Data-driven inversion/depth imaging derived from approximations to one-dimensional inverse acoustic scattering

2005 Inverse Problems 21 1823

(http://iopscience.iop.org/0266-5611/21/6/002)

Institute of Physics Publishing | Inverse Problems

Inverse Problems **21** (2005) 1823–1850 doi:10.1088/0266-5611/21/6/002

Data-driven inversion/depth imaging derived from approximations to one-dimensional inverse acoustic scattering

Lasse Amundsen[1,2], **Arne Reitan**[3,5], **Hans Kr Helgesen**[4] **and Børge Arntsen**[1]

[1] Statoil Research Centre, Postuttak, N-7005 Trondheim, Norway
[2] Department of Petroleum Engineering and Applied Geophysics, The Norwegian University of Science and Technology, N-7491 Trondheim, Norway
[3] Skomakerveien 14, N-4839 Arendal, Norway
[4] Department of Physics, The Norwegian University of Science and Technology, N-7491 Trondheim, Norway

E-mail: lam@statoil.com

Received 20 May 2005, in final form 22 August 2005
Published 3 October 2005
Online at stacks.iop.org/IP/21/1823

Abstract
This paper presents a new mathematical framework based on inverse scattering for the estimation of the scattering potential and its nature of a one-dimensional acoustic layered medium from single scattering data. Given the Born potential associated with constant-velocity imaging of the single scattering data, a closed-form implicit expression for the scattering potential is derived in the WKBJ and eikonal approximations. Adding physical insight, the WKBJ and eikonal solutions can be adjusted so that they conform to the geometrically derived precise solutions of the one-dimensional scattering problem recently suggested by the authors. In a layered medium, the WKBJ and eikonal approximations, in addition to providing an implicit solution for the scattering potential, provide an explicit estimate of the potential, not within the actual potential discontinuities (layer interfaces), but within the Born potential discontinuities derived by the constant-velocity imaging. This estimate of the potential is called the 'squeezed' potential since it mimics the actual potential when the depth axis is squeezed so that the discontinuities of the actual potential match those of the Born potential. It is shown that the squeezed potential can be estimated by amplitude-scaling the Born potential by an amplitude function of the Born potential. The accessibility of the squeezed potential makes the inverse acoustic scattering problem explicit and non-iterative since the estimated squeezed potential can non-linearly be stretched with respect to the depth axis so that the potential discontinuities are moved towards their correct depth location. The non-linear stretch function is a function of the Born potential. The solution

[5] Professor Emeritus.

is fully data-driven in the respect that no information about the medium other than the Born potential is required.

1. Introduction

Scattering is an important tool in studying the structure and dynamics of matter and is used in many disciplines of the physical, mathematical and engineering sciences as an investigative probe. The fields of application are numerous, and diverse as quantum physics, nuclear physics, classical physics such as elasticity, acoustics, electromagnetics, geophysics, to electrical engineering and medical imaging. In all these fields, scattering theory is used in one form or another to extract information about the system under investigation. Forward scattering involves determining the scattered field when the scattering potential is known. The inverse scattering problem, on the other hand, is the reverse: from the measured scattered field satisfying certain boundary conditions, determine the scattering potential. Whereas forward scattering is quite well understood, the inverse scattering problem still attracts attention in many major branches of science and engineering. The common thread underlying every inverse scattering problem, irrespective of the field of application, is the requirement of a *procedure* that identifies from the scattering data the properties of the scattering potential.

Scattering of acoustic waves has undergone a long development. The literature on acoustic scattering is extensive, and reference is made to Morse and Ingard (1968) and Gladwell (1993) and references therein. The problems in the field of acoustics are closely allied to those in elasticity and seismology, in particular. The research on inverse acoustic back-scattering (reflection seismic) reached a new height when Weglein and co-workers in a series of papers (Weglein *et al* 2000, 2002, 2003, Innanen 2003, Innanen and Weglein 2003, Shaw *et al* 2004, Shaw 2005, Zhang and Weglein 2005, Liu *et al* 2005) described a general approach to the problem of inversely reconstructing the potential. The potential to be recovered is expanded in a series, each term of which is determined in terms of the scattering data and a reference Green's function. The approach requires no prior information about the subsurface medium parameters. Further, Weglein and co-workers show the validity of the concept of 'subseries' within the expansion of the potential, where each subseries is associated with a specific inversion task that can be carried out separately. The four subseries associated with the inversion process are (1) free-surface related multiple removal, (2) internal multiple attenuation or elimination, (3) spatial location of reflectors in the subsurface (depth imaging) and (4) identification of changes in medium properties across reflectors (inversion). The two last subseries work on primaries, that is, those events in the data that have experienced a single upward reflection in the subsurface. When all multiples, which is seismic energy which has been reflected more than once, have been eliminated from the reflection data, one is left with primaries. The primaries are those reflection events that are used to determine the locations and properties (contrast in wave velocities) of the reflector that caused their observation (recording at the surface).

The inverse scattering problem to be analysed in the present paper is limited to that of processing primaries, or equivalently, single scattering events. As any data from a layered medium will contain both primaries and multiples, data have to go through a pre-processing step to remove all types of multiples before applying the proposed inversion/depth imaging steps to be presented. The pre-processing is in agreement with the standard practice to seek to attenuate all multiples from seismic data before using primaries for imaging changes in the medium's properties (see, e.g., Claerbout (1971), Berkhout (1982), Weglein (1985), Stolt and

Benson (1986) and Ikelle and Amundsen (2005)). The historical evolution and development of seismic processing and inversion explain the motivation for addressing the inverse acoustic scattering problem as that of inverting primaries. On its own, the single scattering model in a layered medium is an idealization realized only when the incident wave from the distant source is scattered only once at each interface in the medium. One therefore could be led to think that the single scattering model should only be used when the medium changes slowly with depth with no strong discontinuities of the medium parameters. However, the realization is within practical reach with the use of new methods which have been developed to attenuate and eliminate internal multiples (Weglein *et al* 1997, Ramirez and Weglein 2005).

The inverse scattering series is exact (Weglein *et al* 2003). Although all developments based on the inverse scattering series have been very useful for the processing and interpretation of seismic data, it turns out nevertheless to be extremely challenging to identify all high-order terms related to the subseries for imaging primaries at depth, also for the case of a one-dimensional (1D) layered medium. Even though work is in progress to extend the depth imaging and inversion capability of the inverse scattering series (see, e.g., Innanen (2003) and Innanen and Weglein (2003)), the current depth imaging scheme is based on what is called leading-order inverse scattering. As a response to this challenge, influenced by the leading-order closed-form 1D depth imaging/inversion algorithms described by Shaw *et al* (2004) and Innanen (2003), Amundsen *et al* (2005) suggested a closed-form solution of the 1D inverse scattering problem that is precise when the product of reflection coefficients from any three interfaces of the medium is negligible compared to the reflection coefficients themselves. The solution was found from physical intuition and geometrical observations of the relationship between the actual potential and the 'Born potential' associated with constant-velocity imaging of the single scattering data. The solution, however, was not founded on a firm mathematical-physics framework, which is essential and required for extending the solution for the 1D acoustic medium to multidimensional acoustic and elastic stratified media. It is our objective in the present paper to give the mathematical-physics framework for 1D inverse acoustic (or seismic) scattering problem for layered media where velocities are generally discontinuous at layer interfaces. Once the framework is established for 1D acoustic scattering, it will be the fundament required for extension to multidimensional stratified media.

It is well appreciated in physics that approximations play an important role in the understanding of processes that cannot be analytically solved exactly. Depth imaging derived from the inverse scattering series, for 1D media, contains an infinite number of terms of which many, based on physical insight, are neglected to give useful and practical algorithms for imaging (Innanen 2003, Shaw *et al* 2004). Realizing that approximations are inevitable at some stage in the inverse scattering series, one is led to ask: building on the tremendous achievements already obtained by inverse scattering series analysis, can we achieve added understanding of the inverse scattering problem if approximations are introduced into the forward scattering model instead of the inverse scattering series? This is the line of action we take in the present work to investigate what kind of inverse solutions can be obtained. We remark that our mathematical-physics framework for the inverse scattering problem does not make explicit use of the inverse scattering series introduced in exploration seismology by Weglein *et al* (2003). Instead, our approach is to attempt a direct inversion of the forward scattering formula.

The approximations we invoke in the forward scattering model are the WKBJ, eikonal and Born approximations. The WKBJ approximation has its name after Wentzel, Kramers and Brillouin, who independently introduced it in quantum mechanics in 1926, and Jeffreys, who contributed to its development in 1923. The WKBJ approximation is also known as the WKB approximation or the BWK approximation, depending on the number of contributors that are mentioned in which order, or also as the *classical approximation* or the *phase integral method*. For example, this approximation has been widely used in atomic physics. We refer to the textbook by Schiff (1955 and later editions) for a description of the physical basis and applications of this method. Other general references on the WKBJ approximation are Morse and Feshbach (1953), Fröman and Fröman (1965), Bender and Orszag (1978) and Bransden and Joachain (1989). Some references related to geophysical applications of the WKBJ approximation are Bremmer (1951), Aki and Richards (1980), Clayton and Stolt (1981), Robinson (1982, 1986), Bleistein (1984), Ursin (1984, 1987) and Amundsen (1994). To the eikonal approximation (from $\epsilon\iota\kappa\omega\nu$ = image) there is, in our opinion, no better introduction than that of Glauber (1959) in his 1958 Boulder lectures. By an extension of the method to scattering problems beyond those that are described by a potential, the so-called Glauber approximation or assumption of phase additivity, this method was used extensively in the 1960s and 1970s for elementary particle scattering on nuclei, and also for electron scattering on atoms. In fact, one of the present authors contributed to these developments (see, e.g., Reitan (1979)).

The Born approximation is, of course, so widely used in many branches of the sciences that it need only be mentioned here for the sake of completeness. In inverse scattering, the Born approximation is recognized as an important practical tool as long as all scattering involves a single interaction of the probe wavefield with the target. In the nomenclature of reflection seismic, the Born approximation accounts for primary reflections from the subsurface. In scattering studies, the actual medium is divided into a reference medium and a scattering potential that characterizes the difference between the actual and reference media. In a general sense, the Born approximation is known to be good as long as the product of the range of the scattering potential and its average strength is small.

Even though it is well known that the key to a successful application of the Born approximation is that the reference medium must be selected or estimated accurately enough to capture all the long-wavelength information contained in the actual medium, a convenient choice of reference medium is, still, that of a homogeneous one. One reason for the latter choice is that the reference response of a homogeneous medium is known analytically. However, inverse scattering imaging based on a homogeneous reference medium is known to produce incorrect results. The related Born image potential can be far from the actual potential, both in amplitude and in positioning of discontinuities. In this paper, where we consider single scattering, it turns out, nevertheless, very convenient as a first step towards the inverse solution to express the imaging of the primaries through the Born potential.

The full inverse scattering problem of seismic, viewed as a three-dimensional (3D) problem, is one of the most challenging problems of geophysics and mathematical physics. In this paper, therefore, we shall be concerned with 1D inverse acoustic single scattering from layered media. One-dimensional models, although they severely restrict or idealize the system under consideration, when treated rigorously they are rich, intricate and involve many deep results from pure mathematics, and need to be understood before dealing with the general 3D models.

The literature on the 1D inverse problem is extensive. Some of the important works in geophysical inversion from the early literature are Goupillaud (1961), Claerbout (1968), Ware and Aki (1968), Razavy (1975), Gjevik *et al* (1976), Koehler and Taner (1977), Nilsen and Gjevik (1978), Berryman and Greene (1980), Burridge (1980), Carroll and Santosa (1981), Coen (1981), Symes (1981), Santosa (1982), Santosa and Schwetlick (1982), Bube and Burridge (1983) and Bruckstein *et al* (1985). The reader may also consult the reviews by Newton (1981) and Ursin and Berteussen (1986).

The paper is organized as follows: first, we give a brief review of the forward models for acoustic single scattering in the WKBJ, eikonal and Born approximations. This is used as a mathematical framework for relating the Born potential to the primary reflection response of a layered acoustic medium. In a constant-velocity reference medium, the Born potential is simply obtained by data trace integration, by which the primary reflection events are placed at depths computed linearly only using the constant reference velocity and the travel times of primaries. Next, we derive a closed-form implicit relationship between the Born potential and the potential of the medium in the WKBJ and eikonal approximations. Then, we show how the relationship can be made more accurate by conforming it to the geometrical relationship between the Born and WKBJ (eikonal) solutions found by Amundsen *et al* (2005). Instead of embarking on developing an iterative scheme for the estimation of the potential, we develop a closed-form direct solution approach that is data-driven and non-iterative. The data-driven method is split into two steps. From the Born potential itself, we estimate within its layer boundaries what the actual potential would be. In appendix C we show that the estimate is precise for those media where the product of any three reflection coefficients is small compared to the reflection coefficients themselves. Since the layer boundaries are mislocated in the Born potential, this first step obtains good estimates of the amplitude of the actual potential but at wrong depths. What the first step achieves is to provide an amplitude-adjusted scaled Born potential, which we denote the 'squeezed' potential since it mimics the actual potential when the depth axis is squeezed. The second step focuses on the mispositioning in depth of the reflectors. From the estimated squeezed potential we show in the WKBJ approximation that the reflectors can be moved with high precision towards their correct spatial location. Finally, we construct a simple example to show how the procedures introduced in this paper can be applied to obtain the potential in the WKBJ and eikonal approximations from the Born potential.

2. Forward scattering models

We consider a 1D acoustic medium, where the velocity is a function of depth, $c = c(z)$, that is embedded in a homogeneous reference medium with wave velocity c_0. The 1D wave equation for scattering of pressure waves P with angular frequency ω and corresponding wave number $k = \omega/c_0$ in a velocity potential

$$\alpha(z) = 1 - \left(\frac{c_0}{c(z)}\right)^2 \tag{1}$$

that characterizes the difference between the reference and actual media, is described by the equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k^2 - k^2\alpha(z)\right] P(k, z, z_0) = -s(k)\delta(z - z_0). \tag{2}$$

The z-axis is pointing vertically downwards and the source with strength $s(k)$ is located at $z = z_0$. The forward problem associated with equation (2) is stated as follows: given the potential α, find the solution P that satisfies prescribed boundary conditions. In this section, the forward problem is stated in the WKBJ, eikonal and Born approximations. As is usual in scattering theory, the potential α is assumed to vanish asymptotically, i.e., $\alpha \to 0$ as $z \to \pm\infty$, at which limit the wavefunction P is merely a plane propagating wave described by $\exp(\pm \mathrm{i}kz)$. We write the solution of equation (2) as the sum of a term $P_0(k, z, z_0)$ which is the solution of the wave equation in the homogeneous reference medium, i.e., with no potential, and an additional term $\delta P(k, z, z_0)$ caused by the potential $\alpha(z)$, i.e.,

$$P(k, z, z_0) = P_0(k, z, z_0) + \delta P(k, z, z_0), \tag{3}$$

where

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+k^2\right]P_0(k,z,z_0)=-s(k)\delta(z-z_0), \tag{4}$$

and where the additional pressure δP obeys the differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+k^2\right]\delta P(k,z,z_0)=k^2\alpha(z)P(k,z,z_0). \tag{5}$$

Formally, the solution of the differential equation (5) for the additional pressure δP can be expressed as the integral equation

$$\delta P(k,z,z_0)=-k^2\int_{-\infty}^{\infty}\mathrm{d}z' g(k,z,z')\alpha(z')P(k,z',z_0), \tag{6}$$

where the causal Green function in free space

$$g(k,z,z')=-\frac{1}{2\mathrm{i}k}\exp(\mathrm{i}k|z'-z|) \tag{7}$$

is a solution of the scattering-free problem

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+k^2\right]g(k,z,z')=-\delta(z-z'). \tag{8}$$

The incident wave due to the point source at z_0 is

$$P_0(k,z,z_0)=s(k)g(k,z,z_0)=S(k)\exp(\mathrm{i}k|z_0-z|), \tag{9}$$

where $S(k)=(\mathrm{i}/2k)s(k)$ is the initial wavefield from the source. When the source is located at $z_0=0$, the incident wave becomes

$$P_0(k,z,0)\equiv P_0(k,z)=S(k)\exp(\mathrm{i}kz). \tag{10}$$

The Green function entering expression (6) for the additional pressure $\delta P(k,z)\equiv\delta P(k,z,0)$ is $g(k,z,z')$ with $z'>z$, so that

$$\delta P(k,z)=F(k)\exp(-\mathrm{i}kz), \tag{11}$$

where $F(k)$ is the scattering amplitude. We have

$$F(k)=-\frac{\mathrm{i}k}{2}\int_{-\infty}^{\infty}\mathrm{d}z'\exp(\mathrm{i}kz')\alpha(z')P(k,z'), \tag{12}$$

where $P(k,z)\equiv P(k,z,0)$. When the receiver is located at $z=0$, the scattering amplitude is identical to the scattered data,

$$F(k)=\delta P(k,z=0). \tag{13}$$

2.1. The scattering amplitude

It is convenient to introduce the dimensionless scattering amplitude

$$\Phi(k)=\frac{F(k)}{S(k)} \tag{14}$$

and write the expression for $\Phi(k)$ in a form reminiscent of a quantum mechanical matrix element,

$$\Phi(k)=-\frac{\mathrm{i}k}{2}\int_{-\infty}^{\infty}\mathrm{d}z\,\psi_{\mathrm{f}}^*(z)\alpha(z)\psi_{\mathrm{i}}(z), \tag{15}$$

where

$$\psi_{\mathrm{i}}(z) = \frac{P(k,z)}{S(k)}. \tag{16}$$

In the absence of the potential α, the initial and final states ψ_{i} and ψ_{f} are given by the expressions

$$\psi_{\mathrm{i}0}(z) = \psi_{\mathrm{f}0}^{\star}(z) = \exp(\mathrm{i}kz). \tag{17}$$

Introducing the so-called complex phase shift function $\chi(z)$, we write the actual states as

$$\psi_{\mathrm{i}}(z) = \psi_{\mathrm{f}}^{\star}(z) = \psi_{\mathrm{i}0}(z)\exp[\mathrm{i}\chi(z)], \tag{18}$$

and equation (15) becomes

$$\Phi(k) = -\frac{\mathrm{i}k}{2}\int_{-\infty}^{\infty}\mathrm{d}z\,\exp(2\mathrm{i}kz)\alpha(z)\exp[2\mathrm{i}\chi(z)]. \tag{19}$$

The influence of the potential $\alpha(z)$ on the wavefunctions $\psi_{\mathrm{i}}(z)$ and $\psi_{\mathrm{f}}(z)$ is thus contained in the phase shift function $\chi(z)$. Since the scattered wave ψ_{f} travels through the same potential $\alpha(z)$ as the incident wave ψ_{i}, the phase shift function is the same for both cases. Writing ψ_{i} and $\psi_{\mathrm{f}}^{\star}$ in the form (18) is, so far, just a matter of convenience. The physical basis for this choice and various approximations to the phase shift function χ are considered in the next section.

2.2. *The phase shift function*

In this section we present three different approximations to the phase shift function $\chi(z)$ which appears in expressions (18) for the wavefunctions $\psi_{\mathrm{i}}(z)$ and $\psi_{\mathrm{f}}(z)$. These are the Born, WKBJ and eikonal approximations, respectively.

2.2.1. Differential equations. In accordance with (2), the wavefunction (18) for the initial state obeys the differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k^2 - k^2\alpha(z)\right]\psi_{\mathrm{i}}(z) = 0. \tag{20}$$

Factorizing ψ_{i} in the form

$$\psi_{\mathrm{i}}(z) = \psi_{\mathrm{i}0}(z)\phi(z) = \exp(\mathrm{i}kz)\phi(z), \tag{21}$$

we get the differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2\mathrm{i}k\frac{\mathrm{d}}{\mathrm{d}z} - k^2\alpha(z)\right]\phi(z) = 0 \tag{22}$$

for the function $\phi(z)$.

In terms of the phase shift function $\chi(z)$, we write

$$\phi(z) = \exp[\mathrm{i}\chi(z)], \tag{23}$$

where the differential equation for $\chi(z)$ becomes

$$\mathrm{i}\frac{\mathrm{d}^2\chi_{\mathrm{X}}(z)}{\mathrm{d}z^2} - 2k\frac{\mathrm{d}\chi_{\mathrm{X}}(z)}{\mathrm{d}z} - \left(\frac{\mathrm{d}\chi_{\mathrm{X}}(z)}{\mathrm{d}z}\right)^2 - k^2\alpha(z) = 0. \tag{24}$$

2.2.2. The Born approximation. In the Born approximation, the exact pressure wave (17) is replaced by the incident wave (18) in expression (15) for the scattering amplitude, which means that the function $\phi(z)$ defined in equation (21) is

$$\phi_{\mathrm{B}}(z) \equiv 1, \tag{25}$$

the corresponding phase shift function being

$$\chi_{\mathrm{B}}(z) \equiv 0. \tag{26}$$

2.2.3. The WKBJ approximation. We separate the real and imaginary parts of $\chi(z)$,

$$\chi(z) = \chi_{\mathrm{R}}(z) + \mathrm{i}\chi_{\mathrm{I}}(z). \tag{27}$$

According to equation (24) we then, for a real potential $\alpha(z)$, obtain the coupled differential equations

$$\chi_{\mathrm{I}}'' + 2k\chi_{\mathrm{R}}' + (\chi_{\mathrm{R}}')^2 - (\chi_{\mathrm{I}}')^2 + k^2\alpha = 0, \qquad \chi_{\mathrm{R}}'' - 2k\chi_{\mathrm{I}}' - 2\chi_{\mathrm{R}}'\chi_{\mathrm{I}}' = 0, \tag{28}$$

where the primes denote differentiation with respect to z. On the assumption that the phase shift function varies slowly over a wavelength, we can disregard the second derivatives in equations (28) and put $\chi_{\mathrm{I}}(z) \equiv 0$. The first equation of (28) becomes

$$2k\chi_{\mathrm{R}}' + (\chi_{\mathrm{R}}')^2 + k^2\alpha = 0, \tag{29}$$

where the physically acceptable root for the derivative $\chi_{\mathrm{R}}' \equiv \chi_{\mathrm{W}}'$ of the phase shift function is

$$\chi_{\mathrm{W}}'(z) = k\left\{[1-\alpha(z)]^{\frac{1}{2}} - 1\right\}. \tag{30}$$

In the WKBJ approximation, the phase shift function is thus

$$\chi_{\mathrm{W}}(z) = k\int_{-\infty}^{z} \mathrm{d}z'\left\{[1-\alpha(z')]^{\frac{1}{2}} - 1\right\}, \tag{31}$$

where it is assumed that $\alpha(z') < 1$ for all $z' < z$.

Alternatively, we can base the present approximation upon the concept of a local wave number $k_{\mathrm{local}}(z)$, where in accordance with equation (20),

$$k_{\mathrm{local}}^2(z) = k^2 - k^2\alpha(z). \tag{32}$$

It is convenient to express the WKBJ phase shift function χ_{W} in equation (31) in terms of the WKBJ shift function ξ_{W} in the following way:

$$\chi_{\mathrm{W}}(k, z) = -k\xi_{\mathrm{W}}(z). \tag{33}$$

The WKBJ shift function obeys

$$\xi_{\mathrm{W}}'' = 0, \qquad -2\xi_{\mathrm{W}}' + (\xi_{\mathrm{W}}')^2 + \alpha = 0, \tag{34}$$

so that

$$\xi_{\mathrm{W}}(z) = \int_{-\infty}^{z} \mathrm{d}z'\left[1 - (1-\alpha(z'))^{\frac{1}{2}}\right]. \tag{35}$$

2.2.4. The eikonal approximation. The eikonal approximation is based upon an expansion of the square root in equation (31) to first order in the dimensionless potential α. The corresponding phase shift function is then

$$\chi_{\mathrm{E}}(z) = -\frac{k}{2}\int_{-\infty}^{z} \mathrm{d}z'\alpha(z'). \tag{36}$$

Another way of arriving at expression (36) is to use the differential equation (22) for $\phi(z)$ as a starting point. Assuming again a slowly varying deviation from plane-wave behaviour, or, equivalently, a weak potential $\alpha(z)$, we disregard the second derivative and obtain the first-order differential equation

$$\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} = -\frac{\mathrm{i}k}{2}\alpha(z)\phi(z). \tag{37}$$

The solution of this is then $\phi(z) = \exp[\mathrm{i}\chi_{\mathrm{E}}(z)]$, with a phase shift function $\chi_{\mathrm{E}}(z)$ as given by (36).

As for the WKBJ approximation, it is convenient to express the phase shift function χ_{E} in equation (36) in terms of the eikonal shift function ξ_{E} according to

$$\chi_{\mathrm{E}}(k, z) = -k\xi_{\mathrm{E}}(z), \tag{38}$$

so that

$$\xi_{\mathrm{E}}(z) = \frac{1}{2}\int_{-\infty}^{z} \mathrm{d}z'\alpha(z'). \tag{39}$$

2.3. The single scattering forward model

Equation (19) is a non-linear forward model for computing the dimensionless single scattering amplitude $\Phi(k)$ from the potential α. Replacing the phase shift function by the shift function, $\chi = -k\xi$, gives the forward single scattering model

$$\Phi(k) = -\frac{\mathrm{i}k}{2}\int_{-\infty}^{\infty} \mathrm{d}z\,\alpha(z)\exp[2\mathrm{i}k(z - \xi(z))]. \tag{40}$$

The WKBJ, eikonal and Born approximations are obtained by choosing $\xi = \xi_{\mathrm{W}}$, $\xi = \xi_{\mathrm{E}}$ and $\xi = 0$, respectively.

We make the following remarks. The single scattering amplitude is found by performing an integral over depth over the product of an amplitude function and a delay function. The amplitude function is the scattering potential. The delay function consists of the product of two functions, where the first $\exp(2\mathrm{i}kz)$ accounts for two-way wave propagation in the reference medium, whereas the second $\exp[-2\mathrm{i}k\xi(z)]$ accounts for the influence of the potential. For a piecewise-constant layered medium, the delay function in the WKBJ approximation predicts the exact travel times of single scattering events. However, performing the integral over depth, the predicted amplitudes of the single scattering events will not be exact for the piecewise-constant layered medium unless the boundary conditions of continuity of the pressure and the displacement at the interfaces are explicitly introduced. For the sake of forward modelling, the boundary conditions can be easily accounted for. Interfaces or discontinuities in the potential are then treated by correctly coupling the incident wave to the scattered waves. However, for the inverse problem, where the location of interfaces is not known, it would be cumbersome to account for the continuity conditions in an explicit manner. Therefore, we choose to neglect these conditions at the expense of using a forward model that predicts slightly incorrect amplitudes of the single scattering events.

When we later simulate data to test the inverse scattering algorithm to be described in the next section, we do not base the simulation on the single scattering forward model (40), but on an exact forward model for primary reflections in a piecewise-constant layered medium. This model is described in appendix A. The relationship between the forward model (40) and that developed in appendix A with respect to amplitude handling is not discussed further in the present paper. We would like to indicate, however, that the WKBJ forward model (40) converges to the exact forward model when reflection coefficients of the medium are 'small' in the sense that the product of any three of the reflection coefficients is vanishingly small compared to the reflection coefficients.

3. Inverse scattering

The inverse scattering problem consists of reconstructing the potential α from the dimensionless scattering amplitude $\Phi(k)$ (the single scattering data measured at depth $z = 0$). In this section we develop a procedure for reconstructing the potential α from the scattering amplitude. The solution is obtained in two steps. First, the Born potential is computed from the scattered field using a constant reference medium. Second, a relationship between the Born potential and the actual potential obtained in the WKBJ approximation is derived. This relationship is used as a basis for introducing a data-driven estimation approach for the potential, requiring no other information than the Born potential itself. The data-driven approach is divided into two computational steps. First, the Born amplitude is adjusted by scaling the Born amplitude by a WKBJ correction amplitude function that is a function of the Born amplitude. We denote this potential the 'squeezed' potential since it mimics the actual potential when the depth axis is squeezed. Second, the estimated squeezed potential is non-linearly shifted with respect to the depth axis. The non-linear shift function is a function of the Born potential. Thus, the solution for the potential can be considered to be obtained by amplitude and shift adjusting the Born potential. No information other than the Born potential is required.

The inverse Fourier transform over frequency of equation (40) yields

$$4\int_{-\infty}^{2z/c_0} \mathrm{d}t' \Phi(t') = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \alpha(z) \xi^n(z), \tag{41}$$

where $z = c_0 t/2$. Recall that in the Born approximation, the shift function is zero. By considering $\xi = 0$ in the forward model (40) and inverse Fourier transforming over frequency, the Born potential, per definition, is obtained:

$$\alpha_{\mathrm{B}}(z) \equiv 4\int_{-\infty}^{2z/c_0} \mathrm{d}t' \Phi(t'). \tag{42}$$

Equation (42), which amounts to data trace integration of single scattering events, is known as linear migration inversion. The primary events are placed at depths computed linearly using their travel times together with the constant reference velocity. Thus, the trace integration yields the Born approximation of the scattering potential. Equation (41) therefore can be written as

$$\alpha_{\mathrm{B}}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \alpha(z) \xi^n(z). \tag{43}$$

Given the Born potential $\alpha_{\mathrm{B}}(z)$ associated with constant-velocity imaging of single scattering data, our goal is now to use equation (43) as the basis for solving the inverse scattering problem in the WKBJ and eikonal approximations.

Observe that equation (43), on the left-hand side, contains the Born potential and, on the right-hand side, involves the actual potential. For a layered medium where velocities are

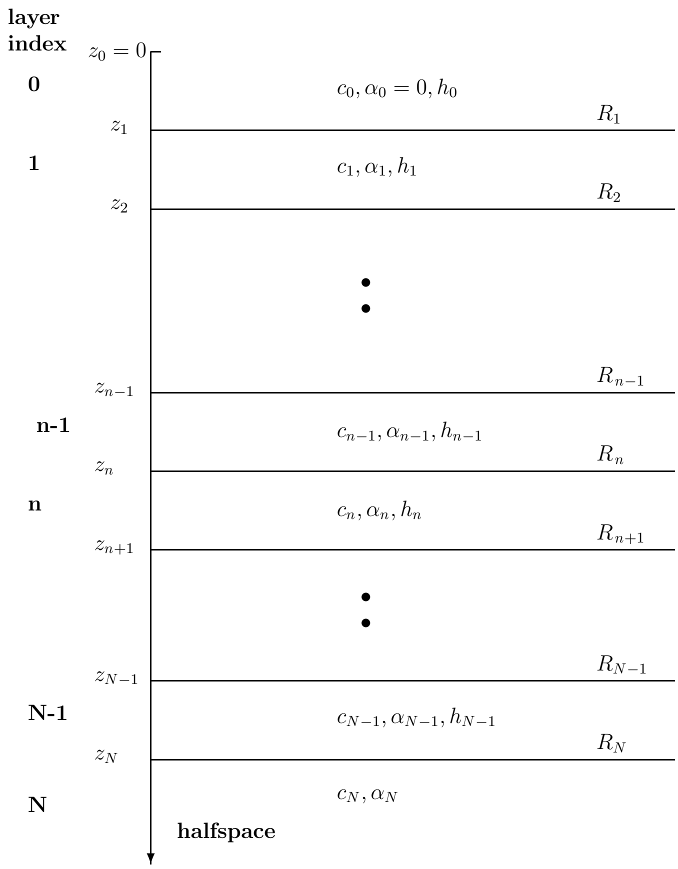

Figure 1. Layered medium. The velocity $c(z)$ and potential $\alpha(z)$ are generally discontinuous at layer interfaces $z_1, z_2, \ldots, z_N$.

generally discontinuous at layer interfaces but constant or slowly varying functions of depth between layer interfaces, any physical solution for the actual potential should not contain more discontinuities (layer interfaces) than the number of discontinuities (layer interfaces) in the Born potential. Before embarking on the inverse solution of equation (43), we will use this requirement to guide us towards the form of the inverse acoustic scattering solution in the WKBJ and eikonal approximations.

3.1. The form of the inverse solution

To this end, it is instructive to consider plane-wave propagation through a medium with $N+1$ homogeneous layers with constant layer velocities c_n and thicknesses h_n as shown in figure 1. To illustrate the concepts to be introduced, we use the high-velocity contrast piecewise-constant ten-layer ($N = 9$) acoustic 1D medium with velocity $c(z)$ shown in figure 2(a). The layer depths $z_n = \sum_{j=0}^{n-1} h_j$ and velocities are also listed in table 1. The source and the receiver are both located at depth $z = 0$ in the zeroth layer which is the reference medium with known velocity c_0. The scattering potential is plotted in figure 2(b). From equation (1) it follows that the potential in layer n is

$$\alpha_n = 1 - \left(\frac{c_0}{c_n}\right)^2. \tag{44}$$

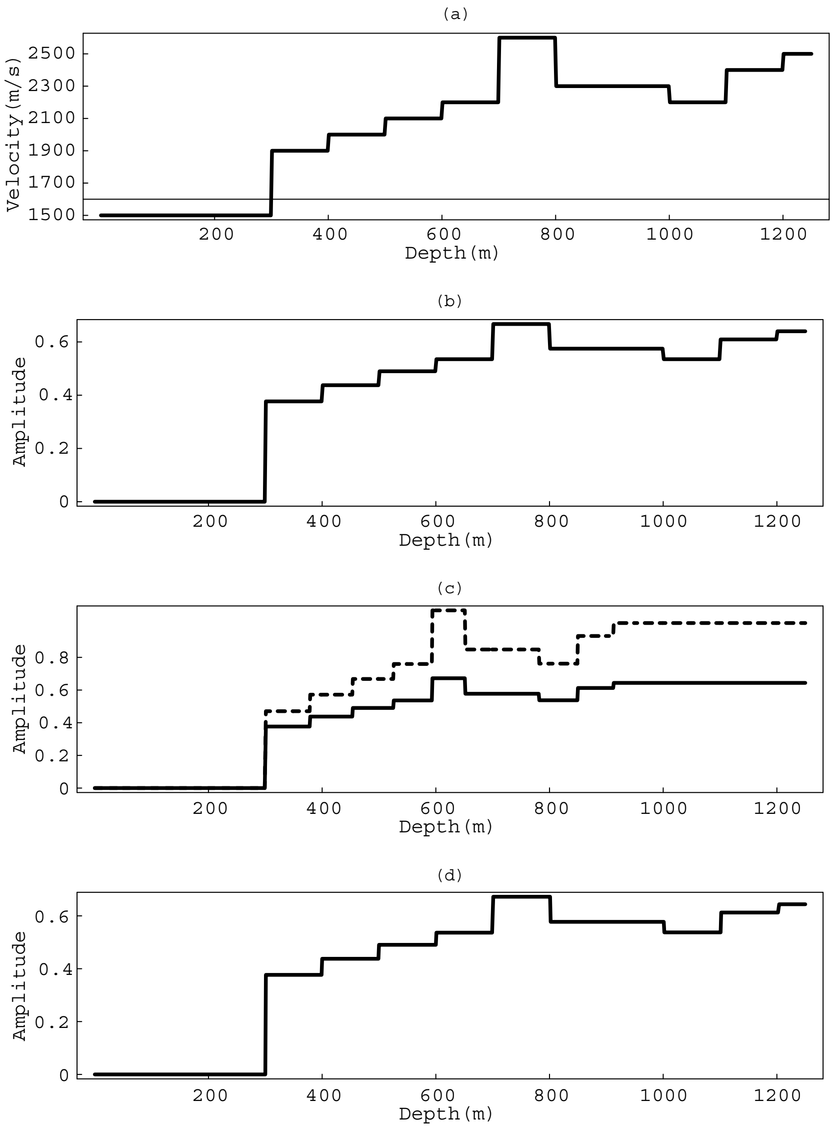

Figure 2. 'Squeezing' and 'stretching' of potentials. (a) The actual velocity model, $c(z)$, which is listed in table 1. (b) The actual potential, $\alpha(z)$. (c) Squeezing of the actual potential according to equation (48). Comparison of the squeezed potential $\hat{\alpha}$ (solid line) and the Born potential α_B (dashed line) shows that the layer boundaries are at identical depths. Note that $\hat{\alpha}$ can be estimated from α_B according to equation (68). (d) Stretching of the squeezed potential $\hat{\alpha}$ according to equation (49) restores the actual potential, α.

The primary pressure data for this example consist of N reflections. As shown in appendix A, the normalized primary reflection response can be represented as

$$\Phi(t) = \sum_{n=1}^{N} \hat{R}_n \delta(t - t_n), \tag{45}$$

where $\hat{R}_n$, defined in equation (A.3), is the amplitude of the reflection event from interface

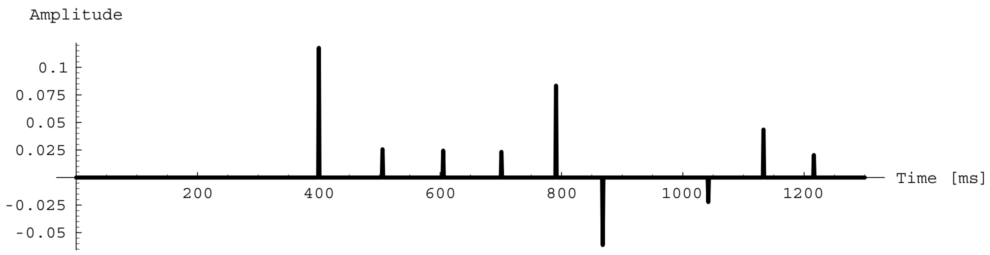

Figure 3. Primary reflection events from the ten-layer (nine-interface) model in figure 2(a).

Table 1. Ten-layer model with reference velocity $c_0 = 1500$ m s^{-1}. Here, z_n is the actual layer depth, $z_{n\mathrm{B}}$ is the layer depth from Born constant-velocity imaging, $\hat{z}_{n\mathrm{W}}$ is the estimated layer depth from depth imaging in the WKBJ approximation, $\hat{z}_{n\mathrm{E}}$ is the estimated layer depth from depth imaging in the eikonal approximation, c_n is the actual layer velocity, $\hat{c}_{n\mathrm{W}}$ is the estimated layer velocity in the WKBJ approximation computed according to equation (C.8), $\hat{c}_{n\mathrm{E}}$ is the estimated layer velocity in the eikonal approximation computed according to equation (C.8), α_n is the actual potential, $\alpha_{n\mathrm{B}}$ is the Born potential and R_n is the reflection coefficient.

n	z_n (m)	$z_{n\mathrm{B}}$ (m)	$\hat{z}_{n\mathrm{W}}$ (m)	$\hat{z}_{n\mathrm{E}}$ (m)	c_n (m s^{-1})	$\hat{c}_{n\mathrm{W}}$ (m s^{-1})	$\hat{c}_{n\mathrm{E}}$ (m s^{-1})	α_n	$\alpha_{n\mathrm{B}}$	R_n
0					1500	1500	1500	0.00	0.00	
1	300	300	300	300	1900	1894	1906	0.38	0.47	0.118
2	400	378.9	400	397	2000	1989	2013	0.44	0.57	0.026
3	500	453.9	499	494	2100	2082	2123	0.49	0.67	0.024
4	600	525.4	599	590	2200	2174	2237	0.54	0.76	0.023
5	700	593.6	697	684	2600	2523	2759	0.67	1.09	0.083
6	800	651.3	795	773	2300	2266	2359	0.57	0.85	−0.061
7	1000	781.7	991	958	2200	2176	2240	0.54	0.76	−0.022
8	1100	849.9	1090	1052	2400	2353	2484	0.61	0.93	0.043
9	1200	912.4	1189	1144	2500	2438	2616	0.64	1.01	0.020

n measured at $z = 0$ at time $t_n = 2\sum_{j=0}^{n-1} h_j/c_j$ and $\delta(t)$ is the Dirac delta function. The primary data corresponding to the ten-layer model are shown in figure 3.

In imaging in the Born approximation, the first reflector is located at its correct depth, $z_{1\mathrm{B}} = z_1$ since $\alpha(z) = 0$ for $z < z_1$. Using that the Born-estimated thickness of layer n is $h_{n\mathrm{B}} = (c_0/c_n)h_n$, it follows that the depth at which the reference velocity images the nth reflector is $z_{n\mathrm{B}} = c_0 \sum_{j=0}^{n-1} h_j/c_j$.

The depth model that would be obtained from constant-velocity Born imaging is shown in figure 4. The Born solution for α is found by substituting the primary reflection response (45) into equation (19), yielding the Born potential

$$\alpha_{\mathrm{B}}(z) = 4\sum_{n=1}^{N} \hat{R}_n H(z - z_{n\mathrm{B}}), \tag{46}$$

where $H(z)$ is the Heaviside function. Thus, the Born potential in layer n becomes

$$\alpha_{n\mathrm{B}} = 4\sum_{j=1}^{n} \hat{R}_j = 4\left(R_1 + \sum_{j=2}^{n} R_j \prod_{i=1}^{j-1}\left(1 - R_i^2\right)\right). \tag{47}$$

For the ten-layer model, the Born potential $\alpha_{\mathrm{B}}(z)$ is shown by the dashed line in figure 2(c).

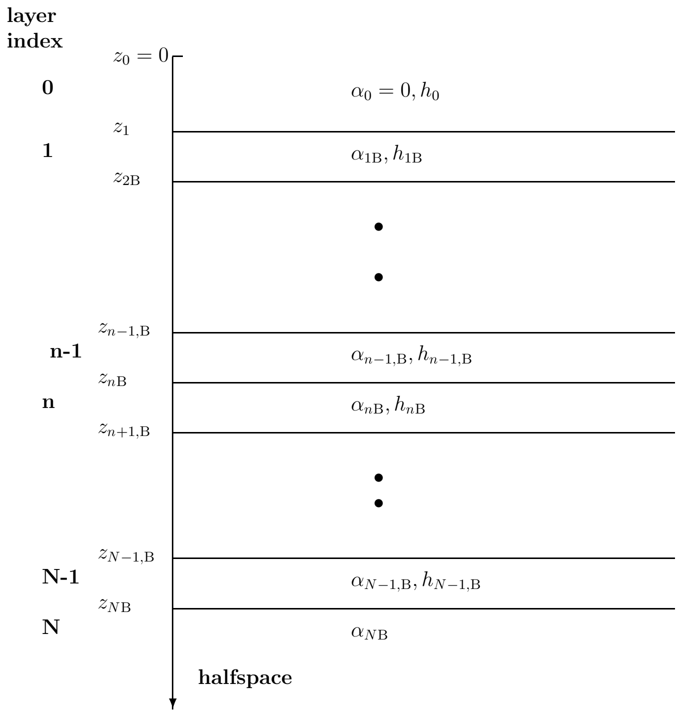

Figure 4. Layered medium that would be obtained from Born constant-velocity imaging.

Amundsen *et al* (2005) observed that it is possible to non-linearly shift the interfaces of the actual potential $\alpha(z)$ onto the interfaces of the Born potential $\alpha_B(z)$ by the transformation

$$\alpha(z) = \hat{\alpha}\left(z - \int_{-\infty}^{z} dz'\left[1 - (1 - \alpha(z'))^{\frac{1}{2}}\right]\right). \tag{48}$$

Equation (48) shows that traversing along the depth axis of the actual potential α, say from depth z_{n-1} to z_n, corresponds to traversing along the depth axis of the shifted potential $\hat{\alpha}$ from depth $z_{n-1,B}$ to z_{nB}. The depth geometry of $\hat{\alpha}$ is sketched in figure 5. Thus, the actual potential α can be shifted in such a way that its layer interfaces coincide with the layer interfaces of the Born potential α_B. The shifted potential $\hat{\alpha}$ is called the 'squeezed' potential since it appears like the actual velocity potential when the depth axis is squeezed. Note that it is only the layer interfaces that are affected by the squeeze operation. For the ten-layer medium, the squeezed potential $\hat{\alpha}(z)$ is displayed in figure 2(c) together with the Born potential. Observe that the interfaces of $\hat{\alpha}(z)$ and α_B match. Inside layer n, the amplitude of $\hat{\alpha}$ equals that of α. The number of interfaces is preserved.

On the other hand, the layer interfaces of the squeezed potential $\hat{\alpha}$ now coinciding with the layer interfaces of the Born potential α_B can be restored, that is, 'stretched' onto the layer interfaces of the actual potential α, by the transformation

$$\hat{\alpha}(z) = \alpha\left(z + \int_{-\infty}^{z} dz'\left[(1 - \hat{\alpha}(z'))^{-\frac{1}{2}} - 1\right]\right). \tag{49}$$

Figure 2(d) shows that the stretch procedure (49) restores the actual potential. The number of interfaces is unaffected. Stated differently, equation (49) shows that traversing along the depth axis of $\hat{\alpha}$, say from depth $z_{n-1,B}$ to z_{nB}, corresponds to traversing along the depth axis of α from depth z_{n-1} to z_n.

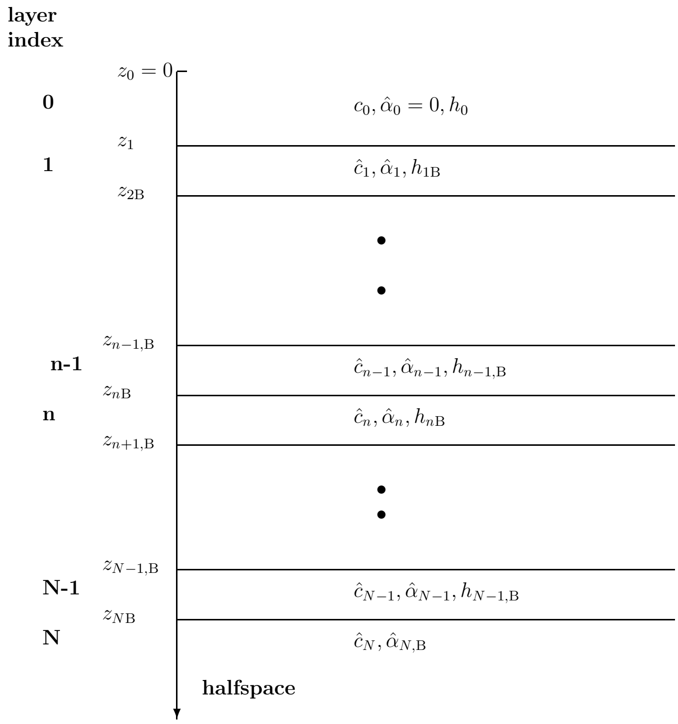

Figure 5. The layer boundaries of the 'squeezed' potential $\hat{\alpha}$ correspond to those of α_{B} shown in figure 4. The amplitudes $\hat{\alpha}_n$ correspond to those of α_n illustrated in figure 1. As shown by equation (68), $\hat{\alpha}_n$ can be estimated from α_{B}, leading to an estimate of layer velocities $\hat{c}_n$ according to equation (C.8).

In the inverse acoustic scattering problem, however, we have at our disposal the scattering amplitude $\Phi(t)$ from which the Born potential $\alpha_{\mathrm{B}}(z)$ can be computed by trace integration. We have argued that any physical solution for the potential should not contain more discontinuities than the number of discontinuities in the Born potential. Furthermore, we have observed that equation (43) on its left-hand side contains the Born potential and on its right-hand side involves the actual potential. First, consider the Born depth profile (left-hand side of equation (43)). When moving from one discontinuity to the next, say from $z_{n-1,\mathrm{B}}$ to $z_{n\mathrm{B}}$ in the Born depth profile, for the number of interfaces to be constant one must move between the corresponding discontinuities z_{n-1} to z_n in the actual depth profile (right-hand side of equation (43)). With reference to equation (49), this requirement is fulfilled when the Born potential is proportional to

$$\alpha_{\mathrm{B}}(z) \propto \alpha\left(z + \int_{-\infty}^{z} \mathrm{d}z'\big[(1 - \hat{\alpha}(z'))^{-\frac{1}{2}} - 1\big]\right). \tag{50}$$

On the right-hand side of equation (50), the innermost α-function is the actual potential depth profile squeezed so that its discontinuities coincide with those of the Born depth profile on the left-hand side. While moving along the Born depth profile from discontinuity $z_{n-1,\mathrm{B}}$ to $z_{n\mathrm{B}}$, the integral on the right-hand side stretches the once squeezed potential back to its original position. In this way, the number of discontinuities is preserved. Considering once more equation (43), it is obvious that the left- and right-hand sides must have equal (or approximately equal, if the WKBJ approximation is invoked) amplitudes. This can be achieved if equation (43) involves on its right-hand side an amplitude correction factor A that adjusts the Born potential amplitude to the actual potential amplitude. Since the exact amplitude correction factor must depend on the actual potential, the form of A must contain that of the squeezed potential since the Born potential and the squeezed potential both have discontinuities at the same depths. Thus, with respect to amplitude, the Born potential should behave as

$$\alpha_{\mathrm{B}}(z) \propto A^{-1}[\hat{\alpha}(z)]. \tag{51}$$

This form of A ensures that whenever there is a discontinuity of $^n\alpha_{\mathrm{B}}$, say at depth $z_{n\mathrm{B}}$, there is a discontinuity in A at $z_{n\mathrm{B}}$. The now obvious, but bold guess suggests that the solution of the inverse scattering problem for the layered model should have the form

$$\alpha_{\mathrm{B}}(z) = A^{-1}[\hat{\alpha}(z)]\alpha\left(z + \int_{-\infty}^{z} \mathrm{d}z'\big[(1-\hat{\alpha}(z'))^{-\frac{1}{2}} - 1\big]\right). \tag{52}$$

Thus, knowing $\alpha_{\mathrm{B}}(z)$, the actual potential can be found by stretching the depth axis of the Born potential solution while at the same time applying an amplitude scaling function A that corrects the Born amplitude onto the actual potential amplitude.

The non-trivial challenge is now first to investigate if wave theory in the WKBJ approximation leads to a similar form as that suggested in equation (52) for the solution of the potential, and secondly to determine the amplitude function A in the WKBJ approximation. The eikonal approximation follows as a special case by an expansion of the square root in equation (52) to first order in the potential α.

3.2. WKBJ approximation: implicit solution for potential

By neglecting terms $\mathrm{d}^n\xi/\mathrm{d}z^n$ for $n = 2, \ldots, \infty$, we can write equation (43) as an infinite sum where the nth term is proportional to the nth power of the derivative of the shift function,

$$\alpha_{\mathrm{B}}(z) \approx \sum_{n=0}^{\infty} \left(\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}\right)^n \sum_{m=n}^{\infty} \frac{1}{(m-n)!}\binom{m}{n} \xi^{m-n}(z) \frac{\mathrm{d}^{m-n}\alpha(z)}{\mathrm{d}z^{m-n}}. \tag{53}$$

Equation (53) is a basis for deriving a closed-form solution for α. To this end, the Fourier representation of $\alpha(z)$ is introduced, which gives

$$\alpha_{\mathrm{B}}(z) \approx \sum_{n=0}^{\infty} \left(\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}\right)^n \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, D_n(k\xi) \exp(-\mathrm{i}kz)\alpha(k), \tag{54}$$

where

$$D_n(k\xi) = \sum_{m=n}^{\infty} \frac{1}{(m-n)!}\binom{m}{n} [-\mathrm{i}k\xi(z)]^{m-n}. \tag{55}$$

The sum D_n can be written as

$$D_n(k\xi) = \exp[-\mathrm{i}k\xi(z)] \sum_{m=0}^{n} \frac{1}{m!}\binom{n}{m} [-\mathrm{i}k\xi(z)]^{n-m}, \tag{56}$$

and the expression for the Born potential then becomes

$$\alpha_{\mathrm{B}}(z) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \left\{ \sum_{n=0}^{\infty} \left(\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}\right)^n \sum_{m=0}^{n} \frac{1}{m!}\binom{n}{m} [-\mathrm{i}k\xi(z)]^{n-m} \right\} \exp[-\mathrm{i}k(z+\xi(z))]\alpha(k). \tag{57}$$

In equation (57), the double sum can be written as a single sum that is recognized as an expression for the exponential function,

$$\sum_{n=0}^{\infty}\left(\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}\right)^n \sum_{m=0}^{n}\frac{1}{m!}\binom{n}{m}[-\mathrm{i}k\xi(z)]^{n-m} = \frac{1}{1-\xi'(z)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\mathrm{i}k\xi(z)\xi'(z)}{1-\xi'(z)}\right)^n$$
$$= \frac{1}{1-\xi'(z)}\exp\left(-\frac{\mathrm{i}k\xi(z)\xi'(z)}{1-\xi'(z)}\right). \tag{58}$$

The Born potential in equation (57) now reads

$$\alpha_{\mathrm{B}}(z) \approx \left(1-\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}\right)^{-1}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\exp\left[-\mathrm{i}k\left(z+\frac{\xi(z)}{1-\xi'(z)}\right)\right]\alpha(k). \tag{59}$$

Using the translation property of the Fourier transform, we obtain a closed-form expression for the Born potential,

$$\alpha_{\mathrm{B}}(z) \approx \left(1-\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}\right)^{-1}\alpha\left(z+\frac{\xi(z)}{1-\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}}\right). \tag{60}$$

However, although close, equation (60) is not exactly of the form proposed by equation (52). We therefore introduce two new approximations, both of which are valid in the WKBJ approximation. First, as shown in appendix B (see equation (B.5)), under the WKBJ assumption that $\xi''(z)$ is negligible, the replacement

$$\frac{\xi(z)}{1-\frac{\mathrm{d}\xi(z)}{\mathrm{d}z}} \to \int_{-\infty}^{z}\mathrm{d}z'\frac{\frac{\mathrm{d}\xi(z')}{\mathrm{d}z'}}{1-\frac{\mathrm{d}\xi(z')}{\mathrm{d}z'}}, \qquad \xi''(z)=0 \tag{61}$$

is justified. Second, as shown in appendix B (see equation (B.11)), when $\xi''(z)$ and all higher order derivatives are disregarded, the following replacement is justified:

$$[1-\alpha(z)]^{-\frac{1}{2}} \to [1-\hat{\alpha}(z)]^{-\frac{1}{2}}, \qquad \frac{\mathrm{d}^n\xi(z)}{\mathrm{d}z^n}=0, \qquad n\geqslant 2. \tag{62}$$

Here, $\hat{\alpha}(z)$, defined in equation (48), represents the actual potential which has layer boundaries at z_n squeezed to new layer boundaries $z_{n\mathrm{B}}$ coinciding with the Born layer boundaries. Inserting the two approximations into equation (60) and recalling equation (35) give the result

$$\alpha_{\mathrm{B}}(z) \approx [1-\hat{\alpha}(z)]^{-\frac{1}{2}}\alpha\left(z+\int_{-\infty}^{z}\mathrm{d}z'\big[(1-\hat{\alpha}(z'))^{-\frac{1}{2}}-1\big]\right), \tag{63}$$

which is consistent with the solution form proposed in equation (52). Observe that WKBJ theory has provided the form of the amplitude scaling function A that corrects the Born potential amplitude onto the actual potential amplitude:

$$A(z) \approx [1-\hat{\alpha}(z)]^{\frac{1}{2}}. \tag{64}$$

Equation (63) is thus the desired inverse scattering solution for the potential α. Since $\hat{\alpha}$ is a function of α (cf equation (48)), equation (63) is an implicit equation for α that can be solved, in principle, by iteration as a data-fitting problem with known α_{B}.

Note that equations (42) and (63) represent a two-step procedure for obtaining the scattering potential. First, the Born potential α_{B} is computed from the dimensionless scattering amplitude according to equation (42). Second, the potential α is solved from equation (63). In this paper, we will not embark on the problem of developing an iterative estimation scheme. Instead, we illustrate the difference between equations (60) and (63), both of which are expressions for the Born potential, by computing the Born potential from the actual potential. The results shown in figure 6 are in agreement with the above discussion. Observe that α_{B} computed from equation (60) gives a reasonable estimate of α_{B} but that the number of interfaces is wrong. On the other hand, equation (63) produces an α_{B}-estimate that is quite precise.

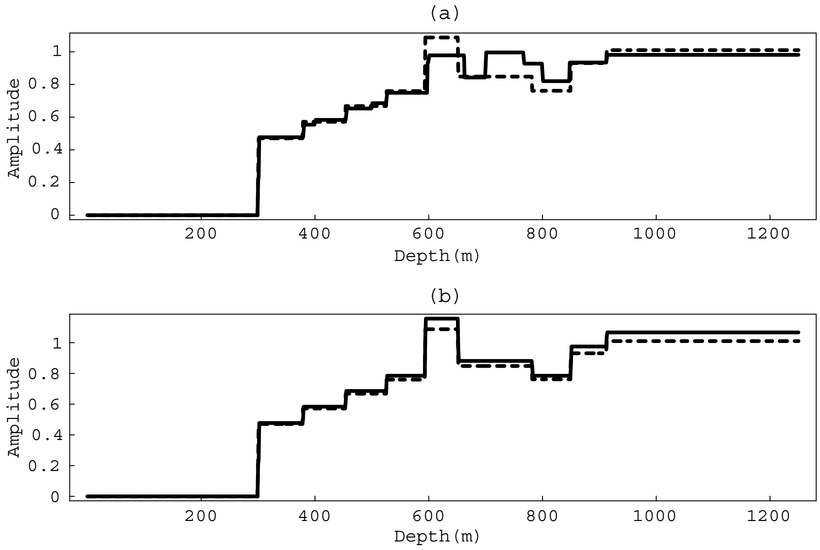

Figure 6. Born potential α_B estimated (solid lines) from (a) equation (60) and (b) equation (63). The exact Born potential is displayed by the dashed line.

3.3. WKBJ approximation: data-driven direct solution for potential

The form of equation (63) suggests another possible approach valid for the case that we can predict from WKBJ theory what the actual layer potentials are, not as a function of their true depth, but as a function of the interface depths provided by the Born potential. Thus, the question we ask is: can we predict $\hat{\alpha}$ from α_B by means of WKBJ theory? As stated earlier, the function $\hat{\alpha}(z)$ represents a mapping of layers of the actual potential onto the layers of the Born potential. The amplitudes of this squeezed potential $\hat{\alpha}$ and the actual potential α are the same, and generally different from the Born amplitudes. On the other hand, the locations of discontinuities (layer boundaries) of the squeezed potential $\hat{\alpha}$ and the Born potential α_B are the same. Inside a layer, velocities are assumed to vary smoothly with depth. It is then reasonable to disregard all derivatives of α, and the WKBJ approximation, it turns out, leads to an approximate relationship between $\hat{\alpha}$ and α_B. The non-zero terms in equation (53) arrive for all $n = m$, giving

$$\alpha_B(z) \approx \hat{\alpha}(z) \sum_{n=0}^{\infty} \left[1 - (1 - \hat{\alpha}(z))^{\frac{1}{2}}\right]^n, \qquad \frac{d^{n+1}\alpha(z)}{dz^{n+1}} = 0. \tag{65}$$

The sum is a geometric series, and we find

$$\alpha_B(z) \approx [1 - \hat{\alpha}(z)]^{-\frac{1}{2}} \hat{\alpha}(z). \tag{66}$$

Inversion of equation (66) yields the admissible solution

$$\hat{\alpha}(z) \approx \hat{\hat{\alpha}}(z) = \left[\left(1 + \tfrac{1}{4}\alpha_B^2(z)\right)^{\frac{1}{2}} - \tfrac{1}{2}\alpha_B(z)\right]\alpha_B(z). \tag{67}$$

Equation (67) shows that once α_B has been derived, then the squeezed potential estimate $\hat{\hat{\alpha}}$ can be obtained,

$$\hat{\hat{\alpha}}(z) = A_W(z)\alpha_B(z), \tag{68}$$

by multiplying α_B by the WKBJ correction amplitude

$$A_W(z) = \left(1 + \tfrac{1}{4}\alpha_B^2(z)\right)^{\frac{1}{2}} - \tfrac{1}{2}\alpha_B(z). \tag{69}$$

The only information that is required for this estimation is the Born potential itself. In appendix B we show that relation (68) is precise for layered media where the product of reflection coefficients from any three interfaces is negligible compared to the reflection coefficients themselves. Therefore, for such media, equation (63) can be rewritten in the following form:

$$\hat{\hat{\alpha}}(z) = \alpha\left(z + \int_{-\infty}^{z} \mathrm{d}z'\left[(1 - \hat{\hat{\alpha}}(z'))^{-\frac{1}{2}} - 1\right]\right). \tag{70}$$

Provided that the Born potential has been computed according to equation (42), then equation (70) suggests a two-step solution for the scattering potential α. First, the Born potential is scaled by the WKBJ correction amplitude (69) to find the squeezed potential estimate $\hat{\hat{\alpha}}$. Then, the potential α is derived as a non-linear shift of $\hat{\hat{\alpha}}$ according to equation (70). The non-linear shift is seen to correspond to stretching the depth axis of the squeezed potential result, $\hat{\hat{\alpha}}$. The effect of shifting is to locate interfaces that are mislocated in $\hat{\hat{\alpha}}$ towards their correct location. Thus, in the absence of the actual velocity function, the scale and shift algorithm extracts the necessary information from the Born depth profile $\alpha_B(z)$.

3.4. Eikonal approximation

The eikonal approximation for the scattering potential follows from equations (63) and (70) by an expansion of the square roots to first order in the potential α. The eikonal approximation data-driven direct solution corresponding to equation (70) is

$$\hat{\hat{\alpha}}(z) = \alpha\left(z + \int_{-\infty}^{z} \mathrm{d}z'\left[\left(1 - \frac{1}{2}\hat{\hat{\alpha}}(z')\right)^{-1} - 1\right]\right), \tag{71}$$

in which the squeezed potential estimate $\hat{\hat{\alpha}}$ in the eikonal approximation,

$$\hat{\hat{\alpha}}(z) = A_E(z)\alpha_B(z), \tag{72}$$

is obtained by multiplying α_B by the eikonal correction amplitude

$$A_E(z) = \left(1 + \tfrac{1}{2}\alpha_B(z)\right)^{-1}. \tag{73}$$

4. Model calculations

As an example of data-driven estimation of the scattering potential, we consider once more the high-velocity contrast piecewise-constant ten-layer acoustic medium listed in table 1 and displayed in figure 2(a). The reference velocity is $c_0 = 1500$ m s^{-1}. The actual potential is plotted in figure 7(a). First, single scattering data are computed according to equation (45) for infinite bandwidth. The scattered data are displayed in figure 3. Second, the Born potential is derived by constant-velocity imaging according to equation (46). The Born depth profile is shown by the blue line in figure 7(b). Third, the squeezed potential estimate $\hat{\hat{\alpha}}$ is calculated from the Born potential by use of equation (68) for the WKBJ approximation and equation (72) for the eikonal approximation. Essentially, $\hat{\hat{\alpha}}$ which for the WKBJ and eikonal cases, respectively, is displayed by red and pink lines in figure 7(b), is obtained by scaling α_B by the proper correction amplitude, A_W or A_E.

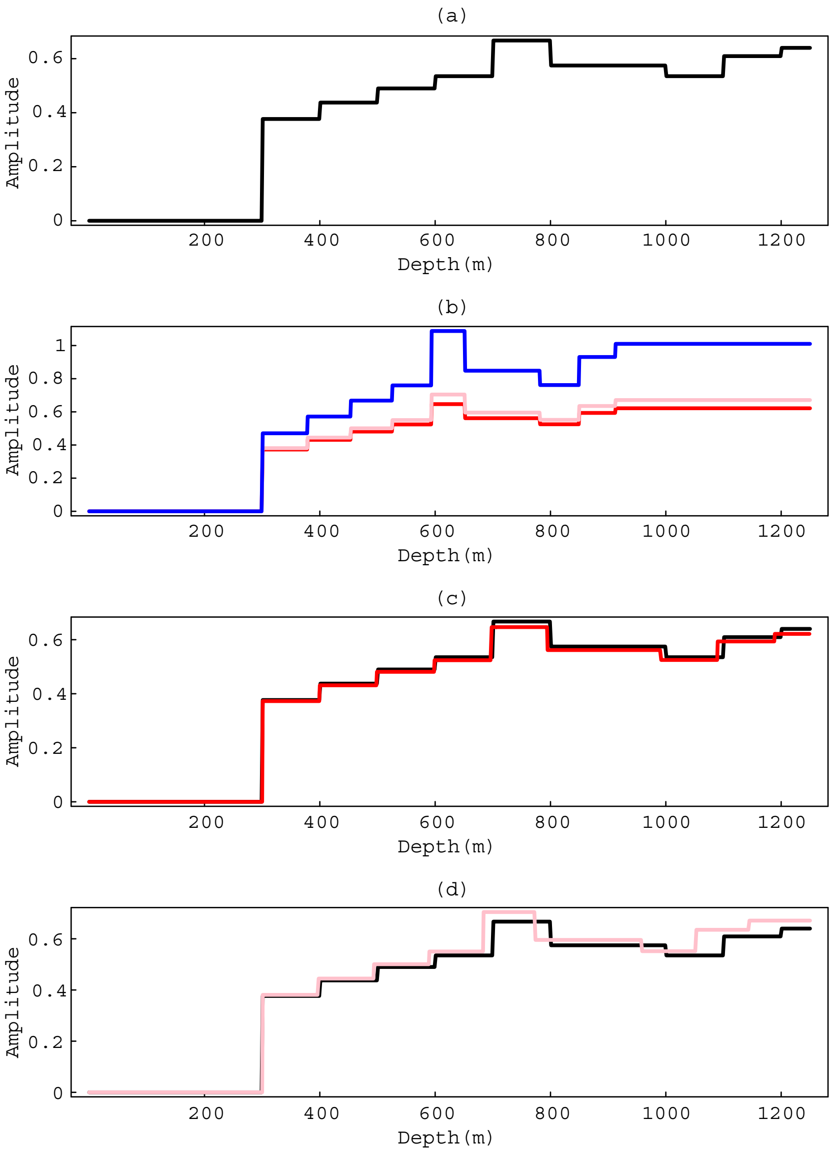

Figure 7. (a) The actual scattering potential. (b) The Born (blue line) and estimated squeezed potentials in the WKBJ (red line) and eikonal (pink line) approximations. (c) WKBJ-derived potential (red line) obtained by stretching the squeezed WKBJ potential in (b, red line) compared to the actual potential (black line). (d) Eikonal-derived potential (pink line) obtained by stretching the squeezed eikonal velocity potential in (b, pink line) compared to the actual potential (black line).

(This figure is in colour only in the electronic version)

Now, the actual potential can be estimated in the WKBJ and eikonal approximations simply by stretching the depth axis of the squeezed potential estimate $\hat{\hat{\alpha}}$ according to formulae (70) and (71), respectively. The WKBJ-derived result is shown in figure 7(c) and the corresponding eikonal-derived potential is plotted in figure 7(d). For the purpose of comparison, the actual potential is re-plotted by the black line. As can be expected, for the high-velocity contrast example, the WKBJ approximation performs better than the eikonal approximation (which essentially is an approximation to the WKBJ approximation) with respect to estimating the potential discontinuities (layer boundaries). The estimated depths of the potential discontinuities are listed in table 1.

5. Future work

In the present paper, we have restricted the inverse scattering analysis to 1D (plane-wave normal-incidence) acoustic scattering. However, many of the elements of the inverse scattering theory are general and extend to higher dimensions as well as to elastic media. Work is in progress to study inverse acoustic and elastic multidimensional single scattering in stratified media and results will be reported as they are obtained. Research related to whether and how the proposed inverse scattering concept can be extended to depth image data from 3D media which vary both vertically and laterally is a topic of future research.

In the practical applications of inverse scattering theory many questions arise, relating for example to the optimum collection of data and to the pre-processing of data. Such questions are being addressed by several university consortia as well as by the seismic contractors and we expect to benefit from their ongoing studies.

6. Conclusions

The ultimate objective of inverse scattering is to determine the medium and its properties from measurements external to the object under investigation. We have given a brief review of the forward models for acoustic single scattering from one-dimensional layered media in the WKBJ, eikonal and Born approximations. From the Born potential associated with constant-velocity imaging of the single scattering data, we derived a closed-form implicit expression for the scattering potential. The inverse scattering solution estimates by iteration the potential by stretching the depth axis of the Born potential solution while at the same time applying an amplitude scaling function that corrects the Born amplitude onto the actual potential amplitude.

For a layered medium we showed that the WKBJ and eikonal approximations, in addition to providing an implicit solution for the scattering potential, provide an explicit estimate of the potential, not within the actual potential discontinuities (layer interfaces), but within the Born potential discontinuities derived by the constant-velocity imaging. The estimate is fed directly into the inverse acoustic scattering solution such that the inversion problem becomes fully explicit and non-iterative. Further, this solution is data-driven in the respect that no other information about the medium other than the Born potential is required. The data-driven method is split into two steps. First, within the layer boundaries provided by the Born potential, it is estimated what the squeezed potential would be. Since the layer boundaries are mislocated in the Born potential, this first step obtains accurate estimates of the amplitude of the actual potential but at wrong depths. The amplitude estimation is simply obtained by scaling the Born potential by a function of itself. In the second step, the mislocated reflectors in the estimated squeezed potential are moved with high precision towards their correct spatial location by applying a non-linear stretch function that is a function of the Born potential only.

A simple model example showed how the potential could be estimated in the WKBJ and eikonal approximations, given the Born potential. For high-velocity contrast media (strong potential), WKBJ theory gives an inverse scattering procedure that with high precision reconstructs the potential and its nature. The eikonal approximation, which is based upon an expansion of the square root in the WKBJ shift function to first order in the potential, and thus involves a weak-discontinuity potential assumption, does not provide the same precise depth location of layer interfaces as that provided by WKBJ theory for high-contrast velocity media.

Acknowledgments

We thank Statoil ASA for allowing us to publish this work. LA is grateful to Professor A B Weglein and Dr S Shaw for introducing him to the inverse scattering series (Weglein *et al* 2003), which spurred the interest in the current work. HKH thanks Statoil for the financial support.

Appendix A. Modelling of the primary reflection response

We consider plane-wave propagation through a medium with $N + 1$ homogeneous layers with constant layer velocities c_n and thicknesses h_n as shown in figure 1. The source and the receiver are both located at depth $z = 0$ in the zeroth layer which is the reference medium with velocity c_0.

The differential equation (2) together with proper boundary conditions yield that the pressure field is made up of an infinite sum of reflections and refractions inside the medium (Bremmer 1951). In the following, we show how to model the exact primary reflection response, that is, the waves that are split off by reflection from the downgoing source wavefield when it is transmitted into the medium. To this end, it is necessary to define the reflection and transmission coefficients in the stack of layers. As is well known, the coefficients can be derived by assuming that the pressure and displacement are continuous fields at every boundary. For a plane wave incident in layer $n - 1$, the reflection coefficient is

$$R_n = \frac{c_n - c_{n-1}}{c_n + c_{n-1}} \tag{A.1}$$

and the transmission coefficient is $T_n^{(\mathrm{D})} = 1 + R_n$. We will also need that a wave transmitted in the opposite direction, upwards from layer n into layer $n - 1$, has transmission coefficient $T_n^{(\mathrm{U})} = 1 - R_n$. Thus, the two-way transmission loss for a plane wave passing down and up through the interface at depth z_n is $T_n^{(\mathrm{D})} T_n^{(\mathrm{U})} = 1 - R_n^2$.

When the source is initiated with unit strength, a plane wave propagates downwards with velocity c_0 into the discontinuous, layered medium. At the boundary of the first layer, at depth $z_1 = h_0$, the incident wave which is represented by

$$D_0(\omega) = \exp(\mathrm{i}\omega h_0/c_0),$$

is split into (i) a refracted wave penetrating into this layer with amplitude $T_1^{(\mathrm{D})}$ and represented by

$$D_1(\omega) = D_0(\omega) T_1^{(\mathrm{D})} \exp[\mathrm{i}\omega(z - z_1)/c_1], \qquad z_1 < z < z_2,$$

and (ii) a reflected wave with amplitude R_1 returning to the receiver level where it is represented by

$$\Phi_1(\omega) = R_1 \exp(2\mathrm{i}\omega h_0/c_0).$$

The downgoing wave $D_1(\omega)$ will be split at the next interface at depth z_2 into a refracted wave

$$D_2(\omega) = D_1(\omega) T_2^{(\mathrm{D})} \exp[\mathrm{i}\omega(z - z_2)/c_2], \qquad z_2 < z < z_3,$$

penetrating into layer 2, and a reflected wave which, after being refracted through the interface at depth z_1, returns to the receiver level with representation

$$\Phi_2(\omega) = R_2\big(1 - R_1^2\big)\exp(2\mathrm{i}\omega h_1/c_1)\exp(2\mathrm{i}\omega h_0/c_0).$$

This procedure of splitting is repeated at each next interface. The chain of wave consisting of the sequence $\Phi_1, \Phi_2, \ldots, \Phi_N$ is called the principal wave by Bremmer (1951) but is called the primary reflection response in reflection seismology.

In the frequency domain, the N events of the dimensionless scattering amplitude can be modelled as

$$\Phi_{\mathrm{X}}(\omega) = \sum_{n=1}^{N}\Phi_n(\omega) = \sum_{n=1}^{N}\hat{R}_n\exp\left(2\mathrm{i}\omega\sum_{m=0}^{n-1}\frac{h_m}{c_m}\right), \tag{A.2}$$

where each wave has the form of the product of an amplitude function and a delay function. The frequency dependency comes only as a complex exponential due to the delay. The amplitude of the wave from the interface at depth z_n is the product of the plane-wave reflection coefficient at z_n and the transmission coefficients encountered by the wave, namely

$$\hat{R}_1 = R_1, \qquad \hat{R}_n = R_n\prod_{j=1}^{n-1}\big(1 - R_j^2\big), \qquad n = 2, 3, \ldots, N. \tag{A.3}$$

Performing an inverse Fourier transform over frequency, the dimensionless scattering amplitude in the time domain becomes

$$\Phi_{\mathrm{X}}(t) = \sum_{n=1}^{N}\hat{R}_n\delta(t - t_n), \tag{A.4}$$

where $\delta(t)$ is the Dirac delta function. The arrival time of the primary reflection from depth z_n is $t_n = 2\sum_{m=0}^{n-1}\frac{h_m}{c_m}$.

Appendix B. Two approximations introduced into the WKBJ approximation

In this appendix, we develop two approximations that are introduced into equation (60).

B.1. First approximation

Writing

$$\xi(z) = \int_{-\infty}^{z}\mathrm{d}z'\xi'(z') \tag{B.1}$$

and

$$\tilde{\xi}(z) = \int_{-\infty}^{z}\mathrm{d}z'\tilde{\xi}'(z'), \tag{B.2}$$

and introducing

$$\tilde{\xi}'(z) = \frac{\xi'(z)}{1 - \xi'(z)}, \tag{B.3}$$

partial integration gives

$$\tilde{\xi}(z) = \left.\frac{\xi(z')}{1 - \xi'(z')}\right|_0^z - \int_{-\infty}^{z}\mathrm{d}z'\frac{\xi(z')\xi''(z')}{(1 - \xi'(z'))^2}. \tag{B.4}$$

In the WKBJ approximation, $\xi''(z) = 0$; therefore,

$$\tilde{\xi}(z) = \frac{\xi(z)}{1 - \xi'(z)} = \int_{-\infty}^{z}\mathrm{d}z'\frac{\xi'(z')}{1 - \xi'(z')}. \tag{B.5}$$

B.2. Second approximation

By writing

$$\frac{1}{\sqrt{1-\alpha(z)}} = \frac{1}{\sqrt{1-\hat{\alpha}(z)}}\sqrt{1+\frac{\alpha(z)-\hat{\alpha}(z)}{1-\alpha(z)}}, \tag{B.6}$$

we readily observe that a sufficient criterion for the validity of the approximation (62) in the WKBJ approximation is

$$\frac{\alpha(z)-\hat{\alpha}(z)}{1-\alpha(z)} \approx 0, \qquad \frac{\mathrm{d}^n\xi(z)}{\mathrm{d}z^n} = 0, \qquad n \geqslant 2. \tag{B.7}$$

By writing equation (49) in the form

$$\hat{\alpha}(z) = \alpha(z+\hat{\hat{\xi}}(z)), \tag{B.8}$$

where

$$\hat{\hat{\xi}}(z) = \int_{-\infty}^{z} \mathrm{d}z'\big[(1-\hat{\alpha}(z'))^{-\frac{1}{2}} - 1\big], \tag{B.9}$$

it can be shown that

$$\frac{\alpha(z)-\hat{\alpha}(z)}{1-\alpha(z)} = -\frac{1}{1-\alpha(z)}\left[\alpha'(z)\hat{\hat{\xi}}(z) + \frac{1}{2}\alpha''(z)\hat{\hat{\xi}}^2(z) + \frac{1}{6}\alpha^{(3)}(z)\hat{\hat{\xi}}^3(z) + \cdots\right]. \tag{B.10}$$

Further, from equation (34) it follows that in the WKBJ approximation where the second-order and all higher order derivatives of the shift function $\xi(z)$ can be disregarded, $\alpha^{(n)}$ for $n \geqslant 1$ can be neglected. Therefore, in the WKBJ approximation, it is allowed to write

$$\frac{1}{\sqrt{1-\alpha(z)}} \approx \frac{1}{\sqrt{1-\hat{\alpha}(z)}}, \qquad \frac{\mathrm{d}^n\xi(z)}{\mathrm{d}z^n} = 0, \qquad n \geqslant 2. \tag{B.11}$$

Appendix C. Analysis of the relationship between the squeezed potential estimate $\hat{\hat{\alpha}}$ and the Born potential α_{B}

The WKBJ-derived equation (68) gives a relationship between the estimated squeezed potential $\hat{\hat{\alpha}}$ and the Born potential α_{B}. In this appendix, we show that $\hat{\hat{\alpha}}$, derived from α_{B}, is a precise estimate of α for those acoustic media where the product of any three reflection coefficients is small compared to the reflection coefficients themselves. Further, the relationship between $\hat{\hat{\alpha}}$ and α_{B} can be used to estimate the acoustic velocity in the layered medium. The corresponding eikonal-derived equation (72) is not analysed in this paper but can be shown to have similar properties.

The relationship is analysed for a layered model by evaluating the accuracy between the n-layer potentials α_n and $\hat{\hat{\alpha}}_n$. The n-layer Born potential $\alpha_{n\mathrm{B}}$ derived in equation (47) is expressed as a function of the reflection coefficients R_1 to R_n. Therefore, $\hat{\hat{\alpha}}_n$ can be represented in terms of the same reflection coefficients. The n-layer actual potential α_n defined in equation (44), on the other hand, is given in terms of the velocity ratio c_0/c_n. For the purpose of comparison with $\hat{\hat{\alpha}}_n$, it needs to be expressed in terms of the reflection coefficients. Equation (A.1) yields

$$\frac{c_{n-1}}{c_n} = \frac{1-R_n}{1+R_n},$$

therefore the velocity ratio c_0/c_n can be written as

$$\frac{c_0}{c_n} = \frac{c_0}{c_1}\frac{c_1}{c_2}\cdots\frac{c_{n-1}}{c_n} = \prod_{j=1}^{n}\frac{1-R_j}{1+R_j}.$$

Hence, the n-layer actual potential α_n can be represented as

$$\alpha_n = 1 - \left(\prod_{j=1}^{n} \frac{1-R_j}{1+R_j}\right)^2. \tag{C.1}$$

Series expanding and subtracting α_n and $\hat{\hat{\alpha}}_n$ show that

$$|\alpha_n - \hat{\hat{\alpha}}_n| = \mathrm{O}_n[R^3], \tag{C.2}$$

where $\mathrm{O}_n[R^3]$ represents the product of any combination of reflection coefficients $R_1, R_2, \ldots, R_n$ to the power of 3. Therefore, the difference between the n-layer actual potential and its estimate from the Born potential is always proportional to the third power of any of the reflection coefficients of the n-layered medium. Thus, when reflection coefficients of the medium are 'small' in the sense that the product of any three of the reflection coefficients is vanishingly small compared to any of the reflection coefficients, it is a good approximation to estimate the squeezed potential from the Born potential by the algorithm

$$\hat{\hat{\alpha}}(z) = A_{\mathrm{W}}(z)\alpha_{\mathrm{B}}(z), \tag{C.3}$$

where the WKBJ correction amplitude A_{W} is given in equation (69).

For example, consider $n = 1$, in which case α_1 and $\hat{\hat{\alpha}}_1$ can be written in terms of reflection coefficients as

$$\alpha_1/4 = \frac{R_1}{1+2R_1+R_1^2} = R_1 - 2R_1^2 + 3R_1^3 - 4R_1^4 + \mathrm{O}\big[R_1^5\big]$$

and

$$\hat{\hat{\alpha}}_1/4 = R_1\big[\big(1+4R_1^2\big)^{\frac{1}{2}} - 2R_1\big] = R_1 - 2R_1^2 + 2R_1^3 + \mathrm{O}_1\big[R_1^5\big],$$

respectively. Subtraction of α_1 and $\hat{\hat{\alpha}}_1$ shows that

$$|\alpha_1 - \hat{\hat{\alpha}}_1|/4 = R_1^3 - 4R_1^4 + \mathrm{O}_1\big[R_1^5\big]. \tag{C.4}$$

Equation (C.4) implies that $\hat{\hat{\alpha}}_1$ is a good approximation to α_1 when R_1^3 is much smaller than R_1. Next, consider $n = 2$, in which case α_2 and $\hat{\hat{\alpha}}_2$ can be written in terms of reflection coefficients as

$$\alpha_2/4 = R_1 + R_2 - 4R_1R_2 - 2R_1^2 - 2R_2^2 + \mathrm{O}_2[R^3]$$

and

$$\hat{\hat{\alpha}}_2/4 = R_1 + R_2 - 4R_1R_2 - 2R_1^2 - 2R_2^2 + \mathrm{O}_2[R^3],$$

respectively. Subtraction of α_2 and $\hat{\hat{\alpha}}_2$ shows that

$$|\alpha_2 - \hat{\hat{\alpha}}_2|/4 = 2R_1R_2^2 + 3R_1^2R_2 + R_1^3 + R_2^3 + 5R_1R_2^3 + 4R_1^3R_2 + 7R_1^2R_2^2 + \mathrm{O}_2[R^5]. \tag{C.5}$$

Equation (C.5) implies that $\hat{\hat{\alpha}}_2$ is a good approximation to α_2 when the product of reflection coefficients R_1 and R_2 of the third power is negligible compared to the reflection coefficients R_1 and R_2. Likewise, the case $n = 3$ yields

$$\alpha_3/4 = R_1 + R_2 + R_3 - 4R_1R_2 - 4R_1R_3 - 4R_2R_3 - 2R_1^2 - 2R_2^2 - 2R_3^2 + \mathrm{O}_3[R^3]$$

and

$$\hat{\hat{\alpha}}_3/4 = R_1 + R_2 + R_3 - 4R_1R_2 - 4R_1R_3 - 4R_2R_3 - 2R_1^2 - 2R_2^2 - 2R_3^2 + \mathrm{O}_3[R^3].$$

Subtraction of the series expansions of α_3 and $\hat{\hat{\alpha}}_3$ gives

$$\begin{aligned}|\alpha_3 - \hat{\hat{\alpha}}_3|/4 = {} & R_1^3 + R_2^3 + R_3^3 + 2R_1R_2^2 + 2R_1R_3^2 + 2R_2R_3^2 + 3R_1^2R_2 + 3R_1^2R_3 \\ & + 3R_2^2R_3 + 4R_1R_2R_3 + \mathrm{O}_3[R^4].\end{aligned} \tag{C.6}$$

Equation (C.6) shows that $\hat{\hat{\alpha}}_3$ is a good approximation to α_3 when the product of any of the reflection coefficients R_1, R_2 and R_3 to the third power is small compared to the reflection coefficients themselves. For a general n, equation (C.2) can be shown to be valid.

C.1. Layer velocities from the estimated potential

From the computation of $\hat{\hat{\alpha}}_n$ the velocity of layer n can be estimated. Equation (44) leads to the algorithm

$$\hat{c}_n = \frac{c_0}{\sqrt{1-\hat{\hat{\alpha}}_n}}. \tag{C.7}$$

In the case that a continuous depth Born profile $\alpha_{\mathrm{B}}(z)$ is derived, the continuous velocity estimate reads

$$\hat{c}(z) = \frac{c_0}{\sqrt{1-\hat{\hat{\alpha}}(z)}}. \tag{C.8}$$

The only information that is required to estimate the layer velocities is the Born potential obtained from constant-velocity imaging.

C.2. Analytic example

Consider the ten-layer medium where layer velocities c_n, actual potentials α_n and reflection coefficients R_n for $n = 1, \ldots, 9$ are listed in table 1. Assume that the Born potentials $\alpha_{n\mathrm{B}}$ corresponding to each of the layers have been found through constant-velocity imaging. Then, compute $\hat{\hat{\alpha}}_n$ from the Born potentials $\alpha_{n\mathrm{B}}$ by use of equation (C.3). Inputting $\hat{\hat{\alpha}}_n$ into equation (C.7) yields the estimated layer velocities, $\hat{c}_{n\mathrm{W}}$. The results presented in table 1 show that the layer velocity estimates are quite precise, the errors being less than 3%. Further, it is observed that the size of the error correlates with the amplitude size of the Born potential, α_{B}. When the same exercise is repeated in the eikonal approximation, one obtains the velocity estimates $\hat{c}_{n\mathrm{E}}$ presented in the table.

References

Aki K and Richards P G 1980 *Quantitative Seismology* (San Francisco: Freeman)

Amundsen L 1994 The propagator matrix related to the Kirchhoff–Helmholtz integral in inverse wavefield extrapolation *Geophysics* **59** 1902–10

Amundsen L, Reitan A and Arntsen B 2005 Geometric analysis of data-driven inversion/depth imaging *J. Seism. Explor.* submitted

Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill)

Berkhout A J 1982 *Seismic Migration, Imaging of Acoustic Wave Energy by Wavefield Extrapolation, A: Theoretical Aspects* 2nd edn (Amsterdam: Elsevier)

Berryman J G and Greene R R 1980 Discrete inverse methods for elastic waves in layered media *Geophysics* **45** 213–33

Bleistein N 1984 *Mathematical Methods for Wave Phenomena* (New York: Academic)

Bransden B H and Joachain C 1989 *Introduction to Quantum Mechanics* (New York: Longman Scientific & Technical)

Bremmer H 1951 The WKB approximation as a first term of a geometric-optical series *The Theory of Electromagnetic Waves: A Symposium* (New York: Interscience) pp 169–79

Bruckstein A M, Levy B C and Kailath T 1985 Differential methods in inverse scattering *SIAM J. Appl. Math.* **45** 312–35

Bube P K and Burridge B 1983 The one-dimensional inverse problem of reflection seismology *SIAM Rev.* **25** 497–559

Burridge R 1980 The Gel'fand–Levitan, the Marchenko, and the Gopinath–Sondhi integral equations in inverse scattering theory, regarded in the context of inverse impulse response problems *Wave Motion* **2** 305–23

Carroll R and Santosa F 1981 Scattering techniques for a one-dimensional inverse problem in geophysics *Math. Methods Appl. Sci.* **3** 145–71

Claerbout J F 1968 Synthesis of a layered medium from its acoustic transmission response *Geophysics* **33** 264–9

Claerbout J F 1971 Toward a unified theory of reflector mapping *Geophysics* **36** 467–81

Clayton R W and Stolt R H 1981 A Born–WKBJ inversion method for acoustic reflection data *Geophysics* **46** 1559–67

Coen S 1981 On the elastic profiles of a layered medium from reflection data: I. Plane wave sources *J. Acoust. Soc. Am.* **70** 172–5

Fröman N and Fröman P O 1965 *JWKB Approximation. Contributions to the Theory* (Amsterdam: North-Holland)

Gjevik B, Nilsen A and Høyen J 1976 An attempt at the inversion of reflection data *Geophys. Prospect.* **24** 492–505

Gladwell G M L 1993 *Inverse Problems in Scattering: An Introduction* (Dordrecht: Kluwer)

Glauber R J 1959 High-energy collision theory *Lectures in Theoretical Physics, Boulder 1958* ed W E Brittin and L G Dunham (New York: Interscience) pp 315–414

Goupillaud P 1961 An approach to inverse filtering of near-surface layer effects from seismic records *Geophysics* **26** 754–60

Ikelle L T and Amundsen L 2005 *Introduction to Petroleum Seismology* (Tulsa, OK: Society of Exploration Geophysics)

Innanen K A 2003 Methods for the treatment of acoustic and absorptive/dispersive wave field measurements *PhD Thesis* University of British Columbia

Innanen K A and Weglein A B 2003 Simultaneous imaging and inversion with the inverse scattering series *Proc. 8th Int. Congr. of the SBGf and 5th Latin American Geophys. Conf.*

Koehler F and Taner M T 1977 Direct and inverse problems relating reflection coefficients and reflection response for horizontally layered media *Geophysics* **42** 1199–206

Liu F, Weglein A B, Innanen K A and Nita B G 2005 Extension of the non-linear depth imaging capability of the inverse scattering series to multidimensional media: strategies and numerical results *Proc. 9th Int. Cong. of the SBGf and 6th Latin American Geophys. Conf.*

Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)

Morse P M and Ingard K U 1968 *Theoretical Acoustics* (New York: McGraw-Hill)

Newton R 1981 Inversion of reflection data for layered media: a review of exact methods *Geophys. J. R. Astron. Soc.* **65** 191–215

Nilsen A and Gjevik B 1978 Inversion of reflection data *Geophys. Prospect.* **26** 421–32

Ramirez A C and Weglein A B 2005 Progressing the analysis of the phase and amplitude prediction properties of the inverse scattering internal multiple attenuation algorithm *J. Seism. Explor.* **13** 283–301

Razavy M 1975 Determination of the wave velocity in an inhomogeneous medium from the reflection coefficient *J. Acoust. Soc. Am.* **58** 956–63

Reitan A 1979 Relativistic Glauber amplitudes for elastic electron and positron scattering by hydrogen atoms and hydrogenlike ions in the ground state *Phys. Rev.* A **20** 1385–92

Robinson E A 1982 Migration of seismic data as WKB approximation *Geoexploration* **20** 7–30

Robinson E A 1986 Migration of seismic data by the WKBJ method *Proc. IEEE* **74** 428–39

Santosa F 1982 Numerical scheme for the inversion of acoustical impedance profile based on the Gelfand–Levitan method *Geophys. J. R. Astron. Soc.* **70** 229–43

Santosa F and Schwetlick H 1982 The inversion of acoustical impedance profiles by the method of characteristics *Wave Motion* **4** 99–110

Schiff L I 1955 *Quantum Mechanics* (London: McGraw-Hill)

Shaw S A 2005 An inverse scattering series algorithm for depth imaging of reflection data from a layered acoustic medium with an unknown velocity model *PhD Thesis* University of Houston

Shaw S A, Weglein A B, Foster D J, Matson K H and Keys R G 2004 Isolation of a leading order depth imaging series and analysis of its convergence properties for a 1D acoustic medium *J. Seism. Explor.* **13** 99–120

Stolt R H and Benson A K 1986 *Seismic Migration: Theory and Practice* (London: Geophysical Press)

Stolt R H and Weglein A B 1985 Migration and inversion of seismic data *Geophysics* **50** 2458–72

Symes W 1981 Stable solution of the inverse reflection problem for a smoothly stratified medium *SIAM J. Math. Anal.* **12** 421–53

Ursin B 1984 Seismic migration using the WKB approximation *Geophys. J. R. Astron. Soc.* **79** 339–52

Ursin B 1987 The plane-wave reflection and transmission response of a vertically inhomogeneous acoustic medium *Deconvolution and Inversion* ed M Bernabini *et al* (Oxford: Blackwell) pp 189–207

Ursin B and Berteussen K A 1986 Comparison of some inverse methods for wave propagation in layered media *Proc. IEEE* **74** 389–400

Ware J A and Aki K 1968 Continuous and discrete inverse-scattering problems in a stratified elastic medium: I. Plane waves at normal incidence *J. Acoust. Soc. Am.* **45** 911–21

Weglein A B 1985 The inverse scattering concept and its seismic application *Developments in Geophysical Exploration Methods* vol 6, ed A A Fitch (Amsterdam: Elsevier) pp 111–38

Weglein A B, Araújo R V, Carvalho P M, Stolt R H, Matson K H, Coates R T, Corrigan D, Foster D J, Shaw S A and Zhang H 2003 Inverse scattering series and seismic exploration *Inverse Problems* **19** R27–83

Weglein A B, Foster D J, Matson K H, Shaw S A, Carvalho P M and Corrigan D 2002 Predicting the correct spatial location of reflectors without knowing or determining the precise medium and wave velocity: initial concept, and analytic and numerical example *J. Seism. Explor.* **10** 367–82

Weglein A B, Gasparotto F A, Carvalho P M and Stolt R H 1997 An inverse scattering series method for attenuating multiples in seismic data *Geophysics* **62** 1975–89

Weglein A B, Matson K H, Foster D J, Carvalho P M, Corrigan D and Shaw S A 2000 Imaging and inversion at depth without a velocity model: theory, concepts, and initial evaluation *Expanded Abstracts, 70th Ann. Int. Mtg., Soc. Expl. Geophys.*

Zhang H and Weglein A B 2005 The inverse scattering series for tasks associated with primaries: depth imaging and direct non-linear inversion of 1D variable velocity and density acoustic media *Expanded Abstracts, 75th Ann. Int. Mtg., Soc. Expl. Geophys.*