partial_j v_i(\mathbf{x}_0', t; \mathbf{x}_0) * \big[\psi_{ijkl}(\mathbf{x}_0', t)\delta_k^p\delta_l^q\big] * h(t) \tag{A-32}$$

$$-\big[\partial_j v_i(\mathbf{x}_0, t; \mathbf{x}_0) * \big[\psi_{ijkl}(\mathbf{x}_0, t)\delta_k^m\delta_l^n\big] * h(t) = 0. \tag{A-33}$$

We now use that fact that $\psi_{ijkl} = \psi_{klij}$ to rewrite equation (A-33):

$$\big[\psi_{ijkl}(\mathbf{x}_0', t; \mathbf{x}_0) * \partial_l v_k(\mathbf{x}_0', t; \mathbf{x}_0)\delta_i^p\delta_j^q\big] * h(t) \tag{A-34}$$

$$-\big[\psi_{ijkl}(\mathbf{x}_0, t; \mathbf{x}_0') * \partial_k v_k(\mathbf{x}_0, 0, t; \mathbf{x}_0')\delta_i^m\delta_j^n\big] * h(t) = 0. \tag{A-35}$$

We can now use the symmetry relations $\psi_{ijkl} = \psi_{jikl} = \psi_{ijlk} = \psi_{jilk}$ to obtain

$$\big[\sigma_{ij}^{mn}(\mathbf{x}_0', t; \mathbf{x}_0)\delta_i^p\delta_j^q\big] * h(t) - \big[\sigma_{ij}^{pq}(\mathbf{x}_0, t; \mathbf{x}_0')\delta_i^m\delta_j^n\big] * h(t) = 0, \tag{A-36}$$

where the constitutive relation (2) has been used. Performing the implied summations over i and j gives

$$\sigma_{pq}^{mn}(\mathbf{x}_0, t; \mathbf{x}_0) * h(t) - \sigma_{mn}^{pq}(\mathbf{x}_0, t; \mathbf{x}_0') * h(t) = 0. \tag{A-37}$$

Equation (A-37) can be Fouriér transformed into

$$\sum_{pq}^{mn}(\mathbf{x}_0', \omega; \mathbf{x}_0)\mathbf{H}(\omega) - \sum_{mn}^{pq}(\mathbf{x}_0, \omega; \mathbf{x}_0')\mathbf{H}(\omega) = 0, \tag{A-38}$$

where $\sum_{pq}^{mn}$, $\sum_{mn}^{pq}$, and H are the Fouriér transforms of σ_{pq}^{mn}, σ_{mn}^{pq}, and h, respectively. It is now clear that equation (A-38) is equivalent to

$$\sum_{pq}^{mn}(\mathbf{x}_0', \omega; \mathbf{x}_0) - \sum_{mn}^{pq}(\mathbf{x}_0, \omega; \mathbf{x}_0') = 0, \tag{A-39}$$

which in the time domain reads

$$\sigma_{pq}^{mn}(\mathbf{x}_0', t; \mathbf{x}_0) = \sigma_{mn}^{pq}(\mathbf{x}_0, t; \mathbf{x}_0'). \tag{A-40}$$

Finally, differentiating both sides with respect to time yields

$$\dot\sigma_{pq}^{mn}(\mathbf{x}_0', t; \mathbf{x}_0) = \dot\sigma_{mn}^{pq}(\mathbf{x}_0, t; \mathbf{x}_0'). \tag{A-41}$$