Validity of the long-wave approximation in layered media

Børge Arntsen, Statoil*

Summary

Wave velocities in a periodically layered medium composed of two different layer types change with scale as the propagation of waves depends strongly on the ratio of the dominant wavelength to layer thickness. The present work attempts to establish a minimum ratio of wavelength to layer spacing for which the long-wave approximation is valid. Analysis of the dispersion relation reveals that the minimum ratio depends on the reflection coefficient and the ratio of travel times through the two layer types. It is found that, depending on the reflection coefficient, the minimum value of the ratio of the dominant wavelength to layer spacing can be as large as approximately 13 and as low as approximately 3.

Introduction

Reservoir characterization requires imaging of heterogeneities in sedimentary rocks such as layering, variation in lithology and pore fluid properties. In general this involves integration of measurements at many different scales. For example is correlation of seismic data with acoustic loggs crucial for geologic interpretation and involves relating measurements of wave velocities at a scale of tens of meters with measurements of velocities at a scale of centimeters.

Wave velocities in for example a periodically layered medium changes with scale as the propagation of waves depends strongly on the ratio of the dominant wavelength, λ_0, to layer thickness, d. When λ_0/d is large, the wave velocity is given by an average of the properties of the individual layers and waves behave as if propagating in an effective anisotropic homogeneous medium (Helbig 1984). On the other hand, when λ_0/d is small, waves can be described using rays with wave velocities that are larger than for the effective medium. For intermediate values of λ_0/d waves are in general dispersive and the velocity changes rapidly with frequency (Marion et al. 1994).

Several workers have tried to establish a minimum value of the dominant wavelength to layer spacing, λ_0/d, at which wave can be treated as if propagating in an anisotropic effective medium. Values ranging from 3 (Helbig 1984) to 5-8 (Carcione et al. (1991)) to 15 (Marion et al (1994)) have been found.

The minimum value of λ_0/d at which wave propagation in a periodically layered medium can be approximated with wave propagation in an anisotropic homogeneous medium is thus still unclear, and it seems worth while to find a relation between λ_0/d and the properties of the medium explaining the apparent different results reached by different authors. To avoid the complications of elastic wave propagation this work is limited to wave propagation normal to the layering of the medium. The medium can then be treated as purely acoustic. Although the anisotropy of the effective medium is an important effect, it will not be further discussed.

In the following section I review wave propagation in a two-component periodically layered medium. I then derive a relation between λ_0/d, the reflection coefficient of the periodic layered medium and the ratio of travel times through each of the two different layer types. In the section on numerical results the formula for λ_0/d is verified with numerical simulations and with laboratory measurements reported in the literature. The formula reveals that the minimum value of λ_0/d depends on two dimensionless numbers; the reflection coefficient r and the ratio of travel times τ_2/τ_1 through the two different constituents of the medium. An upper limit of the minimum value of λ_0/d is also found.

Theory

I consider a horizontally plane layered periodic medium composed by stacking together one layer characterized by wave velocity c_1, density ρ_1 and thickness d_1 with another layer characterized by wave velocity c_2, density ρ_2 and thickness d_2 repatedly. Considering only vertical incidence, the relation between the frequency dependent wave velocity C and the properties of the medium is expressed in the dispersion relation

$$\cos(\omega d/C) = \cos(\omega\tau_1 + \omega\tau_2) - \frac{r^2}{1-r^2}\sin(\omega\tau_1)\sin(\omega\tau_2). \qquad (1)$$

Here ω is the frequency, $d = d_1 + d_2$ is the thickness of one period of the medium and τ_1 and τ_2 are vertical traveltimes defined by $\tau_1 = d_1/c_1$ and $\tau_2 = d_2/c2$. r is the reflection coefficient given by by

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \qquad (2)$$

where the impedance Z_i is equal to by $Z_i = \rho_i c_i, i = 1, 2$.

For sufficiently large wavelength, $\lambda_0/d = 2\pi C_0/(\omega d)$, wave propagation through the periodic layered medium can be approximated with wave propagation through an effective, homogeneous medium with wave velocity C_0 equal to (the long-wave approximation)

$$\frac{1}{C_0^2} = \frac{1}{d^2}\left[(\tau_1+\tau_2)^2 + \left(\frac{4r^2}{1-r^2}\right)\tau_1\tau_2\right]. \qquad (3)$$

The objective is now to try to determine a value of λ_0/d above which the medium must be regarded as a truly layered medium and below which the medium can be regarded as a homogeneous medium with wave velocity given by the average defined by equation (3). This is done by introducing the error ϵ

$$\epsilon = \frac{C_0 - C(\omega)}{C_0}, \tag{4}$$

which measures the relative difference between the velocity C_0 in the long-wave approximation and the velocity $C(\omega)$ for a given finite frequency. By Taylor-expanding the dispersion relation given by equation (1) to fourth order it is possible to express λ_0/d as a function of ϵ

$$\lambda_0/d = \frac{\pi}{\sqrt{3}}\sqrt{\frac{(1-\epsilon)^{-4}-\beta}{(1-\epsilon)^{-2}-1}}. \tag{5}$$

Here the function β is given by

$$\beta = \frac{\left\{(1+\tau_2/\tau_1)^4 + \left(\frac{8r^2}{1-r^2}\right)\left[\tau_2/\tau_1 + (\tau_2/\tau_1)^3\right]\right\}}{\left[(1+\tau_2/\tau_1)^2 + \left(\frac{4r^2}{1-r^2}\right)(\tau_2/\tau_1)\right]^2}. \tag{6}$$

Equation (5) can now be used to define a value for λ_0/d at which the relative difference between the frequency dependent velocity C and the velocity in the long-wave approximation is less than a given limit equal to ϵ.

Since equation (5) is based on a fourth order Taylor-expansion it would be reasonable to question it's validity for small values of λ_0/d. Numerical checks show that equation (5) is a good approximation whenever $\lambda_0/d > 3-4$.

From equation (5) it is immediately seen that λ_0/d depends on the ratio of the vertical traveltimes τ_1 and τ_2 through the two layer types and the reflection coefficient r.

Numerical Results

As an application of the formula given by equation (5) the laboratory experiment described by Marion et al (1994) is considered. The model they used consisted of alternating layers of plastic and steel. The model given in table 1 defines a model similar to the one used by Marion et al (1994), only the thickness of the layers are slightly changed. Figure 1 shows $R = \lambda_0/d$ for

Thickness (m)	Velocity (m/s)	Density (Kg/m^3)
0.0005	2487	1210
0.00100	5535	7900

Table 1: Material properties of periodic layered plastic-steel medium

the model defined in table 1 computed as a function of $\epsilon = \frac{C_0 - C}{C_0}$ using equation (5). From figure 1 it is evident that the value of λ_0/d is relatively insensitive to the value of ϵ when ϵ is larger than 0.01-0.02. However, when ϵ is smaller than approximately 0.01, λ_0/d rapidly increases as ϵ decreases. In the limit when ϵ approaches zero, λ_0/d will approach infinity. This is entirely reasonable since C and C_0 is equal only for infinite wavelength (or zero frequency). It seems clear that equation (5) indicates that at a value of λ_0/d equal to approximately 11.0 ($\epsilon = 0.01$), the periodic layered medium could be treated as a homogeneous medium with velocity and density given by averages.

In order to verify equation (5) numerical simulations were performed using a finite-difference scheme employing optimized spatial differentiators and a second-order time integration method. The scheme is described by Holberg (1987). Figure 2 shows the result of a numerical simulation with the source peak frequency corresponding to $\lambda_0/d = 11.0$ plotted with a thin line. The solid line shows the result of a corresponding numerical simulation where the periodic layered medium was switched with a completely homogeneous medium with wave velocity given by the average defined in equation (3). Although there are differences between the two waveforms, it is a reasonably good verification of equation (5).

In figure 3 a semblance measure has been used to compare additional simulations for the model define in table 1 with simulations for the corresponding homogeneous model using a range of source peak frequencies corresponding to λ_0/d values from 4.0 to 20.0. Figure 3 shows that the semblance value at λ_0/d equal to 11.0 is slightly less than 1.0, becoming almost equal to 1.0 at λ_0/d approximately equal to 15. The main conclusion from the laboratory measurements made by Marion et. al. (1994) was that the transition from short- to long wavelength behavior occurs between $\lambda_0/d = 8$ and $\lambda_0/d = 15$ This coincides reasonably well with the prediction using equation (5).

Figure 4 shows the result of using equation (5) to predict the minimum value of λ_0/d required to make wave propagation through a periodic layered medium behave approximately as wave propagation through a homogeneous medium as a function of volume fraction of steel.

The volume fraction of steel is simply defined by $d2/d$. Here the reflection coefficient has been kept constant and $\epsilon = 0.01$. The input to the calculation is the model defined in table 1 with different relative layer thicknesses. From figure 4 it is seen that the minimum value of λ_0/d does not depend strongly on the the volume fraction of plastic, except for the extreme values of either very small or very large fraction of plastic. The peak value of the minimum value of λ_0/d seems to be approximately located at a volume fraction of 0.7. The conclusion from the laboratory measurements of Marion et al confirms this result, as they found that the minimum value of λ_0/d was almost independent of the volume fraction of plastic, except for the extreme values. They also found that the minimum value of λ_0/d was slightly larger for the midrange of compositions.

Discussion and conclusions

From the numerical results in the preceding section it should be clear that the minimum value of λ_0/d required to make wave propagation through a periodic layered medium behave approximately as wave propagation through an equivalent homogeneous medium is dependent upon the reflection coefficient r and the ratio of the travel times τ_1 and τ_2. The minimum value of λ_0/d is to a good approximation given by equation (5).

Equation (5) can be used to explain the seemingly different results obtained previously for example by Helbig (1984), who found that the minimum value of λ_0/d was equal to 3 and by Carcione et al (1991), who obtained a minimum value of λ_0/d between 5 and 8. Using equation (5) their results can be predicted and the difference explained by the difference in reflection coefficient and the ratio of τ_2 to τ_1.

Inspection of equation (5) reveals that the dimensionless function β approaches 0 whenever the reflection coefficient r becomes close to 1. This implies that λ_0/d approaches

$$\lambda_0/d = \frac{\pi}{\sqrt{6}}\sqrt{\frac{1}{\epsilon}}. \qquad (7)$$

The limit given by equation (7) corresponds to $\lambda_0/d \approx 13$ for $\epsilon = 0.01$.

I have derived an analytical expression for the ratio of the dominant wavelength to the layer spacing for a periodically layered medium. The analysis of this expression leads to the following conclusions:

1. The minimum ratio of wavelength to layer spacing(λ_0/d) required to approximate wave propagation in a periodic layered medium with wave propagation in a corresponding anisotropic homogeneous medium depends on the reflection coefficient (r) and the ratio of travel times (τ_2/τ_1) through each of the two layers. The approximate relation between λ_0/d, r and τ_2/τ_1 is given by equation (5). This is verified by comparison with laboratory measurements and numerical simulations.

2. The minimum value of λ_0/d increases with increasing reflection coefficient and reaches a maximum value for large reflection coefficients at approximately 13. Although a lower limit for the minimum value of λ_0/d has not been found, values as low as 3 has been demonstrated to exist for media with small reflection coefficients.

Acknowledgements

This work was completed while the author was visiting Osservatorio Geofisico Sperimentale in Trieste, Italy. I am greatfull for the hospitality and support given by José Carcione and Geza Seriani.

References

Carcione, J.M., Kosloff, D., and Behle, A., 1991, Long-wave anisotropy in stratified media: A numerical test:: Geophysics, **56**, 245–254.

Helbig, K., 1984, Anisotropy and dispersion in periodically layered media:: Geophysics, **49**, 364–373.

Marion, D., Mukerij, T., and Mavko, G., 1994, Scale effects on velocity dispersion: From ray to effective medium theories in stratified media:: Geophysics, **59**, 1613–1619.

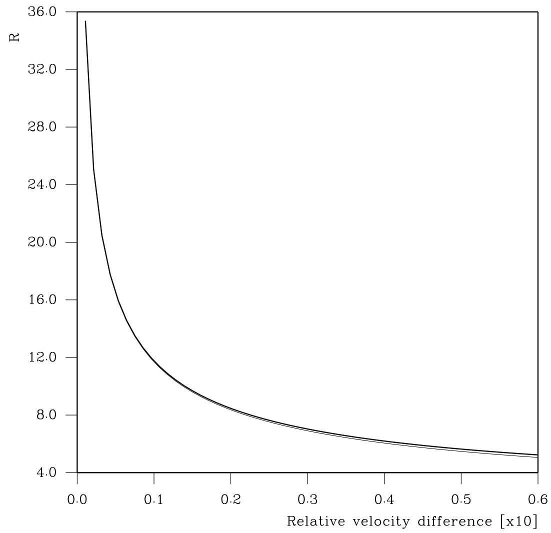

Fig. 1: $R = \lambda_0/d$ as a function of the relative velocity error $\epsilon = (C_0 - C)/C_0$ for the model defined table 1 using equation (5) (thick line). The thin line shows ϵ computed by using the exact phase velocity given by equation (1).

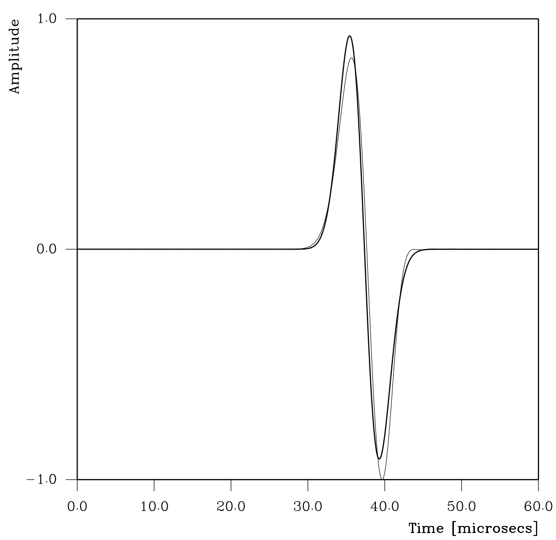

Fig. 2: Numerical simulation of the periodic la yered medium defined in table 1 (thin line) and numerical simulation of a corresponding homogeneous medium (solid line). The dominating wavelength of the source pulse corresponds to $\lambda_0/d = 11.0$

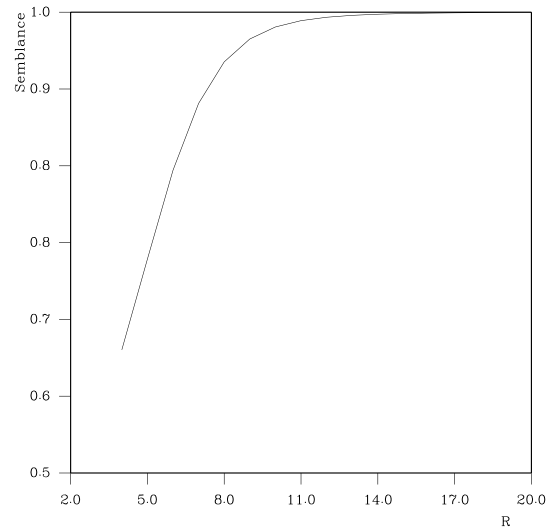

Fig. 3: Sem blance as a function of $R = \lambda_0/d$ for the model defined in table 1

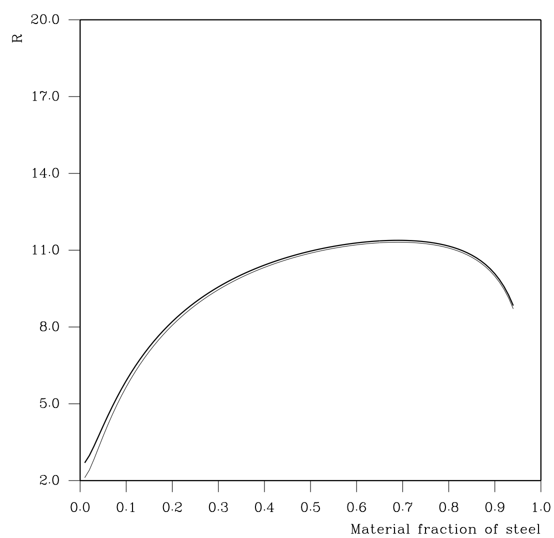

Fig. 4: $R = \lambda_0/d$ as a function of v olume fraction of steel at a relative velocity difference of $\epsilon = (C_0 - C)/C_0 = 0.01$ for the model defined in table 1 using equation (5) (thick line). The thin line is computed using the exact phase velocity given by equation (1).