

# NUMERICAL RAY GENERATION IN THE COMPUTATION OF SYNTHETIC VERTICAL SEISMIC PROFILES<sup>1</sup>

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## ABSTRACT

ARNTSEN, B. 1988. Numerical ray generation in the computation of synthetic vertical seismic profiles. *Geophysical Prospecting* 36, 478–503.

Ray theories are a class of methods often chosen to compute synthetic seismograms due to their efficiency and ability to deal with complex, three-dimensional inhomogeneous media. To deal with the large number of rays needed to compute synthetic seismograms, a ray generation algorithm is given which is capable of generating a numerical code describing each ray. The code describes a subset of all possible rays by considering only pre-critical reflections. In a horizontally plane-layered medium the generation of rays and computation of amplitudes and traveltimes can be efficiently accomplished by grouping the rays into reflection order and dynamic analogue groups. Expressions summing all unconverted rays and rays with a single mode conversion are given for source and receiver located at arbitrary positions within the medium. Examples of zero-offset synthetic VSPs obtained by this method are given.

## INTRODUCTION

Computer modelling of wave propagation plays an increasing role for acquisition, processing and interpretation of vertical seismic profiles (VSPs). Ray theories, which decompose the displacement field into an infinite number of rays, are a class of methods very often chosen to compute synthetic seismograms due to their efficiency and ability to deal with complex three-dimensional (3D) inhomogeneous media (Červený and Hron 1980). Computer programs based on ray theory have several advantages compared to finite-difference programs and programs based on related techniques. The most important advantage is low cost in terms of computer resources. Even small computer systems can be used to obtain seismograms for relatively simple 3D structures, whereas 3D finite-difference computations require hours of CPU-time on large vector computers. Finite-difference methods are, however, able to give a complete wavefield including post-critical reflections, interface waves and other wave phenomena. With the geometries used for VSP, pre-

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critical rays are the most important contributions to the wavefield, and these are handled well by ray theory.

In real numerical computations only a finite number of rays may be considered. One must then select the rays contributing most significantly to the wavefield. This task is considerably simplified if one deals with plane-parallel layers. But even in this case the contribution of each ray depends on many parameters such as wave velocities, thicknesses of layers, density, the length of the raypath and positions of source and receiver. Manual selection of rays is difficult and one is then left with a computer-based algorithm.

Such an algorithm has been formulated by Hron (1971) and an automatic ray generation scheme for the plane-layered case was developed and applied to compute synthetic seismograms for an elastic medium with the source and receiver at the surface (Hron and Kanasewich 1971; Hron 1971, 1972, 1973). Both P- and S-waves, together with certain types of converted waves and head waves were included (Hron, Kanasewich and Alpaslan 1974; Hron, Daley and Marks 1978; Marks and Hron, paper presented at 47th SEG meeting in Calgary, Alberta, Canada, 1981; Hron and Covey 1983). The algorithm was extended to SH-waves propagating in an anelastic medium by Krebes and Hron (1980a, b). Vered and Ben-Menahem (1974) extended the method to a source at depth and a receiver at the surface.

Hron's method is an expansion in the number of ray segments and not in the number of reflections. This leads to an unsystematic inclusion of rays with a different number of reflections depending on depth. The numerical efficiency can often be improved by using an expansion in the number of reflections, as in Vetter (1980a, b).

An algorithm is given here for generating rays with an arbitrary number of reflections for a layered medium with source and receiver placed at arbitrary depths. This geometry is of interest for the computation of synthetic vertical seismic profiles (VSPs). Both unconverted rays and converted rays with a single conversion are included. The algorithm is a generalization of the algorithm given by Vetter (1981) and is based on a classification of rays by the number of reflections.

The first section describes rays in a general layered medium. The second and third sections describe the special case of a horizontally layered medium, for which compact formal expressions, summing all unconverted rays and rays with a single conversion, are given. Chapman (1977) and Abramovici (1984) have given similar expressions for the summation of generalized rays based on Hron's (1971) classification. Their work includes only unconverted waves, but it is a wave method and provides exact formal solutions for SH-waves in a horizontally layered medium with constant wave velocities and densities in each layer. The last section contains details on the numerical implementation of the ray generation algorithm for a horizontally layered medium, and numerical examples of synthetic zero-offset vertical seismic profiles and zero-offset surface seismograms.

## DESCRIPTION OF RAYPATHS IN A LAYERED MEDIUM

According to ray theory the elastic wavefield in a 3D inhomogeneous medium can be decomposed into an infinite number of rays. Each ray extends from the source,

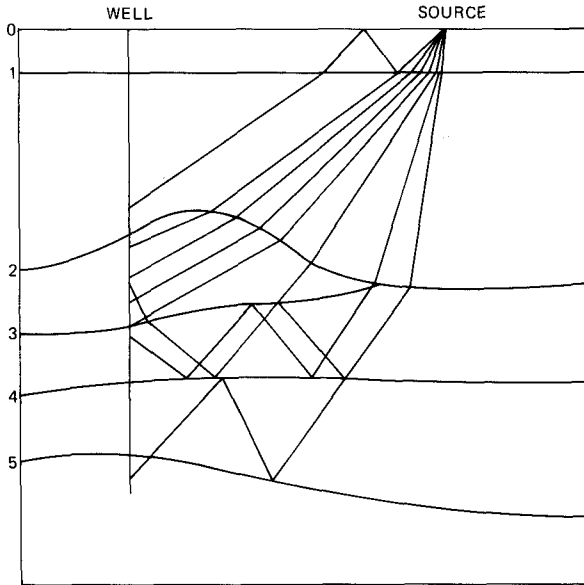


FIG. 1. Raypaths in a layered medium.

located at an arbitrary position within the medium, to the receiver at another position.

The raypaths may in general be quite complex, and the number of rays large. To construct synthetic seismograms resembling real data, the rays contributing most significantly to the wave-field must be selected and included in the calculations. Figure 1 shows a limited number of possible raypaths in a simple model and even here the difficulty of specifying and summing the most significant rays is evident. It is clear that performing the ray selection manually by intuition or trial-and-error is likely to give incomplete synthetic seismograms. A systematic description of the raypaths enabling selection of the most significant rays is needed. The description must also be suitable for handling by a computer, so that the various rays can be generated and enumerated in a systematic way.

To describe the raypaths of seismic waves, a model of the geometry of the subsurface is needed. It is also necessary to know the wave velocities and densities for every point within the model. A general model may consist of layers separated by interfaces. The wave velocities and densities are discontinuous across the interfaces, but change smoothly within the layers. The interfaces extend laterally through the whole model. It is assumed that each point on the interface described by one  $x$ - and one  $y$ -coordinate corresponds to a single  $z$ -value and that no interfaces cross. A 2D cross-section of such a model is shown in Fig. 2. The interfaces are numbered sequentially from 0 to  $L$ , with layer  $k$  above interface  $k$ .

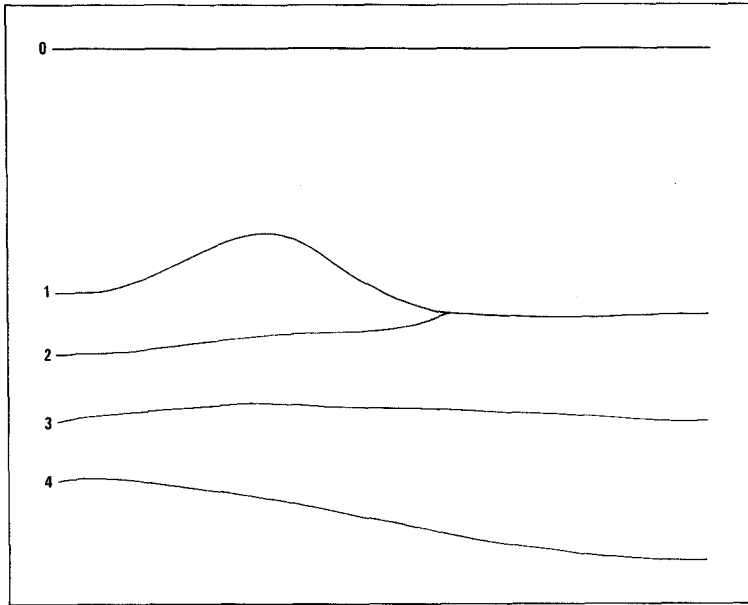


FIG. 2. Layered medium.

The response at the receiver is the sum of terms corresponding to individual rays. The displacement velocity  $v(x, y, z, \omega)$  can be written as

$$v(x, y, z, \omega) = \sum_k V_k, \tag{1a}$$

where the amplitude  $V_k$  of ray number  $k$  is given by

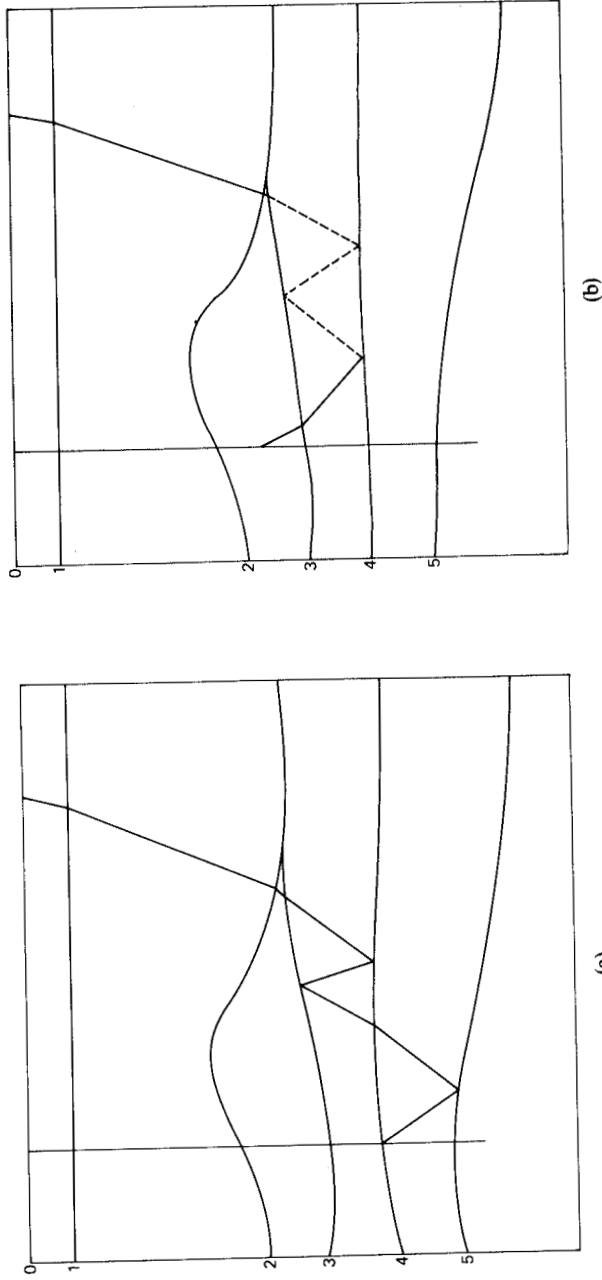
$$V_k = G_k(x, y, z)R_k \exp(i\omega\tau(x, y, z))V_0(\omega). \tag{1b}$$

$\omega$  is the frequency and  $G_k$  is an amplitude function taking into account the geometrical spreading of the ray.  $R_k$  is a product of reflection and transmission coefficients accounting for reflection and refraction of the ray at individual interfaces.  $V_0(\omega)$  is the source function and  $\tau_k(x, y, z)$  is the arrival time of the ray at position  $(x, y, z)$ .

The raypath shown in Fig. 3a can be described by recording the interface numbers where the ray has been reflected. The label on the amplitude is replaced by a sequence of upper and lower indices as follows:

$$V_{3^4 \ 5; \ 0, 4} \tag{2}$$

The upper sequence denotes reflections with the incident ray above the reflecting interface; the lower sequence denotes reflections with the incident ray below the reflecting interface. The sequences will occasionally be referred to as reflection sequences. The last integer in the upper sequence (preceded by a semicolon) denotes the source interface. Similarly the receiver interface is denoted by the last integer in the lower sequence. The source interface is the interface closest to the source in the



(a)  
 (b)  
 FIG. 3. Example of rays in a layered medium. (—) are P-ray segments; (---) are S-ray segments.

direction of the propagating ray, while the receiver interface is the interface closest to the receiver in the direction of the propagation of the ray. To distinguish source and receiver interfaces the former is always followed by a dot (.).

This notation is also used to give the initial direction of the ray at the source and the direction of the ray at the receiver. If source or receiver interface is given in the upper sequence it is interpreted as a downgoing ray at the source or receiver, while the appearance of source or receiver interface in the lower sequence indicates an upgoing ray at the source or receiver.

To include mode conversions at the interfaces interacting with the ray, an additional sequence of numbers is introduced. Figure 3b shows an example of a ray with conversions from P- to S-waves and vice versa. The raypath can be described by

$$V \begin{matrix} \bar{4} & 4; & \bar{3}; & 0. \\ \bar{3} & ; & ; & 2 \end{matrix} \tag{3}$$

A bar indicates an S-wave is emerging from the interface; no bar is used for an emerging P-wave. The additional upper sequence enumerates the interfaces where conversion occurs at transmission. In general there is also an additional lower sequence. The upper sequence denotes conversion where the initial ray is above the interface, while the lower sequence denotes conversion when the ray is below the interface. An initial downgoing ray arriving at the receiver in an upwards direction is in general described by

$$V \begin{matrix} \bar{i}1 & i2 & \dots & i_g; & \bar{p}1 & \dots & \bar{p}h; & i_s. \\ j1 & j2 & \dots & j_r; & i1 & \dots & i_n; & j_r \end{matrix} \tag{4}$$

$\bar{i}k$  denotes a reflecting interface with the incident ray above the interface. The tilde ( $\sim$ ) is used to denote that the emerging ray is either a P-ray or an S-ray.  $\tilde{j}k$  denotes a reflecting interface with the incident ray below the interface.  $\bar{p}k$  and  $\tilde{i}k$  are the interfaces where conversion occurs with the incident ray above and below the interface, respectively.

To generate rays in a computer systematically, it is only necessary to generate numerically the reflection sequences and compute the traveltime and amplitude.

To make numerical calculations more practical a sub-group of all the possible rays described by the reflection sequences in (4) can be chosen. Each raypath is then divided into segments, where each segment connects points on interfaces where the ray has been reflected or refracted. Only rays with segments connecting different interfaces are considered. This requirement will limit considerably the number of rays entering the calculations. Expressions for generating rays in this general case are given in Appendix C, (C1)–(C7).

In the rest of this work only a horizontally stratified medium will be considered. The medium consists of  $L$  horizontal layers numbered from 0 to  $L + 1$ , where layer number 0 and  $L + 1$  are half-spaces at the top and bottom of the medium. The density and wave velocity is constant within each layer. In such a medium, ray generation and the computation of synthetic seismograms can be made much more efficient than in a more general medium. Rays with different raypaths, but equal

amplitude and traveltimes, can be collected into groups called dynamic analogue groups (Hron 1971). This reduces the number of amplitude traveltimes computations which have to be performed, and thereby reduces the CPU-time needed for evaluating synthetic seismograms.

### NON-CONVERTED RAYS

Equations (A1)–(A6) in Appendix A express the displacement velocity  $V$  as follows for P- or S-rays without mode-conversion:

$$V = G(\mu_I, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J)A(\mu_I, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J) \times \exp(i\omega\tau(\mu_I, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J)). \tag{5}$$

Indices  $I$  and  $J$ , respectively, denote the uppermost and deepest interfaces where reflection occurs. Here  $\mu_k$  is the number of reflections at interface  $k$  with the incident ray in layer  $k + 1$ , and  $m_k$  is the number of reflections at interface  $k$  with the incident ray in layer  $k$ . The function  $A$  contains products of complex reflection and transmission coefficients and is a function of the integers  $\mu_k$  and  $m_k$ ,  $k = I, \dots, J$ . The geometrical spreading  $G$  and the phase functions  $\tau$  are functions of the same set of integers as  $A$ .

The relations between the reflection sequences, denoted by  $j\alpha$  and  $i\alpha$ , and  $\mu_k$  and  $m_k$  are

$$\mu_k = \sum_{\alpha=1}^{J-1} \delta_{j\alpha, k}, \tag{6a}$$

and

$$m_k = \sum_{\alpha=I+1}^J \delta_{i\alpha, k}. \tag{6b}$$

$\delta_{i, j}$  is the Kronecker delta.

Taking (6a) and (6b) into account, reflection sequences differing only by permutation will give rise to the same set of integers  $(m_k, \mu_k)$ ,  $k = I, \dots, J$ . Since the amplitude and phase function can be expressed only by these two integers, it means that several reflection sequences describe rays with equal phase function and amplitude. The rays form a dynamic analogue group (Hron 1971). The symbol

$$(s, r; \mu_I, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J) \tag{7}$$

will uniquely describe a dynamic analogue group.  $s$  is the source interface;  $r$  is the receiver interface. An example of a dynamic analogue group is shown in Fig. 4. The raypaths in Figs 4a–d are described by

$$V_{0 \ 1;}^2 \ 4; \ 0.2 \quad V_{0 \ 1;}^4 \ 2; \ 0.2 \quad V_{1 \ 0;}^2 \ 4; \ 0.2 \quad V_{1 \ 0;}^4 \ 2; \ 0.2 \tag{8}$$

This dynamic analogue group containing four rays is described by the symbol given in (7) as

$$(0, 2; 1, 1, 0, 0; 0, 1, 0, 1). \tag{9}$$

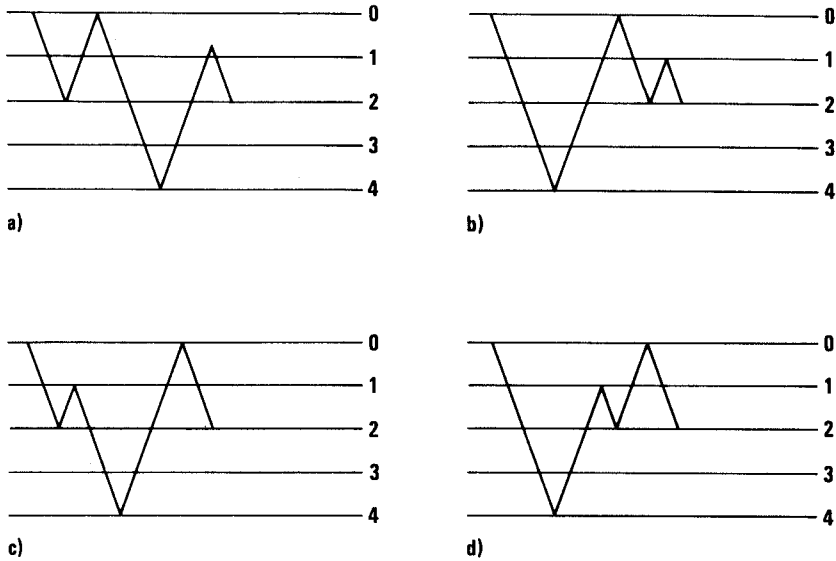


FIG. 4. Dynamic analogue group without conversion.

Another way of describing the group is by selecting one of the rays given in (8). This is done by selecting the ray with non-decreasing reflection sequences (except from the receiver and source indices),

$$V_{0 \ 1; \ 4}^2 \ 0.2 \tag{10}$$

Vetter (1981) calls this ray a “marker ray”. In general a marker ray is selected by using the expressions given in Appendix C.

The advantage of classifying the rays into groups of dynamic analogues is that if the amplitude of one member is known, then the contribution of the whole group can be found by multiplication of the number of members of the group.

By applying the results of Appendix B, (B1)–(B2), the number of rays within a group of dynamic analogues may be calculated for a typical VSP-geometry with the source at the surface and a receiver at arbitrary depth.

In the following formulae half the number of ray segments,  $n_k$ , is defined in terms of the number of ray segments  $h_k$  as

$$n_k = h_k/2, \text{ for } h_k = 2l, \ l = 1, 2, \dots, \tag{11}$$

and

$$n_k = (h_k - 1)/2, \text{ for } h_k = 2l + 1, \ l = 1, 2, \dots \tag{12}$$



$n_k$  is related to the integers  $\mu_k$  and  $m_k$  through (A4c) and (A6c) in Appendix A. For a source at the surface the equations reduce to

$$n_k = \sum_{l=0}^{k-1} (\mu_l - m_l + \delta_{l,r} - \delta_{l,0}) \quad k = I + 1, \dots, J. \tag{13}$$

The function  $N(s, r; \mu_I, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J)$  denotes the ray count for the dynamic analogue group described by the arguments of  $N$  and is given by:

$$N(0, r; \mu_I, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J) = \prod_{i=I+1}^{r-1} D(n_i, n_{i+1}, m_i) \prod_{i=r+1}^{J-1} D(n_i - 1, n_{i+1} - 1, m_i) M, \tag{14a}$$

where

$$D(n_i, n_{i+1}, m_i) = C_{m_i}^{n_i} C_{n_i - m_i}^{n_{i+1}}, \tag{14b}$$

and

$$C_m^n = \frac{n!}{m!(n-m)!}. \tag{14c}$$

The first factor in (14a) accounts for the raypath above the receiver; the second factor for the raypath below the receiver. The function  $M$  takes into account special cases associated with the layer above the receiver layer, and is defined as

$$M = D(n_r + E, n_{r+1} + F, m_r), \tag{14d}$$

where

$$F = \begin{cases} -1 & \text{for } \sigma_r = +1 \text{ and } r \neq 0, \\ 0 & \text{for } \sigma_r = -1 \text{ and } r \neq 0, \\ -1 & \text{for } \sigma_r = +1 \text{ and } r = 0, \\ 0 & \text{for } \sigma_r = -1 \text{ and } r = 0, \end{cases} \tag{14e}$$

and

$$E = \begin{cases} 0 & \text{for } r \neq 0, \\ -1 & \text{for } r = 0. \end{cases} \tag{14f}$$

$\sigma_r = +1$  for a downgoing ray at the receiver;  $\sigma_r = -1$  for an upgoing ray at the receiver.

An equation equivalent to (14) was derived by Vered and Ben-Menahem (1974). For the receiver and source both at arbitrary depths, (14) must be replaced by the more general expression found in Appendix A. By using (14) the number of rays described by (7), or by its equivalent (4), can easily be calculated. For the numerical calculation of synthetic seismograms this is important since the evaluation of the right-hand side of (14a) will often be substantially faster than the calculation of a ray

amplitude. To illustrate the application of the formula, consider the dynamic analogue group defined in (9). The number of rays is equal to

$$N(0, 2; 1, 1, 0, 0; 0, 1, 0, 1) = C_0^1 C_1^2 C_0^0 C_0^0 C_1^2 C_0^0 = 4. \tag{15}$$

With the source at the surface the complete field of non-converted rays for either only upgoing or only downgoing rays at the receiver can be written in a compact way (omitting source and receiver indices):

$$v = \sum_{u=0}^{\infty} V_u, \tag{16}$$

$$V_u = \sum_{g, \gamma, i_1, \dots, i_g} \dots \sum_{j_1, \dots, j_\gamma} N(0, r; \mu_1, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J) V_{j_1}^{i_1} \dots V_{j_\gamma}^{i_g}.$$

Here  $u$  is the number of reflections.  $g$  and  $\gamma$  denote the number of reflections with the incident ray above and below the reflecting interface, respectively. The summation over  $g$  and  $\gamma$  must be carried out under the restriction that

$$g + \gamma = u. \tag{17}$$

Appendix C contains detailed expressions for the lower and upper limits of the sums.

### CONVERTED RAYS

The expressions given in the previous section could, in principle, be extended to waves with an arbitrary number of conversions from P- to S-rays or from S- to P-rays. The classification of rays into groups of dynamic analogues is, however, not efficient for rays with many mode conversions, as the necessary formulae are complicated. This saves little in terms of CPU-time. The number of rays also grows rapidly with the number of conversions, making numerical calculations impractical. Therefore only rays with one mode conversion will be considered. In the following a bar ( $\bar{\phantom{x}}$ ) indicates S-ray quantities, while a double bar ( $\bar{\bar{\phantom{x}}}$ ) indicates quantities connected with the point of conversion. The amplitude of such a ray may be written as

$$V = A(\mu_1, \dots, \mu_{J-1}; \bar{\mu}_1, \dots, \bar{\mu}_{J-1}; m_{I+1}, \dots, m_J; \bar{m}_{I+1}, \dots, \bar{m}_J; \bar{\bar{l}}_c, \bar{\bar{l}}_c, \bar{\bar{m}}_c, \bar{\bar{\mu}}_c) \times G(\mu_1, \dots, \mu_{J-1}, \bar{\mu}_1, \dots, \bar{\mu}_{J-1}; m_{I+1}, \dots, m_J; \bar{m}_{I+1}, \dots, \bar{m}_J) \times \exp(i\omega t(\mu_1, \dots, \mu_{J-1}; \bar{\mu}_1, \dots, \bar{\mu}_{J-1}; m_{I+1}, \dots, m_J; \bar{m}_{I+1}, \dots, \bar{m}_J)). \tag{18}$$

$c$  is the interface where the mode conversion takes place.

The integers  $\bar{m}_k$  and  $\bar{\bar{\mu}}_k$  are straightforward generalizations of the definitions for unconverted rays.  $\bar{l}_c$ ,  $\bar{\bar{l}}_c$ ,  $\bar{\bar{m}}_c$  and  $\bar{\bar{\mu}}_c$  are either zero or one and indicate the transmission/reflection coefficient to be used at the point where mode conversion occurs.

- $\bar{l}_c = 1$  for transmission through interface  $c$  with the incident ray in layer  $c$ ,
- $\bar{\bar{l}}_c = 1$  for transmission through interface  $c$  with the incident ray in layer  $c + 1$ ,
- $\bar{m}_c = 1$  for reflection at interface  $c$  with the incident ray in layer  $c$ , and
- $\bar{\bar{\mu}}_c = 1$  for reflection at interface  $c$  with the incident ray in layer  $c + 1$ .

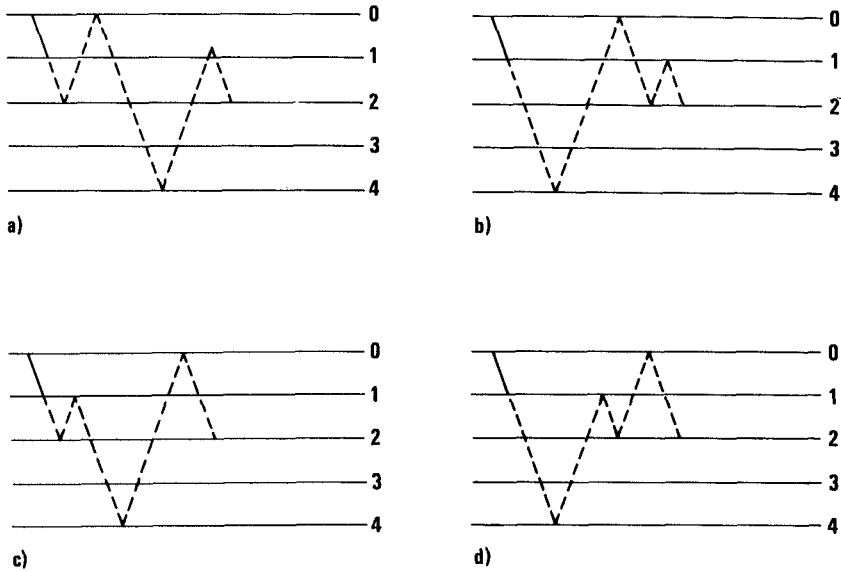


FIG. 5. Dynamic analogue group with conversion. (—) are P-ray segments; (---) are S-ray segments.

The results of Appendix A relate  $\mu_k, \bar{\mu}_k, m_k$  and  $\bar{m}_k$  to the reflection sequences, so that a dynamic analogue group may be specified in much the same way as unconverted rays. Symbol (7) may be directly generalized to include rays with a single conversion.

Figure 5 shows an example of a dynamic analogue group with wave conversion, which takes place at interface number 1. The rays are given by

$$V_0^{\frac{1}{2}} \frac{\frac{1}{2}}{1}; 1; 0.2 \quad V_0^{\frac{1}{2}} \frac{\frac{2}{2}}{1}; 1; 0.2 \quad V_1^{\frac{1}{2}} \frac{\frac{1}{2}}{0}; 1; 0.2 \quad V_1^{\frac{1}{2}} \frac{\frac{2}{2}}{0}; 1; 0.2 \quad (19)$$

The group can be described by a single ray, as for unconverted waves, by selecting the ray with non-decreasing reflection sequences, except for source and receiver indices.

The general condition for selecting a marker ray for a specific dynamic analogue group is given in Appendix C. The ray count for a dynamic analogue group is given by

$$N(0, r; \mu_1, \dots, \mu_{J-1}; \bar{\mu}_1, \dots, \bar{\mu}_{J-1}; m_{I+1}, \dots, m_J; \bar{m}_{I+1}, \dots, \bar{m}_J) \\ = N(0, c; \mu_1, \dots, \mu_{J-1}; m_{I+1}, \dots, m_J) N(c, r, \bar{\mu}_1, \dots, \bar{\mu}_{J-1}; \bar{m}_{I+1}, \dots, \bar{m}_J). \quad (20)$$

Equation (20) is valid with the source at the surface and the receiver at arbitrary depth. The function  $N$  is defined in Appendix B ((B1) and (B2)), and is similar to the function given in (14). The first factor on the right-hand side of (20) accounts for the raypath between the surface and the interface where the conversion takes place. The

second factor accounts for the raypath between the interface where the conversion takes place and the receiver.

Generally, with source and receiver at arbitrary depths the results of Appendix B must be used.

By using the above formulae the number of rays belonging to the dynamic analogue group displayed in Fig. 5 can be calculated as

$$N(0, 2; 0, 0, 0, 0; \bar{1}, \bar{1}, 0, 0; 0, 0, 0, 0; 0, \bar{1}, 0, \bar{1}) = C_0^0 C_0^0 C_0^0 C_0^0 C_0^0 C_0^0 C_0^1 C_1^2 C_1^2 C_1^0 C_0^0 C_0^0 = 4.$$

With the source at the surface, the complete field of converted rays with one conversion for either only upgoing or only downgoing rays at the receiver can be written compactly (omitting source and receiver indices):

$$v = \sum_{u=0}^{\infty} V_u, \tag{21}$$

$$V_u = \sum_c \sum_{\bar{l}, \bar{\lambda}} \sum_{\substack{g, \gamma \\ \bar{g}, \bar{\gamma}}} \sum_{\substack{i_1, \dots, i_g \\ j_1, \dots, j_h}} \dots \sum_{\substack{i_1, \dots, i_g \\ j_1, \dots, j_h}} \dots \sum NV_{j_1^{i_1} \dots j_g^{i_g} j_1^{i_1} \dots j_g^{i_g}; \bar{j}_1^{\bar{i}_1}} \tag{21}$$

$u$  is the total number of reflections and  $g$  and  $\bar{g}$  are the number of P- and S-ray reflections with the incident ray above the reflecting interface.  $\gamma$  and  $\bar{\gamma}$  are the number of P- and S-ray reflections with the incident ray below the reflecting interface. The summation over  $c$  allows the conversion to take place at any interface, while summations over  $\bar{l}$  and  $\bar{\lambda}$  take into account the direction of the rays at the point of conversion. The summation over  $g, \bar{g}, \gamma$  and  $\bar{\gamma}$  must be performed under the restriction:

$$g + \bar{g} + \gamma + \bar{\gamma} = u. \tag{22}$$

Detailed expressions for the limits of the sums are given in Appendix C.

NUMERICAL RAY GENERATION

The actual computation of a synthetic seismogram using ray theory requires, in principle, evaluation of an infinite number of rays. Since this is not feasible, a method of selecting significant rays is needed. Waves reflected many times will often make a smaller contribution to the total wavefield than waves reflected only a few times. Consequently, a possible selection would be to classify the rays according to the number of reflections and then pick all the rays with a reflection number less than a certain threshold. In the computer algorithm for numerical ray generation the sums given in (21) or (16) are evaluated for a given maximum number of reflections. The number of unconverted rays grows rapidly with the number of reflections and can be given by a polynomial in the number of layers,  $L$ , of order  $u$  or  $u - 1$ , where  $u$  is the number of reflections. The number of converted rays grows even faster. An additional selection rule, picking only the rays with mode conversion at a

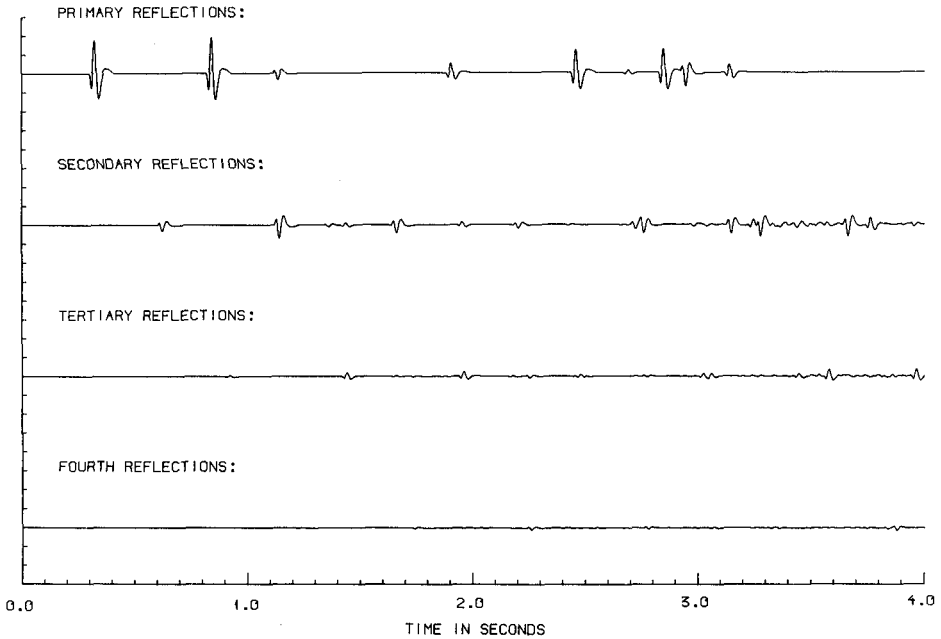


FIG. 6. Synthetic zero-offset surface seismogram.

reflection point, will greatly reduce the number of converted rays. This is a reasonable selection rule, since transmission coefficients for conversions tend to be quite small.

Synthetic zero-offset seismograms computed for simple models will be shown. The classification of rays by the number of reflections has been used. For plotting

TABLE 1. Model used for computation of synthetic seismograms.

Layer number	Velocity (m/s)	Dimensions (m)	Density (g/cc)
0	0.0	—	0.0
1	1500.0	225.0	1.09
2	1615.0	419.0	1.46
3	2050.0	300.0	1.86
4	1950.0	750.0	1.77
5	2160.0	600.0	1.90
6	3050.0	350.0	2.20
7	3165.0	250.0	2.25
8	5350.0	260.0	2.57
9	3600.0	350.0	2.35
10	4770.0	—	2.47

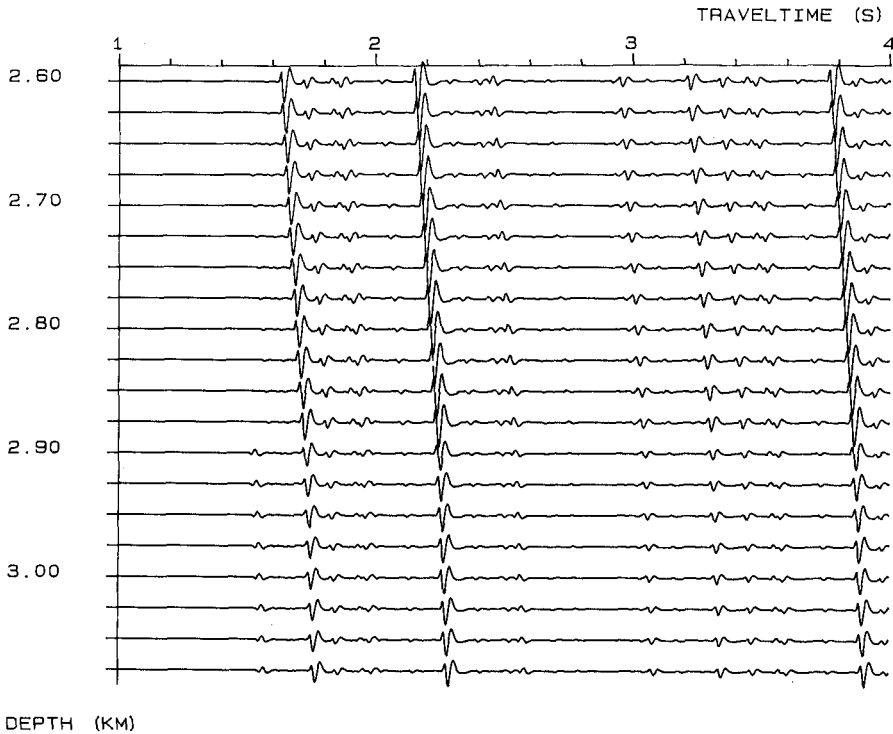


FIG. 7. Synthetic zero-offset seismogram displaying downgoing waves with two reflections.

purposes the seismograms have been scaled by a gain function proportional with time. Figure 6 shows seismograms computed using an airgun source and recorded with a hydrophone, both located just below the surface. The nine-layer model used is given in Table 1. At the top of Fig. 6 the response due to primary reflections is plotted, while the next response is the contribution from all rays reflected three times. As expected the contribution to the total wavefield decreases with increasing number of reflections. This is even more evident from the two bottom responses of Fig. 6 containing all rays reflected five and seven times.

Synthetic zero-offset VSPs have also been calculated for the model in Table 1, and the result is shown in Figs 7–10. The source was located just below the surface. In Figs 7 and 8 the downgoing waves reflected two and four times are displayed. Figures 9 and 10 show the upgoing waves reflected one and three times. Further examples of synthetic VSPs are given by Ursin and Arntsen (1985).

The classification of rays in dynamic analogue groups is only meaningful if the ratio of the actual number of rays to the number of dynamic analogue groups is greater than one. An upper limit for this ratio, for rays with  $u$  reflections is

$$e \leq \left( \sum_k m_k \right)! \left( \sum_k \mu_k \right)!, \tag{23}$$

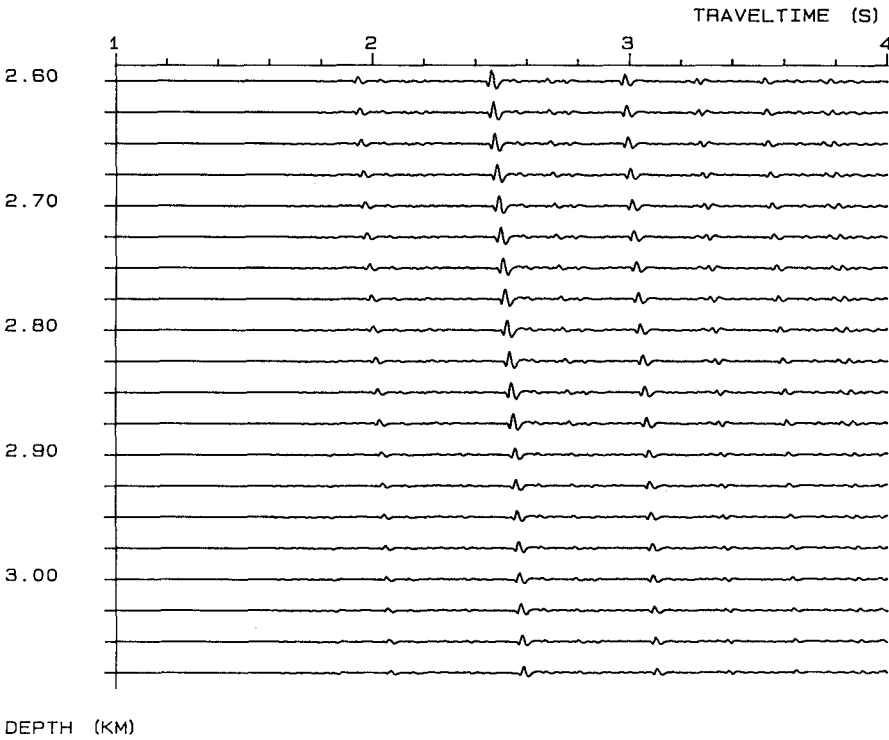


FIG. 8. Synthetic zero-offset VSP displaying downgoing waves with four reflections.

where

$$\sum_k m_k + \sum_k \mu_k = u.$$

The efficiency of grouping the rays into dynamic analogues can be seen from Tables 2 and 3, which shows the actual number of rays, the number of dynamic analogue groups and the ratio between them. The gain in CPU-time needed to evaluate these seismograms is directly proportional to this ratio, as the time needed

TABLE 2. The number of rays and dynamic analogues in the calculation of Fig. 6.

Number of reflections	Cumulative number of rays	Dynamic analogues	Number of rays/ number of dynamic analogues
1	9	9	1.00
3	256	147	1.74
5	4239	1065	3.98
7	38880	4418	8.80

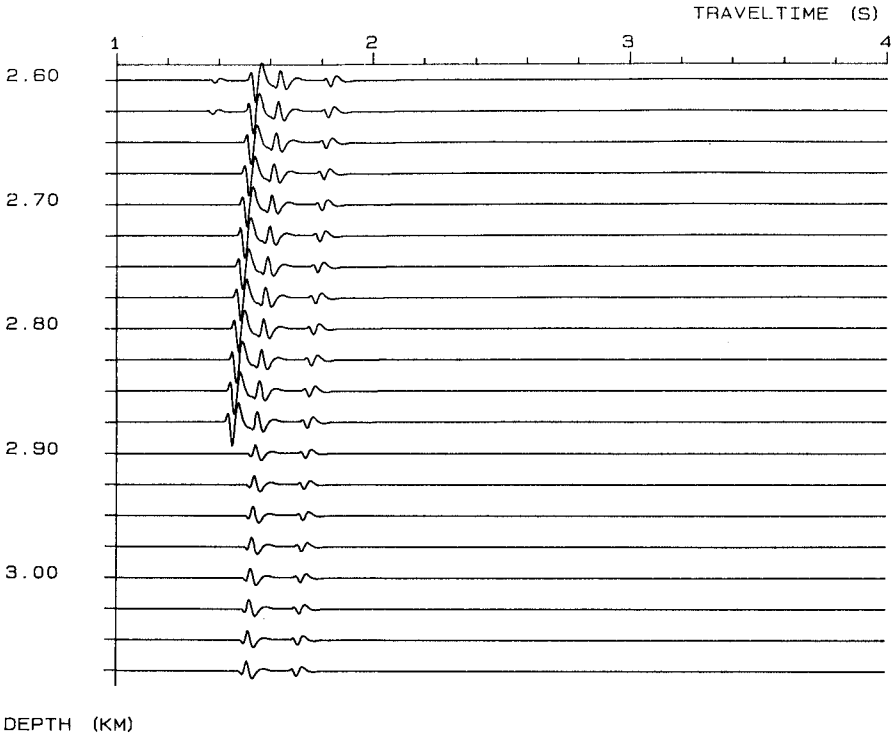


FIG. 9. Synthetic zero-offset VSP displaying upgoing waves with one reflection.

for the computation of the number of rays within a dynamic analogue group is small compared to the amplitude- and travelttime-computations.

Table 3 shows that it is much less efficient to group the rays into dynamic analogue groups for geometries with the receiver at depth than for geometries where the receiver and the source are at the surface.

TABLE 3. The number of rays and dynamic analogues in the calculations of Figs 7–10.

Number of reflections	Cumulative number of rays	Dynamic analogues	Number of rays/ number of dynamic analogues
1	54	54	1.00
2	795	795	1.00
3	2727	2459	1.11
4	13326	9561	1.39
5	41496	24047	1.73



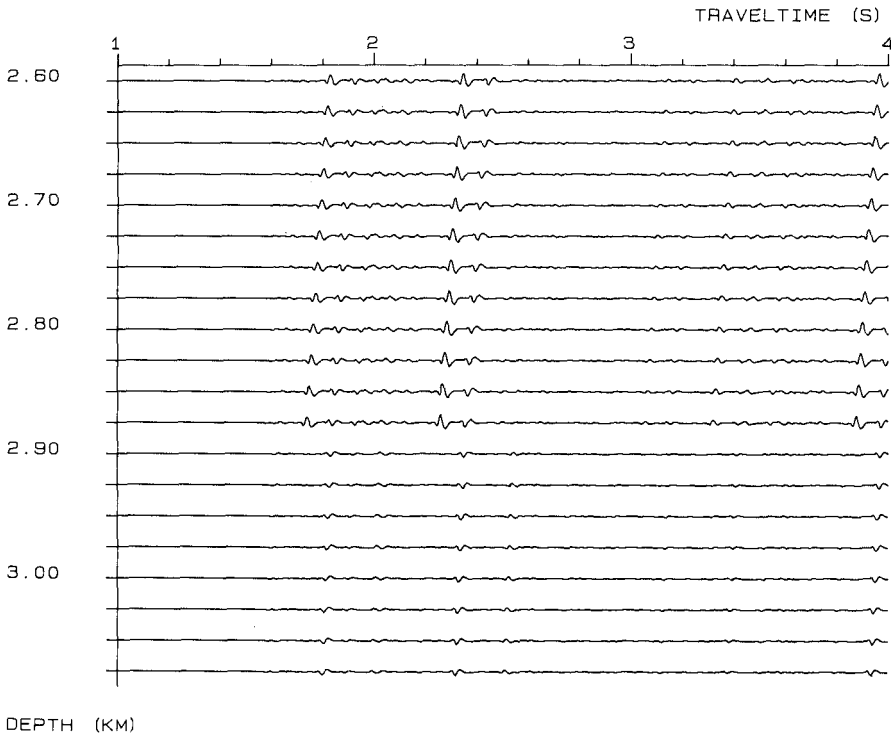


FIG. 10. Synthetic zero-offset VSP displaying upgoing waves with three reflections.

The number of actual rays to the number of dynamic analogue groups for all rays with  $u$  reflections and a single conversion is limited by

$$e \leq \left( \sum_k m_k \right)! \left( \sum_k \mu_k \right)! \left( \sum_k \bar{m}_k \right)! \left( \sum_k \bar{\mu}_k \right)! \quad (24)$$

where

$$\sum_k (m_k + \mu_k + \bar{m}_k + \bar{\mu}_k) = u.$$

It is seen that the efficiency of grouping the rays into dynamic analogue groups is less for converted rays than for unconverted rays.

## CONCLUSIONS

An automatic ray generation scheme selecting the most significant rays is important if ray theory methods are used in the computation of synthetic seismograms resembling real data. An algorithm has been given for generating rays classified systematically according to the number of times they have been reflected. The algorithm is

sufficiently general to allow for the receiver and source to be placed arbitrarily within the medium. Compact expressions for the summation of all unconverted rays, and all rays with a single conversion, have also been given.

The numerical examples show that, for the particular model studied, high-order multiples do not contribute significantly to the total wavefield.

ACKNOWLEDGEMENT

I wish to thank the referees for many helpful suggestions.

APPENDIX A  
COMPUTATION OF DISPLACEMENT VELOCITY IN A  
HORIZONTALLY LAYERED MEDIUM

Consider a horizontally layered medium with constant density and wave velocity in each layer. In the following only P- or S-rays without mode-conversion, or rays with a single mode conversion, will be considered. All S-ray quantities will be indicated by a ( $\bar{\quad}$ ), while a double bar ( $\bar{\bar{\quad}}$ ) is used to label quantities dealing with ray conversion. A tilde ( $\tilde{\quad}$ ) will be used to indicate either P- or S-ray quantities.

According to ray theory the displacement velocity at the receiver can be expressed as a sum of terms of the form

$$V = GA(p)\bar{A}(p)\bar{\bar{A}}(p)V_0(\omega) \exp(i\omega\tau), \tag{A1}$$

where  $A$  and  $\bar{A}$  are P- and S-wave amplitude factors.  $\bar{\bar{A}}$  is an amplitude factor for conversion between P- and S-waves or between S- and P-waves.

$\bar{A}$  is a function of the ray parameter  $p$  and is given by

$$\bar{A} = \prod_{i=1}^J (\bar{R}_i)^{\bar{m}_i(p)} (\bar{S}_i)^{\bar{\mu}_i(p)} (\bar{T}_i)^{\bar{l}_i(p)} (\bar{U}_i)^{\bar{\lambda}_i(p)}. \tag{A2}$$

$I$  and  $J$  label the uppermost and deepest interfaces where reflection occurs.

$V_0(\omega)$  is the source function.  $\bar{R}_i(p)$ ,  $\bar{S}_i(p)$ ,  $\bar{T}_i(p)$  and  $\bar{U}_i(p)$  are reflection and transmission factors associated with interface  $i$ .  $\bar{m}_i$  is the number of reflections at the  $i$ th interface with the incident ray in the  $i$ th layer.  $\bar{l}_i$  is the number of transmissions through interface  $i$  with the incident ray in the  $i$ th layer. Likewise  $\bar{\mu}_i$  is the number of reflections at interface  $i$  with the incident ray in the  $i + 1$  layer.  $\bar{\lambda}_i$  is the number of transmissions through interface  $i$  with the incident ray in the  $i + 1$  layer.

Only rays which have been converted once will be included in the calculations, so that  $\bar{m}_i$ ,  $\bar{l}_i$ ,  $\bar{\mu}_i$  and  $\bar{\lambda}_i$  will be equal to zero or one.  $G$  accounts for geometrical spreading and  $\tau$  is the traveltime. Both are a function of the number of ray segments  $\bar{h}_{I+1}, \dots, \bar{h}_J$ .

Detailed expressions of the factors in (A1) can be found in Hron and Kanasevich (1971) for a medium with constant parameters in each layer, and in Červený (1985) for the general case.

For convenience, the number of pairs of segments in each layer  $\tilde{n}_i$  are introduced:

$$\tilde{n}_i = \tilde{h}_i/2 \quad \tilde{h}_i = 2k, \quad k = 1, 2, \dots, \tag{A3a}$$

$$\tilde{n}_i = (h_i - 1)/2 \quad \tilde{h}_i = 2k + 1, \quad k = 1, 2, \dots \tag{A3b}$$

to simplify numerical evaluation of the displacement velocity given in (A1)  $\tilde{n}_i$ , (and thereby  $\tilde{h}_i$ ,  $\tilde{l}_i$  and  $\tilde{\lambda}_i$  can be expressed by  $\tilde{m}_i$  and  $\tilde{\mu}_i$ ).

This is most easily done by splitting the raypath into two parts, one part containing only P-ray segments, and the other part containing only S-ray segments. It is assumed that the source emits P-waves, so that each ray is converted into an S-ray at interface  $c$ . To treat the opposite case, with the source emitting S-waves which is converted into P-waves, requires only interchange of P- and S-ray quantities in the formulae given below.

$$l_k = \begin{cases} n_k - m_k & \text{for } s > k > I \text{ or } J > k > c \\ n_k - m_k + 1 & \text{for } c > k > s \\ n_c - m_c + H(\sigma_c) + \delta_{c,s}(-H(\sigma_c) - H(-\sigma_c)) & \text{for } c = r, \end{cases} \tag{A4a}$$

$$\lambda_k = \begin{cases} n_k - m_k \\ n_s - m_s - H(-\sigma_s) \end{cases} \quad \text{for } s > k > I \text{ or } J > k > c, \tag{A4b}$$

$$n_k = \begin{cases} \sum_{i=I}^{k-1} (\mu_i - m_i + \delta_{i,c} - H(-\sigma_s)\delta_{i,s}) & s \neq c \\ \sum_{i=I}^{k-1} (\mu_i - m_i + \delta_{i,s}\delta_{\sigma_c,1}\delta_{\sigma_s,-1}) & \text{for } s = c \text{ and } I > 0 \\ \sum_{i=0}^{k-1} (\mu_i - m_i + \delta_{i,0}) & \text{for } s = c \text{ and } I = 0. \end{cases} \tag{A4c}$$

The index  $I$  is the interface where the uppermost (shallowest) reflection takes place, while  $J$  is the interface where the deepest reflection occurs.  $\sigma_s$ ,  $\sigma_r$  and  $\sigma_c$  give the direction of a ray leaving the source, a ray arriving at the receiver and a ray arriving at the point of conversion between P- and S-wave mode.  $\sigma = +1$  for a downgoing ray, and  $\sigma = -1$  for an upgoing ray.

$H$  and  $\delta_{i,j}$  are the step-function and Kronecker delta defined by

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0, \end{cases} \tag{A5a}$$

$$\delta_{i,j} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{A5b}$$

The formulae for the S-ray quantities are

$$\bar{l}_k = \begin{cases} \bar{n}_k - \bar{m}_k & \text{for } c > k > I \text{ or } J > k > r \\ \bar{n}_k - \bar{m}_k + 1 & \text{for } r > k > c \\ \bar{n}_s - \bar{m}_s + H(\sigma_s) + \delta_{r,c}(-H(\sigma_r) - H(-\sigma_r)) & \text{for } k = c, \end{cases} \quad (\text{A6a})$$

$$\bar{\lambda}_k = \begin{cases} \bar{n}_k - \bar{m}_k \\ \bar{n}_c - \bar{m}_c - H(-\sigma_c) & \text{for } c > k > I \text{ or } J > k > r, \end{cases} \quad (\text{A6b})$$

$$\bar{n}_k = \begin{cases} \sum_{l=I}^{k-1} (\bar{\mu}_l - \bar{m}_l + \delta_{l,r} - H(-\sigma_c)\delta_{l,c}) & \text{for } c \neq r \\ \sum_{l=I}^{k-1} (\bar{\mu}_l - \bar{m}_l + \delta_{l,c} \delta_{\sigma_r,1} \delta_{\sigma_c,-1}) & \text{for } c = r \text{ and } I > 0 \\ \sum_{l=0}^{k-1} (\bar{\mu}_l - \bar{m}_l + \delta_{l,0}) & \text{for } c = r \text{ and } I = 0. \end{cases} \quad (\text{A6c})$$

These formulae apply for P-waves without mode-conversion by using only (A4) and replacing  $c$  with  $r$ . Equation (A6) is valid for S-waves without mode-conversion by replacing  $c$  with  $s$ .

### APPENDIX B THE NUMBER OF ELEMENTS IN A GROUP OF DYNAMIC ANALOGUES

The formulae for the number of rays within a dynamic analogue group can be derived for the receiver and source placed arbitrarily within a horizontally layered medium, in basically the same way as for the source and receiver located at the surface (Hron 1972).

Without loss of generality it is assumed that the receiver is positioned below the source, and also that the source emits P-waves. When the source emits S-waves the P- and S-ray quantities should be interchanged in the formulae below.

The number of rays in a dynamic analogue group with one mode conversion is

$$N(s, r; n_{I+1}, \dots, n_J; \bar{n}_{I+1}, \dots, \bar{n}_J; m_{I+1}, \dots, m_{J-1}; \bar{m}_{I+1}, \dots, \bar{m}_{J-1}) \\ = N(s, c; n_{I+1}, \dots, n_J; m_{I+1}, \dots, m_{J-1})N(c, r; \bar{n}_{I+1}, \dots, \bar{n}_J; \bar{m}_{I+1}, \dots, \bar{m}_{J-1}). \quad (\text{B1})$$

$s$  and  $r$  are the source and receiver interfaces, while  $c$  is the interface where the conversion takes place. The quantities  $n_k$ ,  $m_k$ ,  $\bar{n}_k$  and  $\bar{m}_k$  are defined in Appendix A. The symbols on the right-hand side of (B1) are given as

$$N(k, l; \bar{n}_{I+1}, \dots, \bar{n}_J; \bar{m}_{I+1}, \dots, \bar{m}_{J-1}) \\ = \prod_{i=I+1}^k D(\bar{n}_i - 1, \bar{n}_{I+1}, \bar{m}_i) \prod_{i=k+1}^{l-1} D(\bar{n}_i, \bar{n}_{i+1}, \bar{m}_i) \prod_{i=I+1}^{J-1} D(\bar{n}_i - 1, \bar{n}_{i+1} - 1, \bar{m}_i)M \quad (\text{B2})$$

where

$$D(\bar{n}_i, \bar{n}_{i+1}, \bar{m}_i) = C_{\bar{m}_i}^{\bar{n}_i} C_{\bar{n}_i - \bar{m}_i}^{\bar{n}_{i+1}}.$$

The function  $M$  is defined as

$$M = D(n_r + E, n_{r+1} + F, m_r)$$

where

$$F = \begin{cases} -1 & \text{for } \sigma_r = +1 \text{ and } k \neq 1 \\ 0 & \text{for } \sigma_r = -1 \text{ and } k \neq 1 \\ -1 & \text{for } \sigma_r = \sigma_s \text{ and } k = 1 \\ -2 & \text{for } \sigma_r = +1, \sigma_s = -1 \text{ and } k = 1 \\ 0 & \text{for } \sigma_r = -1, \sigma_s = +1 \text{ and } k = 1, \end{cases}$$

and

$$E = \begin{cases} 0 & \text{for } k \neq 1 \\ -1 & \text{for } k = 1. \end{cases}$$

These formulae also apply to non-converted P-rays by replacing  $c$  with  $r$  and putting all  $\bar{n}_i$ 's and  $\bar{m}_i$ 's equal to zero. In the same manner the formulae can be applied for S-rays without mode-conversion by replacing  $c$  with  $s$  and putting all  $n_i$ 's and  $m_i$ 's equal to zero.

### APPENDIX C

#### SUMMATION OF RAYS IN A HORIZONTALLY LAYERED MEDIUM

Using the results of Appendix A and Appendix B the wavefields at the receiver can be written as a sum over reflection counts. In general, with an arbitrary number of conversion points the total displacement velocity is given by

$$v = \sum_{u=0} V_u, \tag{C1}$$

where  $u$  is the number of reflections and  $V_u$  is the total displacement velocity for all rays with  $u$  reflections.  $V_u$  is given by:

$$V_u = \sum_{g, \gamma} \sum_{i_1, \dots, i_g} \dots \sum_{j_1, \dots, j_\gamma} V_{j_1 \dots j_\gamma; i_1 \dots i_g}^{i_1 \dots i_g; \bar{i}_1 \dots \bar{i}_\gamma}. \tag{C2}$$

$g$  is the number of reflections with the incident ray above the reflecting interface and  $\gamma$  is the number of reflections with the incident ray below the reflecting interface. The directions of the rays at source and receiver interfaces are assumed to be general and are not stated explicitly in (C2). The total number of reflections  $u$  is equal to the sum over the total number of reflections with the ray incident from above and below the reflecting interface. This means that the leftmost summation must be carried out under the restriction

$$u = g + \gamma \tag{C3}$$

In the sums over the reflecting interfaces  $jk$  and  $\bar{i}k$ , summation must also be carried out over all raypaths consisting of rays with the same points of reflection and transmission but different distribution and number of conversion points. This is easily

done by a method which is described by Vered and Ben-Menahem (1974), where all raypaths with a fixed number of conversions are generated from an unconverted raypath. To generate the unconverted raypaths one can use the fact that each raypath must connect the source and receiver. The ordered sequence of numbers enumerating the reflection points are given by the expressions in (C4)–(C7). The expressions depend on the direction of the ray at the source and at the receiver. This gives four possibilities, depending on whether the ray is upgoing or downgoing at the source and the receiver.

*Case I*

The ray is downgoing at the source and upgoing at the receiver.

$$\begin{aligned}
 i1 &= \text{Max} (is + 1, jr + 1), \dots, L && \text{for } g = 1 \\
 i1 &= is + 1, \dots, L && \text{for } g > 1 \\
 ik &= j(k - 1) + 1, \dots, L && \text{for } k = 2, \dots, g - 1 \\
 ig &= \text{Max} (j\gamma + 1, jr + 1), \dots, L && \text{for } g > 1 \\
 jk &= 0, \dots, ik - 1 && \text{for } k = 1, \dots, \gamma.
 \end{aligned} \tag{C4}$$

$L$  is the number of layers.

*Case II*

The ray is downgoing at the source and the receiver.

$$\begin{aligned}
 i1 &= is + 1, \dots, L \\
 ik &= j(k - 1) + 1, \dots, L && \text{for } k = 2, \dots, g \\
 jk &= 0, \dots, ik - 1 && \text{for } k = 1, \dots, \gamma - 1 \\
 j\gamma &= 0, \dots, \text{Min} (ig - 1, ir - 1).
 \end{aligned} \tag{C5}$$

*Case III*

The ray is upgoing at the source and the receiver

$$\begin{aligned}
 j1 &= 0, \dots, js - 1 \\
 jk &= 0, \dots, i(k - 1) - 1 && \text{for } k = 2, \dots, \gamma \\
 ik &= jk + 1, \dots, L && \text{for } k = 1, \dots, g - 1 \\
 ig &= \text{Max} (j\gamma + 1, jr + 1), \dots, L.
 \end{aligned} \tag{C6}$$

## Case IV

The ray is upgoing at the source and downgoing at the receiver.

$$\begin{aligned}
 j_1 &= 0, \dots, \text{Min}(j_s - 1, i_r - 1) && \text{for } \gamma = 1 \\
 j_1 &= 0, \dots, j_s - 1 && \text{for } \gamma > 1 \\
 j_k &= 0, \dots, i(k - 1) - 1 && \text{for } k = 2, \dots, \gamma - 1 \\
 j_\gamma &= 0, \dots, \text{Min}(i_g - 1, i_r - 1) && \text{for } k = 2, \dots, \gamma - 1 \\
 i_k &= j_k + 1, \dots, L && \text{for } k = 1, \dots, g \\
 i_g &= j(\gamma - 1) + 1, \dots, L. && \text{(C7)}
 \end{aligned}$$

If the number of conversions is restricted to one, the rays may be aggregated in dynamic analogue groups. Summation of all possible rays with  $u$  reflections can then be performed most easily by splitting the raypath into two parts. The first part extends from the source to the conversion point. The second part extends from the conversion point to the receiver. It is assumed that the source emits P-waves. If the source emits S-waves the appropriate expressions can easily be obtained by interchanging the P- and S-ray segments. The total displacement field can now be written as:

$$V_u = \sum_c \sum_{\bar{l}, \bar{\lambda}} \sum_{\bar{g}, \bar{\gamma}} \sum_{i_1, \dots, i_g} \dots \sum_{j_1, \dots, j_h} \sum_{i_1, \dots, i_g} \dots \sum_{j_1, \dots, j_h} N V_{j_1 \dots j_\gamma}^{i_1 \dots i_g; \bar{p}_1} \quad \text{(C8)}$$

In the first sum on the right-hand side of (C8) label  $c$  is used to indicate the interface where conversion occurs.

$\bar{l}$  is equal to one when a ray incident from above interface  $c$  is converted on transmission. Similarly  $\bar{\lambda}$  is equal to one when a ray incident from below interface  $c$  is converted on transmission. In all other cases  $\bar{l}$  and  $\bar{\lambda}$  are equal to zero.  $g$  and  $\bar{g}$  is the number of P- and S-wave reflections with the incident ray above the reflecting interface.  $\gamma$  and  $\bar{\gamma}$  are the number of P- and S-wave reflections with the incident ray below the reflecting interface.  $N$  is the number of rays within a dynamic analogue group, and is given by the expressions in Appendix B.

The upper and lower limits for the sums over the reflecting interfaces are given below for the first part of the raypath, extending from the source to the point of conversion. It is always assumed that the source is above the point of conversion. If this is not valid it is possible to redefine the point of conversion as the source and vice versa. It is further assumed that the number of reflections for the first part of the raypath is greater than or equal to three. For rays reflected once or twice the expressions given in (C4)–(C7) can be used since the number of rays in each dynamic analogue group is equal to one.

*Case I*

The ray is downgoing at the source and upgoing at the receiver.

$$\begin{aligned}
 i1 &= is + 1, \dots, L \\
 i2 &= \text{Max}(i1 + 1, jc + 1), \dots, L && \text{for } g = 2 \\
 i2 &= j1 + 1, \dots, is, i1, \dots, L && \text{for } g \geq 2 \\
 ik &= i(k - 1), \dots, L && \text{for } k = 3, \dots, g - 1 \\
 ig &= \text{Max}(i(g - 1), jc + 1), \dots, L && \text{for } g \geq 3 \\
 j1 &= 0, \dots, i1 - 1 \\
 jk &= j(k - 1), \dots, ik - 1 && \text{for } k = 2, \dots, \gamma.
 \end{aligned} \tag{C9a}$$

*Case II*

The ray is downgoing at the source and the receiver.

$$\begin{aligned}
 i1 &= is + 1, \dots, L \\
 i2 &= j1 + 1, \dots, is, i1, \dots, L \\
 ik &= i(k - 1), \dots, L && \text{for } k = 3, \dots, g \\
 j1 &= 0, \dots, i1 - 1 \\
 jk &= j(k - 1), \dots, ik - 1 && \text{for } k = 3, \dots, \gamma - 1 \\
 j\gamma &= 0, \dots, \text{Min}(ic - 1, ig - 1) && \text{for } j(\gamma - 1) \geq ic, \gamma \geq 2 \\
 j\gamma &= j(\gamma - 1), \dots, \text{Min}(ic - 1, ig - 1) && \text{for } j(\gamma - 1) < ic, \gamma \geq 2.
 \end{aligned} \tag{C9b}$$

*Case III*

The ray is upgoing at the source and the receiver.

$$\begin{aligned}
 j1 &= 0, \dots, js - 1 \\
 j2 &= j1, \dots, i1 - 1 \\
 jk &= j(k - 1), \dots, i(k - 1) - 1 && \text{for } k = 3, \dots, \gamma \\
 i1 &= j1 + 1, \dots, L \\
 i2 &= i1, \dots, \text{Max}(jc + 1, j2 + 1) && \text{for } g = 2 \\
 i2 &= j1 + 1, \dots, is, i1, \dots, L && \text{for } g \geq 3 \\
 ik &= i(k - 1), \dots, L && \text{for } k = 3, \dots, g - 1 \\
 ig &= \text{Max}(i(g - 1), jc + 1), \dots, L.
 \end{aligned} \tag{C9c}$$



## Case IV

The ray is upgoing at the source and downgoing at the receiver.

$$j1 = 0, \dots, js - 1$$

$$j2 = j1, \dots, \text{Min}(ic - 1, i1 - 1) \quad \text{for } \gamma = 2$$

$$j2 = j1, \dots, i1 - 1 \quad \text{for } \gamma \geq 3$$

$$jk = j(k - 1), \dots, i(k - 1) - 1 \quad \text{for } k = 3, \dots, \gamma - 1$$

$$j\gamma = 0, \dots, \text{Min}(ic - 1, ig - 1) \quad \text{for } j(\gamma - 1) \geq ic, \gamma \geq 3$$

$$j\gamma = j(\gamma - 1), \dots, \text{Min}(ic - 1, ig - 1) \quad \text{for } j(\gamma - 1) < ic, \gamma \geq 3$$

$$i1 = j1 + 1, \dots, L$$

$$i2 = j1 + 1, \dots, js, i1, \dots, L. \quad (\text{C9d})$$

The expressions for the S-wave part of the raypath are similar to (C9). They can be obtained by replacing  $ik, jk, js, is, jc$  and  $ic$  with  $\bar{ik}, \bar{jk}, \bar{jc}, \bar{ic}, \bar{jr}$ , and  $\bar{ir}$ .

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