Wave equation depth migration - a new method of solution

Lasse Amundsen, Børge Arntsen, Arne Reitan, Eirik Ø Dischler, and Bjørn Ursin*
StatoilHydro Reseach Center, and NTNU

SUMMARY

We present a wave propagation method rigorous in one-way and two-way wave theory for complex velocity varying media with new solutions. In the horizontal wavenumber domain, the first-order differential system that governs acoustic wave propagation can be written in terms of field vectors that are coupled in the wavenumber variables through convolutions between the medium and the fields. The differential system can be uncoupled by introducing a reference system with reference velocity equal to the reciprocal of the rms slowness. The uncoupled system of equations has propagator solutions that are coupled in the wavenumber variables. These solutions can be decoupled by introducing simple approximations.

This scheme can be exploited for wave equation depth migration. It then is convenient to introduce new field variables that relate to upgoing and downgoing waves in the reference medium. One-way and various two-way wave equations for the laterally varying medium then can be derived by introducing the down-up wave interaction (DUWI) model. The differential equation for the downgoing (incident) field is derived in the zero-order DUWI model, which neglects the interactions with the upgoing field, resulting in a pure one-way wave equation for the downgoing field. Similarly, the zero-order DUWI model yields a one-way wave equation for the upgoing field. In the first-order DUWI model, the downgoing field from the zero-order DUWI model is used as a source for the upgoing field. This solution gives a quasi two-way wave equation which may be used to migrate overturning waves.

Noteworthy, the differential equations we derive have analytical solutions for migration in the wavenumber domain. Simple approximations lead to numerically fast migration schemes that can be implemented in a manner like the split-step Fourier migration schemes.

INTRODUCTION

Traditionally, seismic migration is based on one-way or two-way wave equations. One-way wave equations in their simplest form are derived by transforming various approximations to a specified dispersion relation to a partial differential equation in the space domain (Claerbout, 1985). Such equations face limitations in the presence of complex, steeply dipping reflectors. Another constraint is the decoupling of the upgoing primaries from the downgoing source field. Two-way wave equations are sometimes used to avoid the steep-dip limitation of one-way equations. However, full two-way equations can not handle upgoing and downgoing fields in a controlled coupling manner, they are expensive in terms of computing cost and storage, and they are very sensitive to velocity errors.

In addition to finite-difference migration, classic migration techniques are Stolt migration, phase-shift migration, Kirchhoff migration and split-step Fourier migration (Secrest, 1975; Stolt, 1978; Gazdag, 1978; Schneider, 1978; Stoffa *et al.*, 1990). An under-explored mathematical model for seismic migration is the WKBJ approximation model discussed by Bremmer (1951), Robinson (1982, 1986) and Ursin (1984, 1987). The main reason it was not much explored during the 1980-90's is that the model was presented under the assumption of a vertically layered, laterally homogeneous earth, thereby leading to a slight generalization of the phase-shift migration method.

Recently, Zhang *et al.* (2005, 2006) have published one-way wave equations with WKBJ type solutions by modifying the above referenced model to heterogeneous media. Amundsen *et al.* (2006, 2008) have published papers using the model with zero-order and first-order interactions between downgoing and upgoing acoustic or elastic fields to derive solutions to the inverse scattering problem for layered media.

It is noted that a similar model was used by Glauber (1959) in the study of elementary particle scattering on nuclei, and also for electron scattering on atoms.

In this paper we first use that the first-order differential equations for the pressure field P and the vertical component of particle velocity V_z have a simple solution when the acoustic velocity is expressed in terms of a reference velocity being the reciprocal of the rms slowness and a perturbation (the difference between the full velocity and the reference velocity). A similar model was introduced by Pai (1988). Next, we introduce field variables that relate to upgoing and downgoing waves. The full differential equations for the upgoing and downgoing waves are the start for deriving one-way and various two-way wave equations for the upgoing and downgoing components by the use of the down-up wave interaction (DUWI) model. The DUWI model generalizes the Bremmer-Robinson-Ursin model referenced above to treat arbitrary complex velocity media, but importantly, does not invoke any WKBJ type solutions. The solutions are exact.

THE DIFFERENTIAL SYSTEM

Let ω denote circular frequency and (x,y,z) the Cartesian coordinates. The depth axis is positive downwards. The acoustic medium is described by the varying velocity $c = c(x,y,z)$ and the depth dependent density $\rho_0(z)$.

First-order differential equations for P and V_z

The first-order wave equation for pressure P and vertical component of particle velocity V_z can be written

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}z}(x,y,z,\omega) = \mathbf{A}(x,y,z,\omega)\mathbf{b}(x,y,z,\omega) \tag{1}$$

with field vector $\mathbf{b} = (P, V_z)^T$ and system matrix

$$\mathbf{A} = \begin{bmatrix} 0 & \mathrm{i}\omega\rho_0 \\ -(\mathrm{i}\omega\rho_0)^{-1}(\omega^2/c^2 + \partial_x^2 + \partial_y^2) & 0 \end{bmatrix} \quad (2)$$

We introduce the depth-dependent reference velocity $c_0(z)$ and split the system matrix in two parts: the reference model system matrix

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \mathrm{i}\omega\rho_0 \\ -(\mathrm{i}\omega\rho_0)^{-1}(\omega^2/c_0^2 + \partial_x^2 + \partial_y^2) & 0 \end{bmatrix} \quad (3)$$

and the perturbation matrix

$$\mathbf{A}_1 = \mathbf{A} - \mathbf{A}_0 = \begin{bmatrix} 0 & 0 \\ -(\mathrm{i}\omega\rho_0)^{-1}T & 0 \end{bmatrix} \quad (4)$$

The function T is a measure of the lateral velocity inhomogeneity at depth z relative to the reference velocity. It is given as

$$T(x,y,z,\omega) = -K_0^2(z)\alpha(x,y,z) \quad (5)$$

where $K_0 = \omega/c_0$ is the reference wavenumber, and

$$\alpha(x,y,z) = 1 - \left(\frac{c_0(z)}{c(x,y,z)}\right)^2 \quad (6)$$

is the velocity potential.

A Fourier transform over the horizontal coordinates yields the wavenumber domain equation

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}z}(k_x,k_y,z,\omega) = \mathbf{A}(k_x,k_y,z,\omega)\mathbf{b}(k_x,k_y,z,\omega) \quad (7)$$

with system matrix $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$, where

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \mathrm{i}\omega\rho_0 \\ -(\mathrm{i}\omega\rho_0)^{-1}k_z^2 & 0 \end{bmatrix} \quad (8)$$

and

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ -(\mathrm{i}\omega\rho_0)^{-1}T* & 0 \end{bmatrix} \quad (9)$$

The squared vertical wavenumber is defined as

$$k_z^2(k_x,k_y,z,\omega) = \omega^2/c_0^2(z) - k_x^2 - k_y^2 \quad (10)$$

Observe that $T = T(k_x,k_y,z)$ and that $*$ denotes 2D convolution,

$$f_1(k_x,k_y) * f_2(k_x,k_y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}k_x'\mathrm{d}k_y' f_1(k_x-k_x',k_y-k_y') f_2(k_x',k_y')$$

Equation (7) is a first-order differential equation for the field vector $\mathbf{b}$ with wavenumber variables k_x and k_y. Due to the convolution term appearing in the perturbation matrix $\mathbf{A}_1$, the field vectors are *coupled* in the wavenumber variables.

By choosing the reference velocity c_0 as the reciprocal of the rms slowness (cfr. Pai, 1988), that is,

$$1/c_0^2(z) = [1/c^2](k_x=0,k_y=0,z) = \int\int \mathrm{d}x\mathrm{d}y 1/c^2(x,y,z) \quad (11)$$

then $T(k_x=0,k_y=0,z)=0$, and we observe that the term $\mathbf{A}_1\mathbf{b}$ is independent of the wavenumber variables k_x and k_y (but depends on wavenumbers $k_x' \neq k_x$ and $k_y' \neq k_y$). Thus, equation (7) can be written as the *uncoupled* differential system

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}z}(k_x,k_y,z,\omega) = \mathbf{A}_0(k_x,k_y,z,\omega)\mathbf{b}(k_x,k_y,z,\omega) + [\mathbf{A}_1\mathbf{b}](z,\omega) \quad (12)$$

Its solution in terms of propagators thus is simple, but not investigated further here. Instead we go on to derive uncoupled first-order differential equations for a set of new field variables that relate to upgoing and downgoing waves in the reference medium. Since the differential equations are derived from the system (12) they too will have simple solutions. The solutions are *coupled*, but for numerical calculations simple approximations lead to an *uncoupling* that is precise for migration.

First-order differential equations for U and D

To derive one-way and two-way wave equations for migration, we introduce new variables $\mathbf{w} = (U,D)^T$ so that $\mathbf{b} = \mathbf{L}_0\mathbf{w}$. By choosing $\mathbf{L}_0$ to be the eigenvector matrix of the system matrix $\mathbf{A}_0$ related to the reference medium, equation (12) becomes

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}z} = \Lambda_0\mathbf{w} + \mathbf{L}_0^{-1}\frac{\mathrm{d}\mathbf{L}_0}{\mathrm{d}z}\mathbf{w} + \mathbf{L}_0^{-1}\mathbf{A}_1\mathbf{L}_0\mathbf{w} \quad (13)$$

with eigenvalue matrix $\Lambda_0 = \mathbf{L}_0^{-1}\mathbf{A}_0\mathbf{L}_0 = \mathrm{diag}(-\mathrm{i}k_z, \mathrm{i}k_z)$. Eigenvectors are not unique, and both amplitude and flux normalization are common choices. With amplitude normalization, the general differential equations for U and D are

$$\frac{\mathrm{d}U}{\mathrm{d}z} = -\mathrm{i}k_zU - s[D-U] - gT*[U+D] \quad (14)$$

and

$$\frac{\mathrm{d}D}{\mathrm{d}z} = \mathrm{i}k_zD + s[D-U] + gT*[U+D] \quad (15)$$

with scattering function

$$s = -\frac{1}{2}\frac{1}{k_z}\frac{\mathrm{d}k_z}{\mathrm{d}z}$$

and Green's function radiation pattern

$$g = \frac{\mathrm{i}}{2k_z}$$

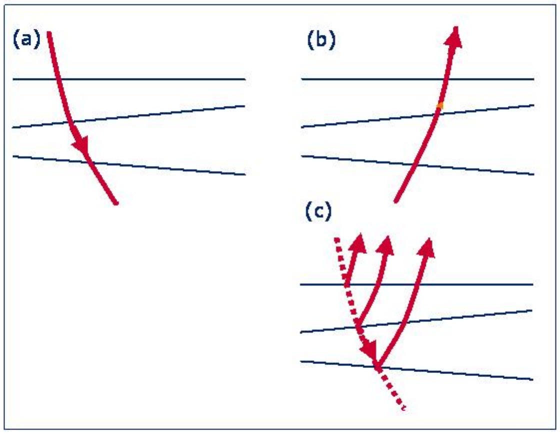

Figure 1: (a) and (b): The zero-order DUWI models for the downgoing and upgoing fields, respectively. (c) In the first-order DUWI model for the upgoing wave, the influence of the downgoing wave on the upgoing wave is included.

THE DUWI MODEL

Now, we dust the Bremmer-Robinson-Ursin model and extend it to treat laterally varying velocity media, however, without invoking the WKBJ approximation. This new mathematical model for migration is called the down-up wave interaction (DUWI) model.

The zero-order DUWI model neglects the interaction with the field traveling in the opposite direction, resulting in a pure one-way wave equations. In the first-order DUWI model, the downgoing field from the zero-order DUWI model is used as a source for the upgoing field. This solution gives a quasi two-way wave equation.

The differential equation for the downgoing wavefield D_0 in the zero-order DUWI model follows from equation (15) as

$$\frac{\mathrm{d}D_0}{\mathrm{d}z}(k_x,k_y,z) - [\mathrm{i}k_z(k_x,k_y,z)+s(k_x,k_y,z)]D_0(k_x,k_y,z) = g(k_x,k_y,z)T(k_x,k_y,z)*D_0(k_x,k_y,z) \quad (16)$$

It describes the one-way propagation of the incident field through the medium, as depicted in Figure 1a. The source term on the right side of equation (16) has the effect of coupling plane waves due to the velocity inhomogeneities contained in the function T, thus modifying the plane wave solution in the rms slowness background medium described by the vertical wavenumber.

The differential equation for the scattered, upgoing field U_0 in the zero-order DUWI model (cfr Figure 1b) follows from equation (14) as

$$\frac{\mathrm{d}U_0}{\mathrm{d}z}(k_x,k_y,z) + [\mathrm{i}k_z(k_x,k_y,z)-s(k_x,k_y,z)]U_0(k_x,k_y,z) = -g(k_x,k_y,z)T(k_x,k_y,z)*U_0(k_x,k_y,z) \quad (17)$$

Equations (16) and (17) are one-way wave equations in laterally varying media, and they are accurate for steep dips.

The first-order DUWI model for the scattered field is described by the differential equation

$$\begin{aligned}\frac{\mathrm{d}U_1}{\mathrm{d}z}(k_x,k_y,z) + [\mathrm{i}k_z(k_x,k_y,z)-s(k_x,k_y,z)]U_1(k_x,k_y,z) &= -g(k_x,k_y,z)T(k_x,k_y,z)*U_1(k_x,k_y,z)\\ &-[s(k_x,k_y,z)+g(k_x,k_y,z)T(k_x,k_y,z)*]D_0(k_x,k_y,z) \quad (18)\end{aligned}$$

which follows from equation (14). In this model, the downgoing field from the zero-order DUWI model acts as a source for the upgoing field. This quasi two-way wave equation can be used to image turning waves, which travel first downward and then curve upwards.

In seismic migration, the wavefield is extrapolated from z to $z+\Delta z$, where Δz is assumed to be small. To simplify our analysis, we assume in this small depth interval that the rms slowness is vertically constant so that the vertical wavenumber k_z is constant. Then, $s=0$. At $z+\Delta z$ transmission effects can be accounted for. In the wavenumber domain, the exact solutions for the fields at $z+\Delta z$ are

$$\begin{aligned} D_0 &= \Phi_0^{(D)} + aT*D_0\\ U_0 &= \Phi_0^{(U)} + aT*U_0\\ U_1 &= \Phi_1^{(U)} + aT*U_1 \quad (19)\end{aligned}$$

where $a=g\Delta z/2$, and $\Phi_0^{(D)},\Phi_0^{(U)},\Phi_1^{(U)}$ are known fields. For instance,

$$\begin{aligned}\Phi_0^{(D)}(k_x,k_y,z+\Delta z) &= \exp(\mathrm{i}k_z\Delta z)\\ &[D_0(k_x,k_y,z)+a(k_x,k_y)T(k_x,k_y,z)*D_0(k_x,k_y,z)] \quad (20)\end{aligned}$$

The solutions (19) show that wavefield extrapolation for a given wavenumber is accomplished by using the whole spectrum of horizontal wavenumbers simultaneously, which is necessary to respect the lateral variations in velocity at every depth step. The solutions (19) can be approximated so that they can be fast numerically solved.

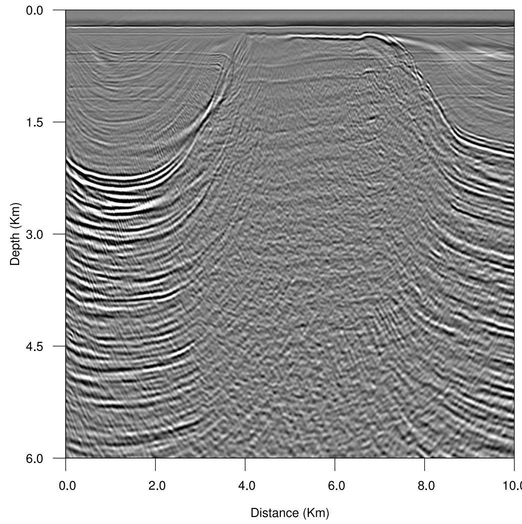

Figure 2: Real data from a salt basin imaged with an approximation to DUWI migration.

The final step of migration is imaging, where the downgoing and upgoing fields at depth are used to estimate the earth's reflectivity.

The method we have presented is valid for acoustic fields in velocity-inhomogeneous media. The density may vary with depth. It is possible to extend the differential equations and corresponding solutions to depth extrapolate fields propagating in elastic or electromagnetic isotropic/transverse isotropic media (Amundsen and Reitan, patent application, 2008). The elastic DUWI model then yields differential equations for P-P, P-SV, SV-P, SV-SV, and SH-SH migration. We expect that, say, the P-P and P-SV differential equations and associated solutions can be used to properly depth extrapolate ocean-bottom seismic data. Finally, we remark that higher-order interactions can be introduced to treat multiples during the migration.

NUMERICAL EXAMPLE

One of several approximations to the solutions (19) is to neglect the wavenumber dependence of the radiation characteristics of the Green's radiation, so that $g = g_0$, $a = a_0 = \mathrm{i}\Delta z/(4K_0)$. The solution for the downgoing field in the space domain then becomes

$$D_0(x,y,z+\Delta z) = \Phi_0^{(D)}(x,y,z+\Delta z)/[1-a_0T(x,y,z)] \qquad (21)$$

where $\Phi_0^{(D)}$ is calculated in the wavenumber domain according to equation (20) and inverse Fourier transformed to the space domain. The solution for the upgoing field is similar, when $\mathrm{i} \to -\mathrm{i}$.

In the case that the medium is "smooth" so that the its derivatives can be neglected, the DUWI migration bears a close relationship to split-step Fourier migration.

A migration algorithm based on the simple approximation $a = a_0$ has been applied to a dataset from the Nordkapp Basin, located in the Barents Sea. The basin is an exploration area with complex geology. It contains several salt diapirs with shallow crests immediately below the seabed, which make imaging of seismic data very difficult. Especially deeper parts of the salt flanks below the Base Cretaceous and also the base of the salt are badly imaged or not imaged at all. We have selected data from a 2D survey which exhibits the base salt imaging problem described in Haugen *et al.* (submitted to SEG 2008). The image from the DUWI migration based on the approximation (21) is shown in Figure 2.

CONCLUSIONS

We have proposed a new mathematical model for wave-equation depth migration that gives rigorous one-way and two-way differential equations for the downgoing and upgoing wave-fields. The differential equations are uncoupled in the wavenumber variables due to the specific choice of reference medium. The differential equations have exact coupled solutions that can be uncoupled by well-defined approximations leading to numerically fast migration schemes.

A similar procedure can be defined for elastic and electromagnetic differential systems.

ACKNOWLEDGEMENTS

We thank StatoilHydro for allowing us to present DUWI migration.

EDITED REFERENCES

Note: This reference list is a copy-edited version of the reference list submitted by the author. Reference lists for the 2008 SEG Technical Program Expanded Abstracts have been copy edited so that references provided with the online metadata for each paper will achieve a high degree of linking to cited sources that appear on the Web.

REFERENCES

Amundsen, L., A. Reitan, B. Arntsen, and B. Ursin, 2006, Acoustic nonlinear amplitude versus angle inversion and data-driven depth imaging in stratified media derived from inverse scattering approximations: Inverse Problems, **22**, 1921–1945.

———2008, Elastic nonlinear amplitude versus angle inversion and data-driven depth imaging in stratified media derived from inverse scattering approximations: Inverse Problems, **24**, in print.

Bremmer, H., 1951, The WKB approximation as a first term of a geometric-optical series, *in* The Theory of Electromagnetic Waves: A Symposium, 169–179.

Claerbout, J. F., 1985, Imaging the Earth's Interior: Blackwell Scientific.

Gazdag, J., 1978, Wave-equation migration by phase shift: Geophysics, **43**, 1342–1351.

Glauber, R. J., 1959, High-energy collision theory, *in* W. E. Brittin and L. G. Dunham, eds., Lectures in theoretical physics, 315–414.

Ikelle, L. T., and L. Amundsen, 2005, Introduction to petroleum seismology: SEG.

Pai, D. M., 1988, Generalized *f-k* (frequency-wavenumber) migration in arbitrarily varying media: Geophysics, **53**, 1547–1555.

Robinson, E. A., 1982, Migration of seismic data as WKB approximation: Geoexploration, **20**, 7–30.

———1986, Migration of seismic data by the WKBJ method: Proceedings of the IEEE, **74**, 428–439.

Schneider, W. A., 1978, Integral formulation for migration in two and three dimensions: Geophysics, **43**, 49–76.

Secrest, B. G., 1975, Migration techniques: Texas Instruments TR 08-75-35.

Stoffa, P. L., J. T. Fokkema, R. M. de Luna Freire, and W. P. Kessinger, 1990, Split-step Fourier migration: Geophysics, **55**, 410–421.

Stolt, R. H., 1978, Migration by Fourier transform: Geophysics, **43**, 23–48.

Ursin, B., 1984, Seismic migration using the WKB approximation: Geophysical Journal of the Royal Astronomy Society, **79**, 339–352.

———1987, The plane-wave reflection and transmission response of a vertically inhomogeneous acoustic medium, *in* M. Bernabini et al., Deconvolution and inversion, 189–207.

Zhang, Y., X. Sheng, and G. Zhang, 2006, Imaging complex salt bodies with turning-way one-way wave equation: 76th Annual International Meeting, SEG, Expanded Abstracts, 2323–2326.

Zhang, Y., G. Zhang, and N. Bleistein, 2005, Theory of true-amplitude one-way wave equations and true amplitude common-shot migration: Geophysics, 70, E1–E10.