Sensitivity Analysis and Uncertainty in EFWI Using the Hessian Matrix

ROSE meeting, 24th April 2018

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Outline

- Brief introduction to (Elastic) Full Waveform Inversion
- Present the Frechét derivative
- Adjoint theory for calculating the Hessian
- Calculate the Hessian
  - Gradient model
  - Gullfaks model

Motivation

- Measure the resolution of FWI
- Quantify parameter cross-talk
- Investigate possibility of a Newton solver

Elastic wave propagation

Elastic waves can be described by the wave-operator $\mathbf{L}(\mathbf{u},\mathbf{m})$:

$$\mathbf{L}(\mathbf{u},\mathbf{m}) = \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) - \nabla\boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t), \tag{1}$$

- Space-coordinates $\mathbf{x} \in G \subset \mathbb{R}^3$
- Time $t \in [0,T] \subset \mathbb{R}$
- Displacement field $\mathbf{u}(\mathbf{x},t)$
- Driving force $\mathbf{f}(\mathbf{x},t)$
- Model $\mathbf{m}(\mathbf{x},t)$
- Density ρ
- Stress tensor $\boldsymbol{\sigma}(\mathbf{x},t)$
- Stiffness tensor $\sigma_{ij} = C_{ijkl}\partial_k u_l$

Full Waveform inversion[1]

1. Guess a starting model (**m**) based on other information.

[1] Tarantola 1984; Mora 1987.

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3. Compare the synthetic recording to the real.

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Full Waveform inversion[1]

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5. Apply the model update.

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Full Waveform inversion[1]

1. Guess a starting model (**m**) based on other information.
2. Do a synthetic run.
3. Compare the synthetic recording to the real.
4. Use this information to calculate a gradient update.
5. Apply the model update.
6. Repeat.

[1] Tarantola 1984; Mora 1987.

Misfit – How good is our model?

A measure of how our model performs compared to a reference model, the misfit function

$$\Psi = \Psi(\mathbf{u}(\mathbf{m}, \mathbf{x}_r), \mathbf{d}_0) \tag{2}$$

- Receiver location $\mathbf{x}_r$
- Reference recording $\mathbf{d}_0$

Misfit

Introduce the Jacobian $\mathbf{J}$ as

$$\nabla_m \Psi(\mathbf{m} + \delta\mathbf{m}) = \mathbf{J}(\mathbf{m} + \delta\mathbf{m}) \tag{3}$$

Linearise around $\mathbf{m}$ resulting in

$$\mathbf{J}(\mathbf{m} + \delta\mathbf{m}) \simeq \mathbf{J}(\mathbf{m}) + \nabla_m \mathbf{J}(\mathbf{m})\delta\mathbf{m} = 0. \tag{4}$$

Thus introducing the Hessian

$$\mathbb{H}(\mathbf{m}) = \nabla_m \mathbf{J}(\mathbf{m}) = \nabla_m \nabla_m \Psi(\mathbf{m}). \tag{5}$$

Iterative methods[2]

- The model update $\delta \mathbf{m}$ can then be obtained by solving

$$\mathbb{H}(\mathbf{m})\delta \mathbf{m} = -\mathbf{J}(\mathbf{m}) \tag{6}$$

- Iff **H** is invertible we can “simply” solve

$$\delta \mathbf{m} = -\mathbb{H}^{-1}\mathbf{J}.$$

[2] Virieux and Operto 2009; Métivier et al. 2012; Epanomeritakis et al. 2008.

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- A common approximation is

$$\delta \mathbf{m} \simeq \alpha \mathbf{J},$$

and a line search for the optimal $\alpha \in \mathbf{R}$.

[2] Virieux and Operto 2009; Métivier et al. 2012; Epanomeritakis et al. 2008.

Jacobian

By backward propagating the misfit kernels we can obtain the adjoint field $\mathbf{u}^{\dagger}$ which we can use to calculate the Jacobian using the Frechét derivative

$$\mathbf{J} = \mathbf{J}(\mathbf{u}^{\dagger}, \mathbf{u}) = \int_{T} \mathbf{u}^{\dagger} \nabla_{m} \mathbf{L}(\mathbf{u}, \mathbf{m})\, \mathrm{d}t, \tag{7}$$

which boils down to cross-correlating the adjoint and forward fields.

Perturbed wavefields

Perturbed forward field

$$\delta \mathbf{u} = \lim_{\nu \to 0} \frac{1}{\nu}\Big[\mathbf{u}(\mathbf{m} + \nu\delta\mathbf{m}) - \mathbf{u}(\mathbf{m})\Big], \tag{8}$$

Perturbed adjoint field

$$\delta \mathbf{u}^{\dagger} = \lim_{\nu \to 0} \frac{1}{\nu}\Big[\mathbf{u}^{\dagger}(\mathbf{m} + \nu\delta\mathbf{m}) - \mathbf{u}^{\dagger}(\mathbf{m})\Big], \tag{9}$$

Hessian[3]

The Hessian acting on a model perturbation $\delta\mathbf{m}$ can be split up into three components

$$\mathbb{H}\delta\mathbf{m} = \mathbf{H}_1(\mathbf{u}^\dagger, \delta\mathbf{u}) + \mathbf{H}_2(\delta\mathbf{u}^\dagger, \mathbf{u}) + \mathbf{H}_3(\mathbf{u}^\dagger, \mathbf{u}). \tag{10}$$

These components can be written out as

$$\mathbf{H}_1(\mathbf{u}^\dagger, \delta\mathbf{u}) = \int_T \mathbf{u}^\dagger \nabla_m \mathbf{L}(\delta\mathbf{u}, \mathbf{m})\, \mathrm{d}t, \tag{11}$$

$$\mathbf{H}_2(\delta\mathbf{u}^\dagger, \mathbf{u}) = \int_T \delta\mathbf{u}^\dagger \nabla_m \mathbf{L}(\mathbf{u}, \mathbf{m})\, \mathrm{d}t, \tag{12}$$

$$\mathbf{H}_3(\mathbf{u}^\dagger, \mathbf{u}) = \int_T \mathbf{u}^\dagger \nabla_m \nabla_m \mathbf{L}(\mathbf{u}, \mathbf{m})(\delta\mathbf{m})\, \mathrm{d}t. \tag{13}$$

[3] Fichtner and Trampert 2011.

Hessian

The first two terms $\mathbf{H}_1$ and $\mathbf{H}_2$ can be calculated in a similar way as the Jacobian by replacing the relevant fields.
The last term $\mathbf{H}_3$ can be calculated by recycling the Jacobian calculations as

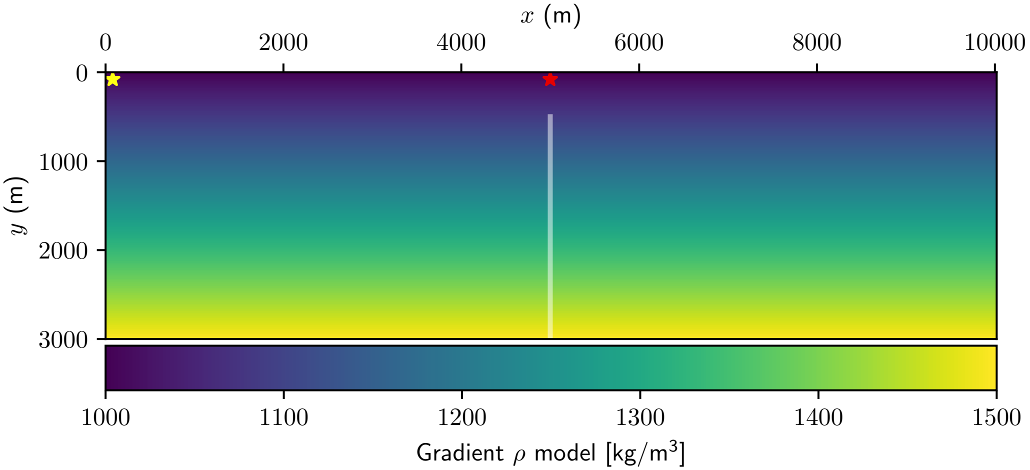
$$\mathbf{H}_3 = \begin{bmatrix} 0 & \rho^{-1} J_{v_p} & \rho^{-1} J_{v_s} \\ \rho^{-1} J_{v_p} & v_p^{-1} J_{v_p} & 0 \\ \rho^{-1} J_{v_s} & 0 & v_s^{-1} J_{v_s} \end{bmatrix} \begin{bmatrix} \delta\rho \\ \delta v_p \\ \delta v_s \end{bmatrix} \tag{14}$$

Construcing the Hessian

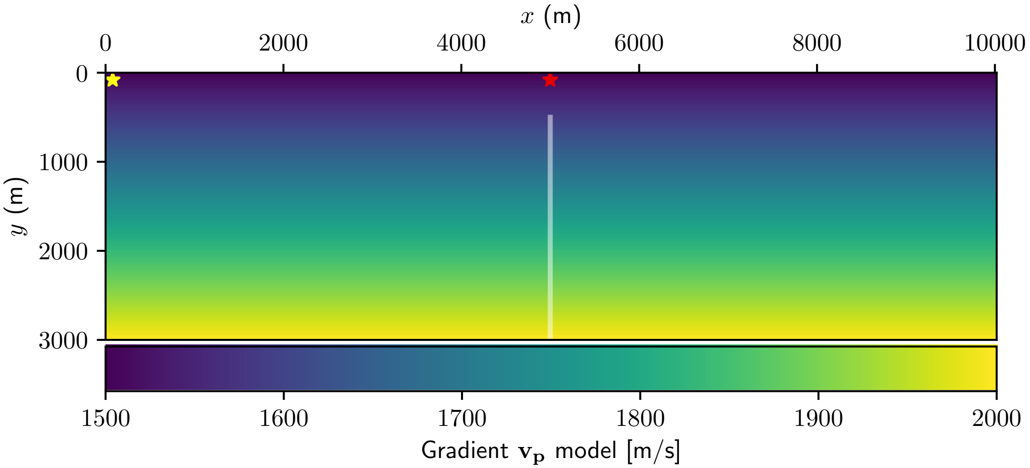
$$\mathbf{H}^m(x_i)\delta m(x_j) = \mathbf{H}_i^m \delta_j^m$$

$$\left[\begin{array}{ccc|cc|cc}
\mathbf{H}_0^\rho \delta_0^\rho & \mathbf{H}_0^\rho \delta_1^\rho & \cdots & \mathbf{H}_0^\rho \delta_0^{v_p} & \cdots & \mathbf{H}_0^\rho \delta_0^{v_s} & \cdots \\
\mathbf{H}_1^\rho \delta_0^\rho & \mathbf{H}_1^\rho \delta_1^\rho & \ddots & \vdots & \ddots & \vdots & \ddots \\
\mathbf{H}_2^\rho \delta_0^\rho & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\hline
\mathbf{H}_0^{v_p} \delta_0^\rho & \cdots & \cdots & \mathbf{H}_0^{v_p} \delta_0^{v_p} & \cdots & \mathbf{H}_0^{v_p} \delta_0^{v_s} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\hline
\mathbf{H}_0^{v_s} \delta_0^\rho & \cdots & \cdots & \mathbf{H}_0^{v_s} \delta_0^{v_p} & \cdots & \mathbf{H}_0^{v_s} \delta_0^{v_s} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots
\end{array}\right] \tag{15}$$

Gradient model



Gradient model



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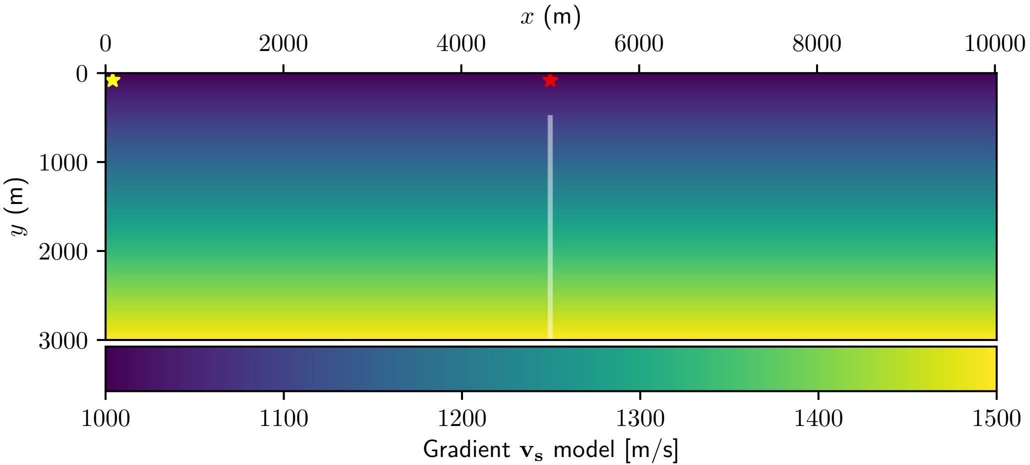
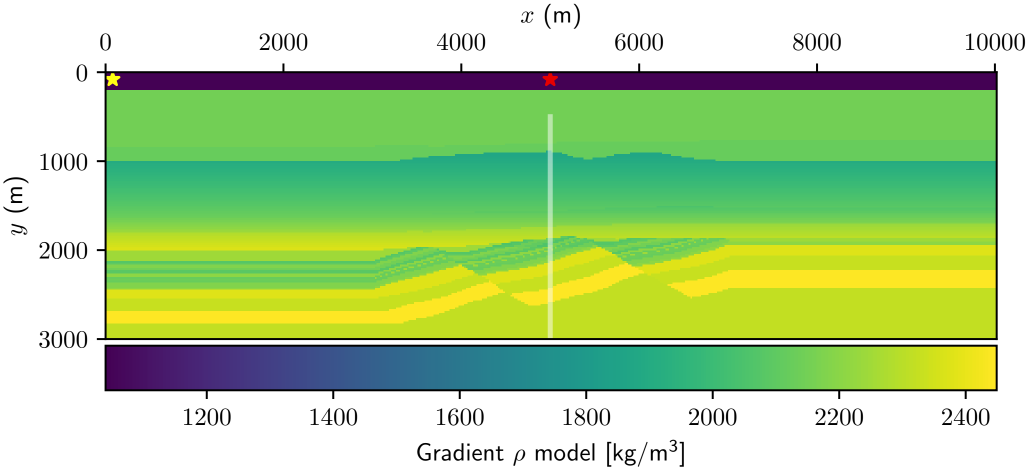


Gullfaks model



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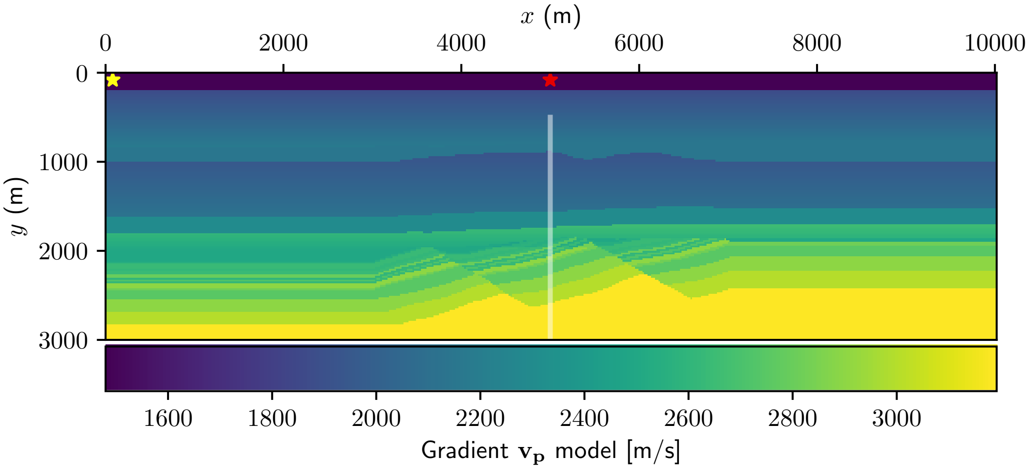

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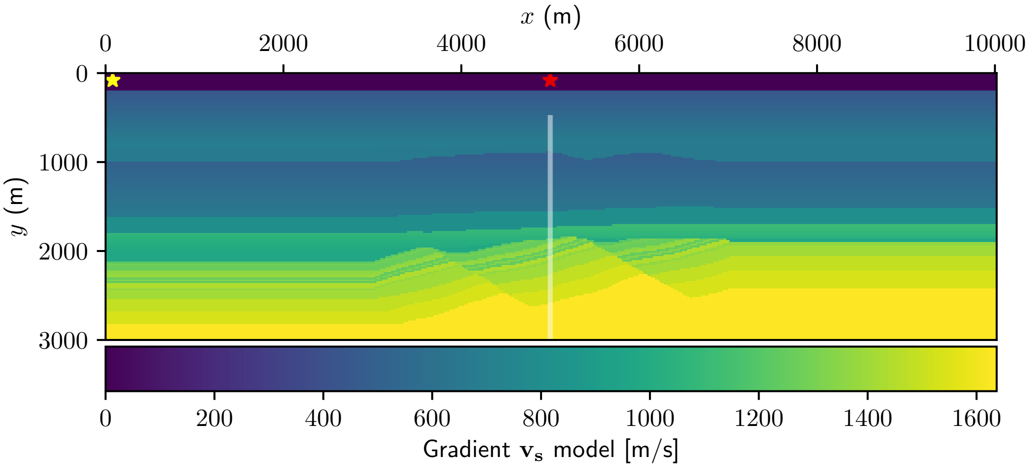
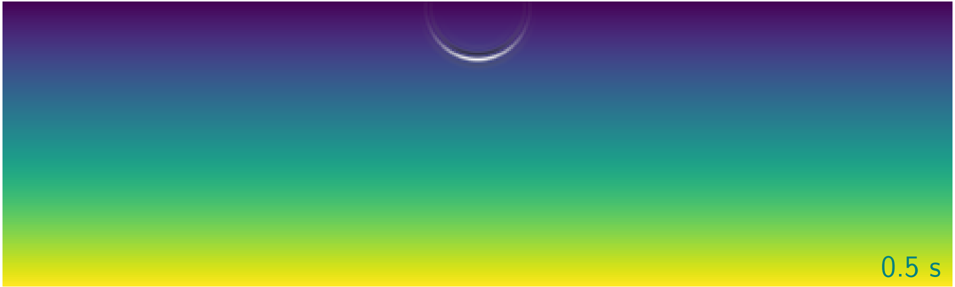


Gradient model — Shot 1



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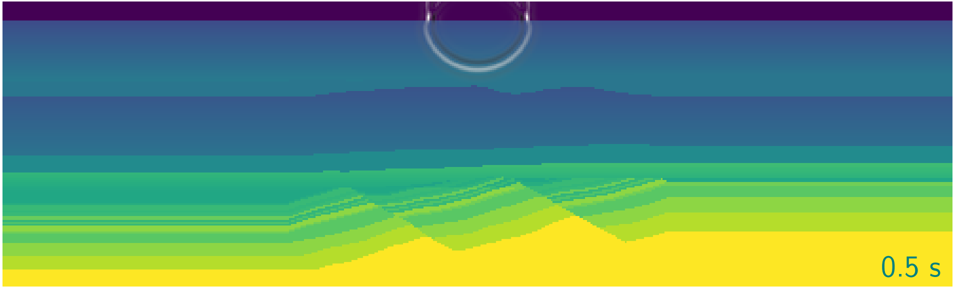

Gullfaks model — Shot 1

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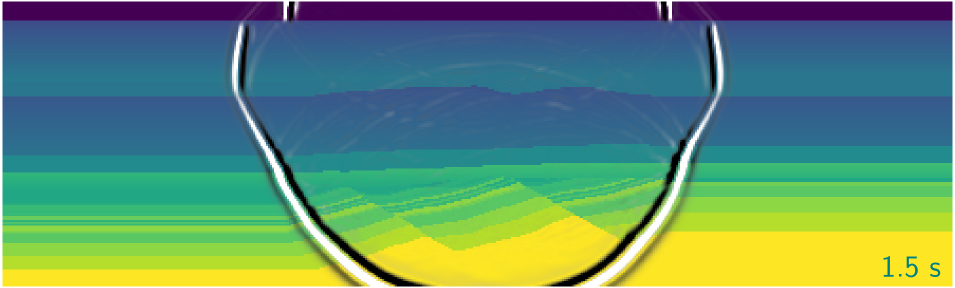

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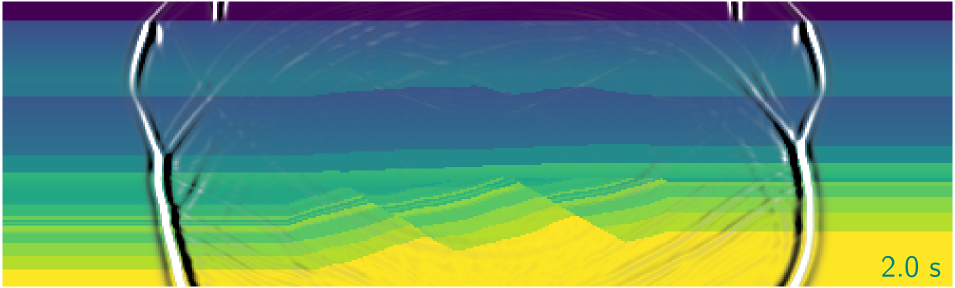

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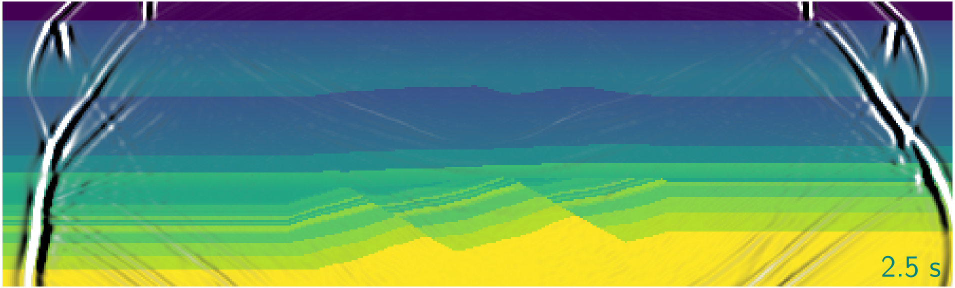

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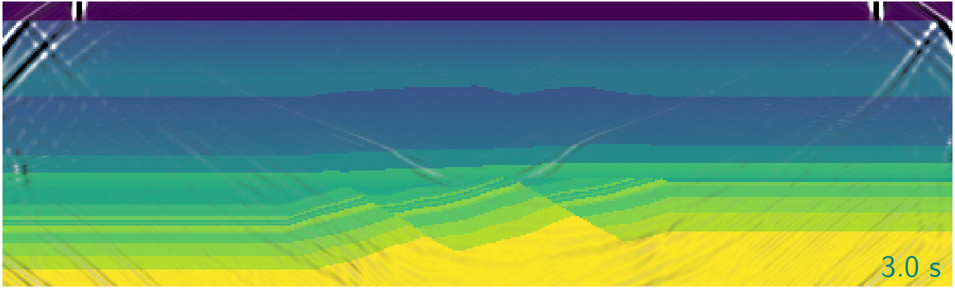

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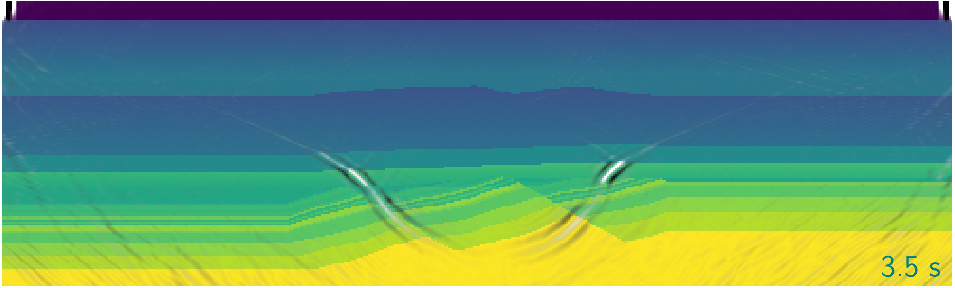

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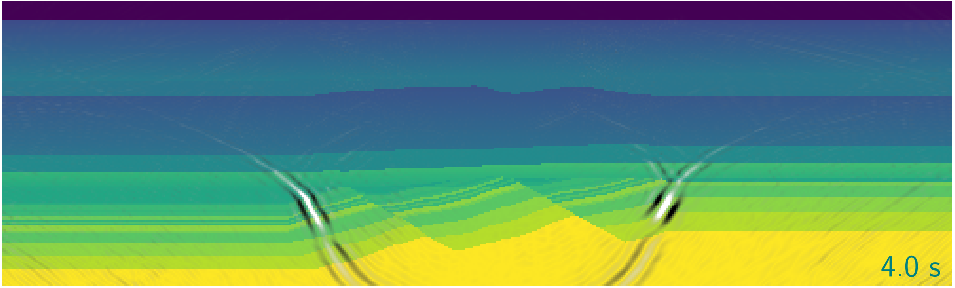

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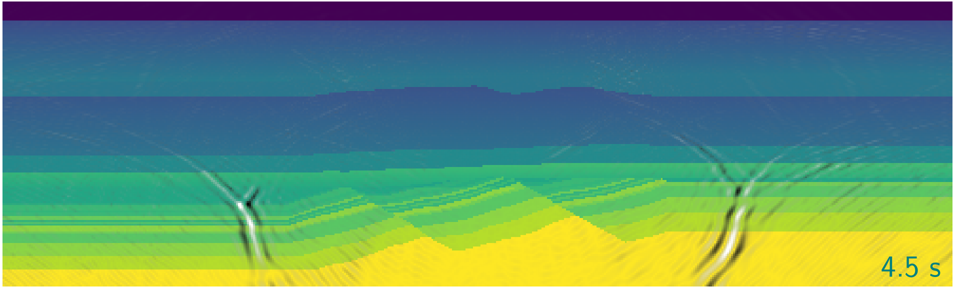

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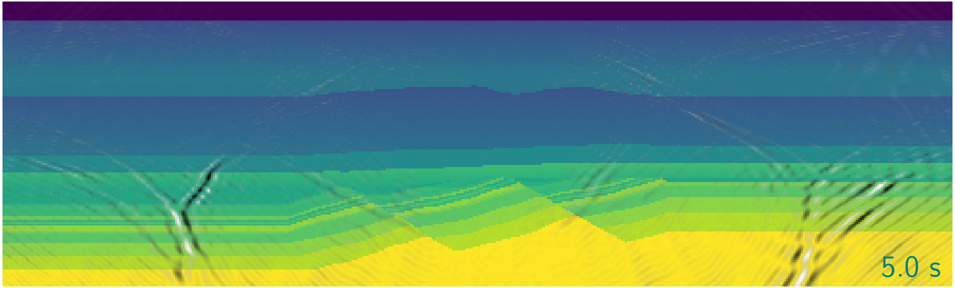

Gradient model — Shot 2

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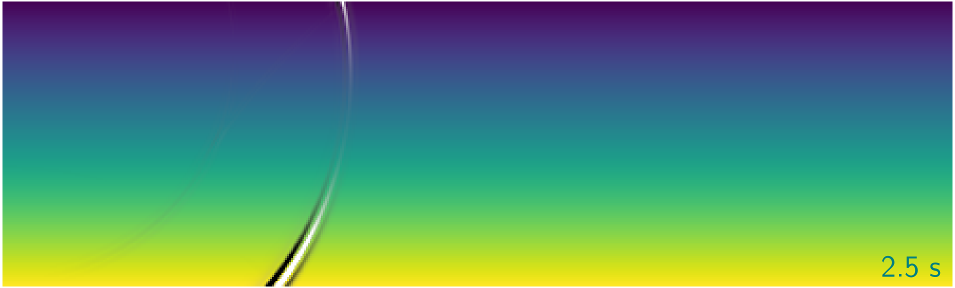

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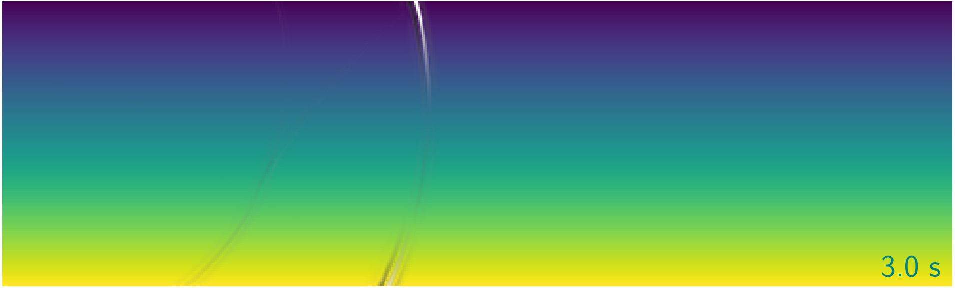

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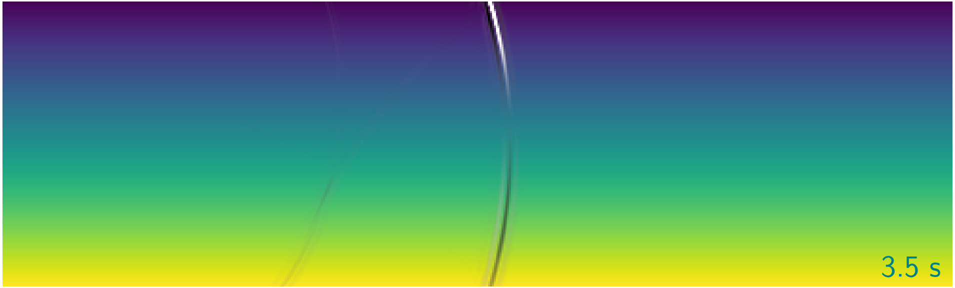

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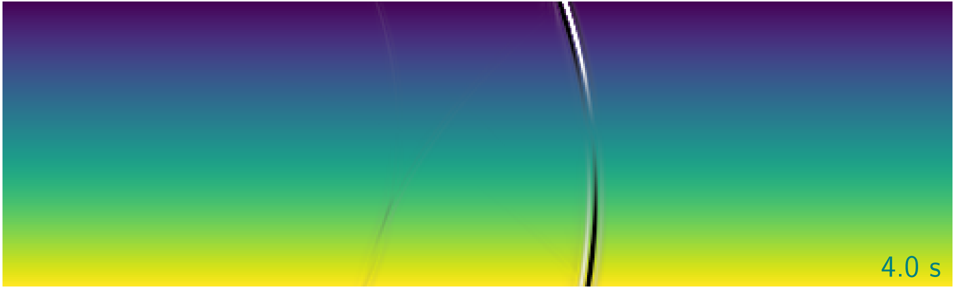

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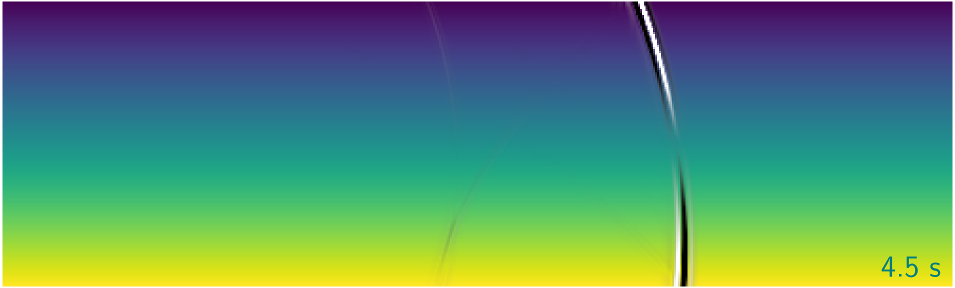

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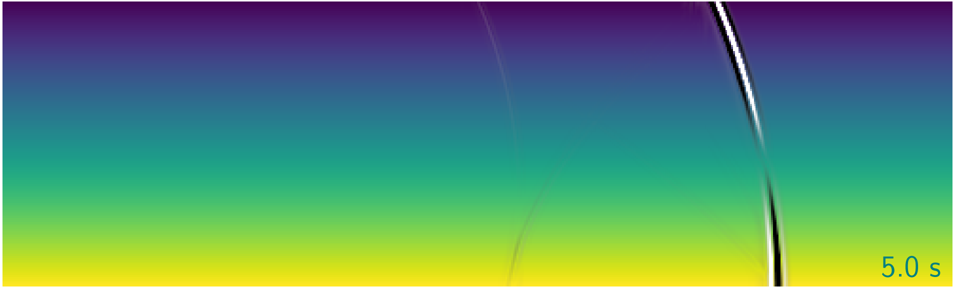

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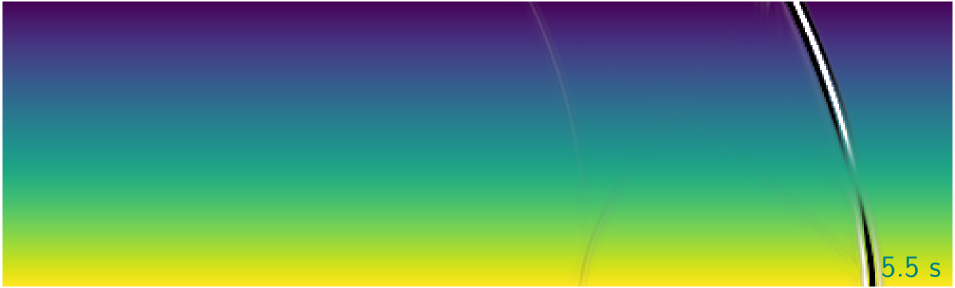

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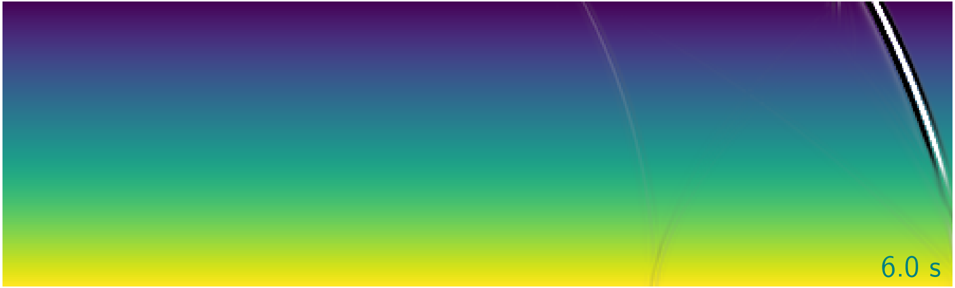

Gradient model — Shot 2

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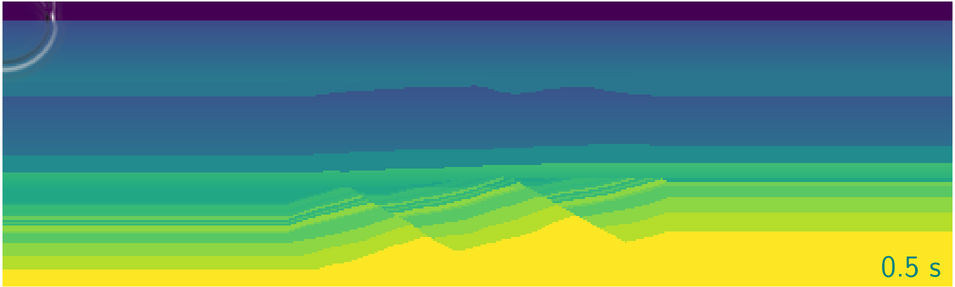
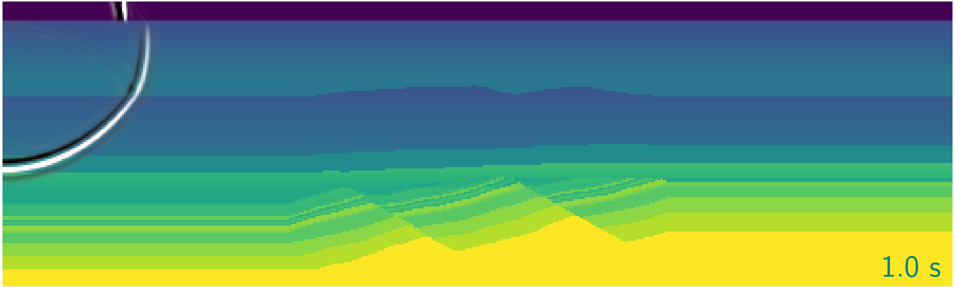
Gullfaks model — Shot 2



Gullfaks model — Shot 2



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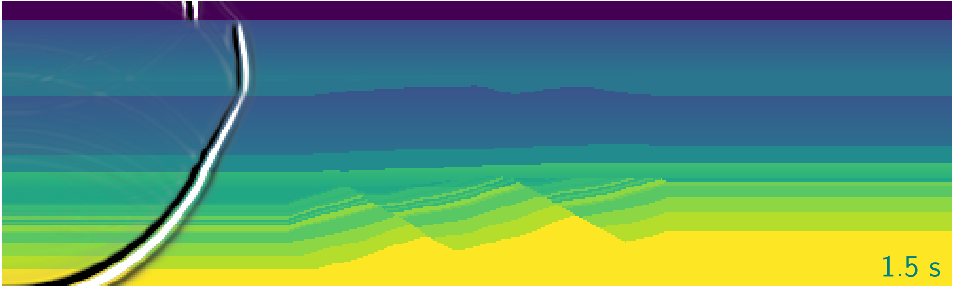

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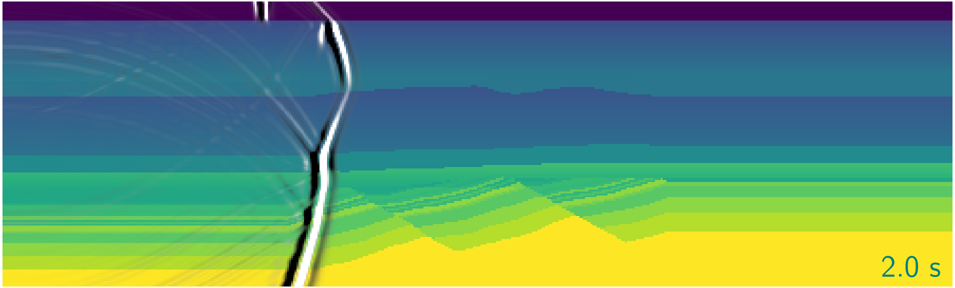

Gullfaks model — Shot 2

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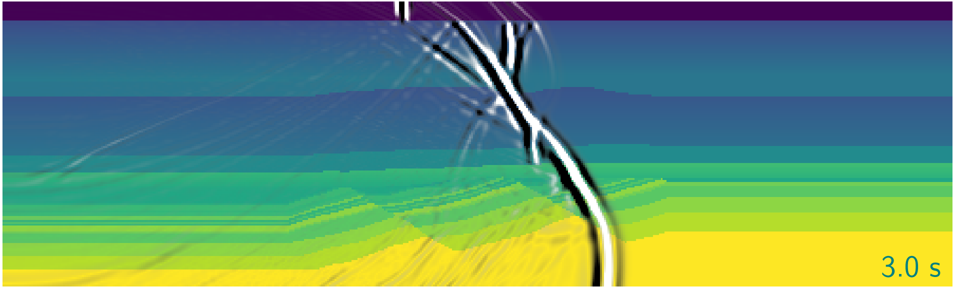

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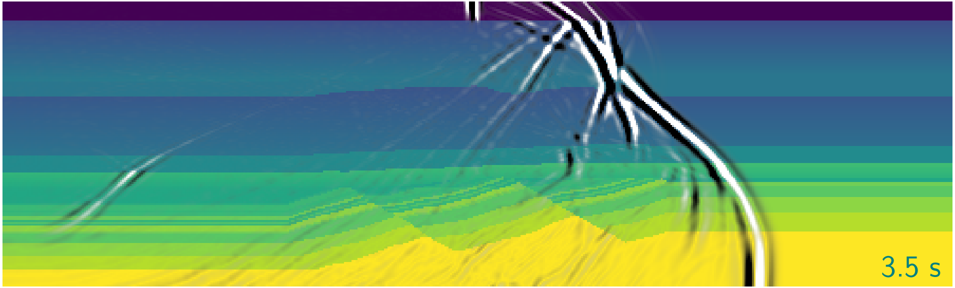

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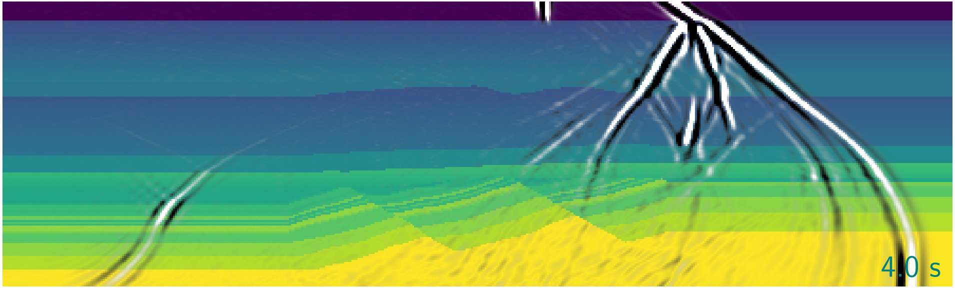

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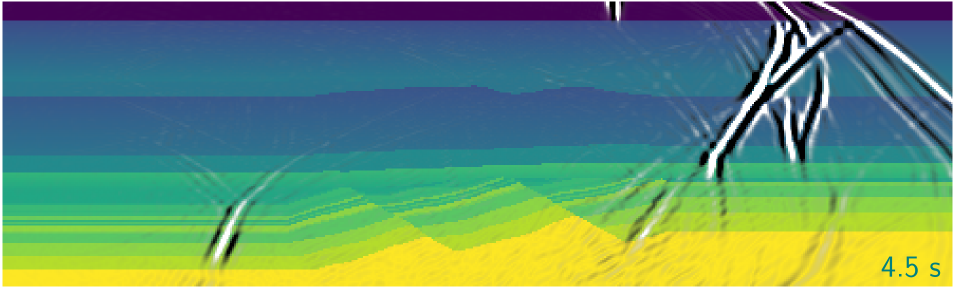

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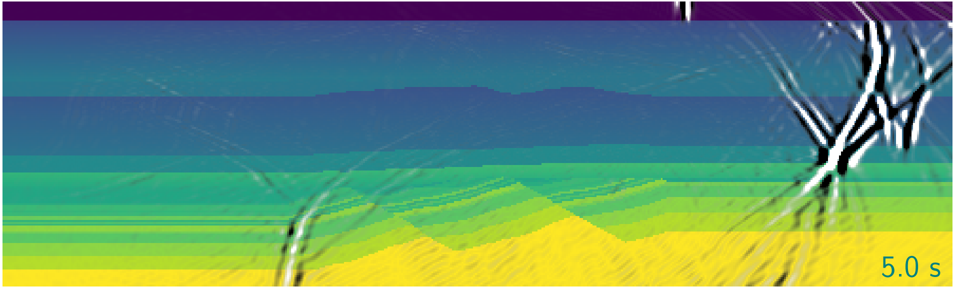

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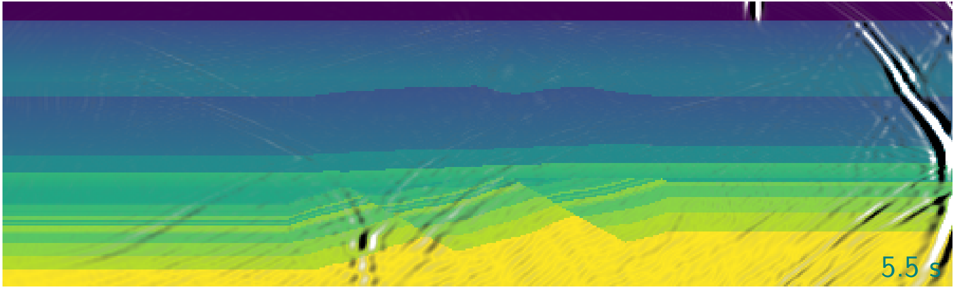

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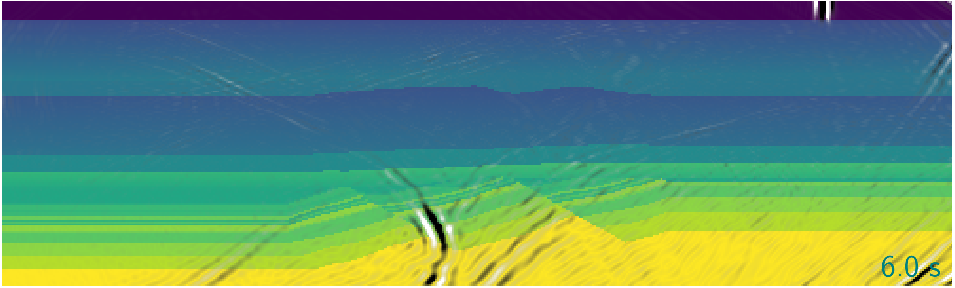

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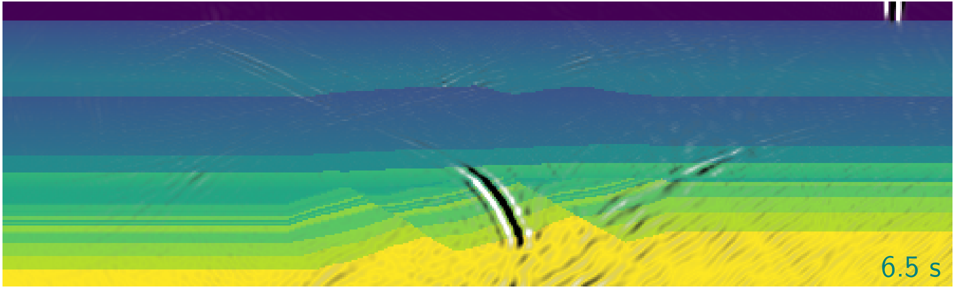

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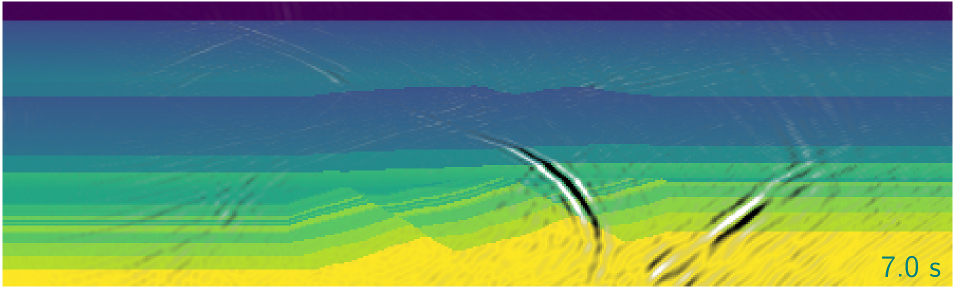

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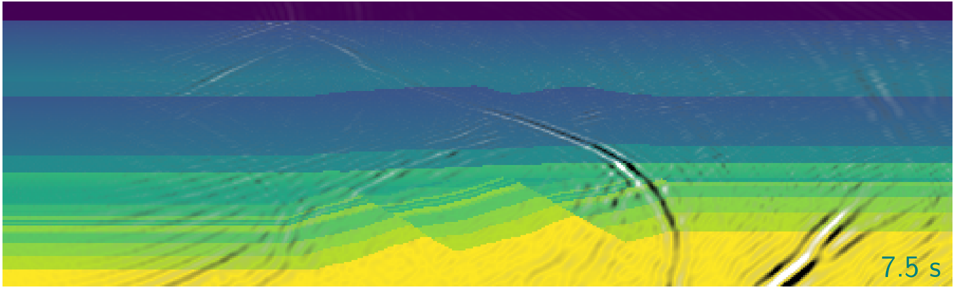

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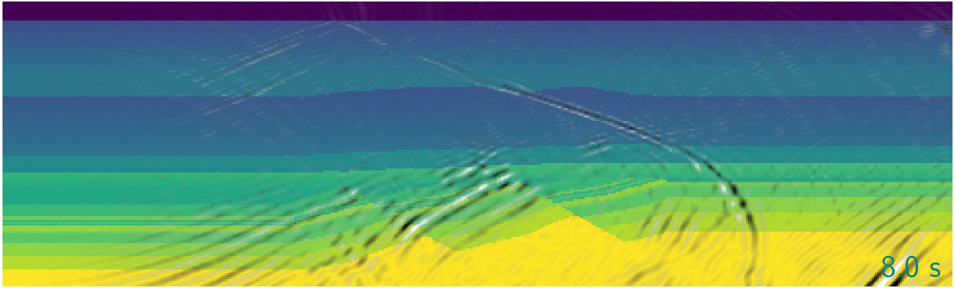

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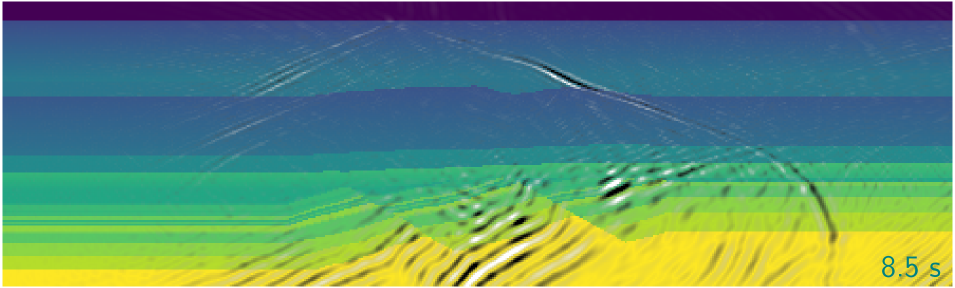

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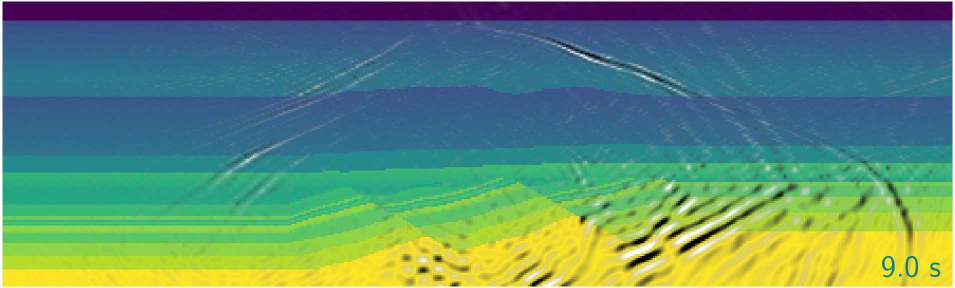
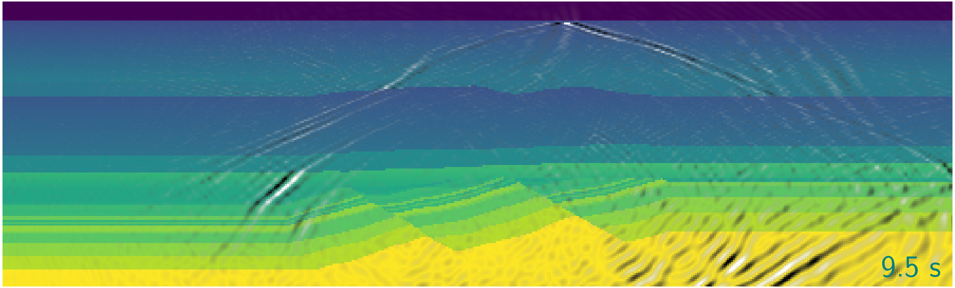
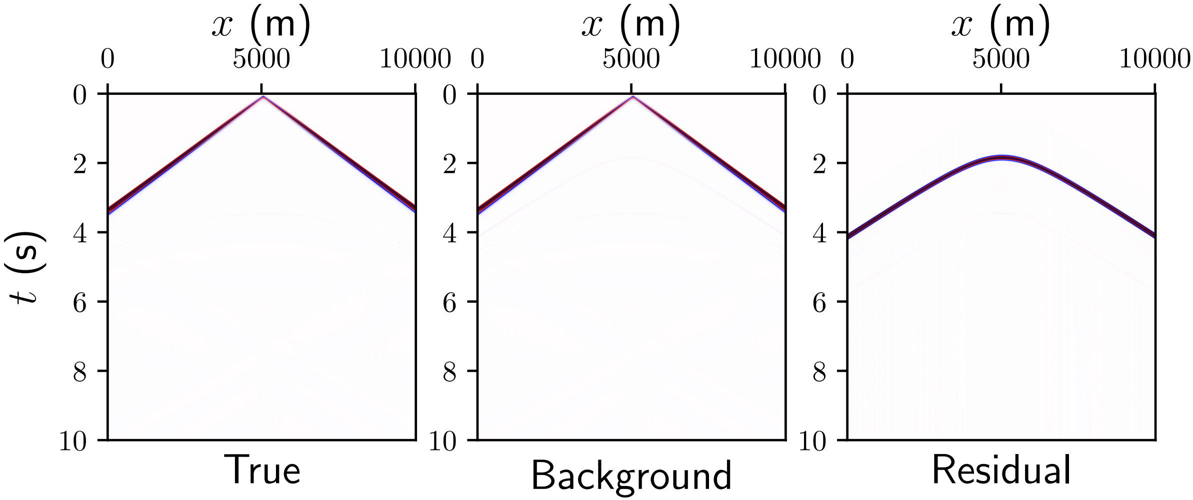


Gullfaks model — Shot 2



Gradient recording shot 1, perturbation at 1500 m



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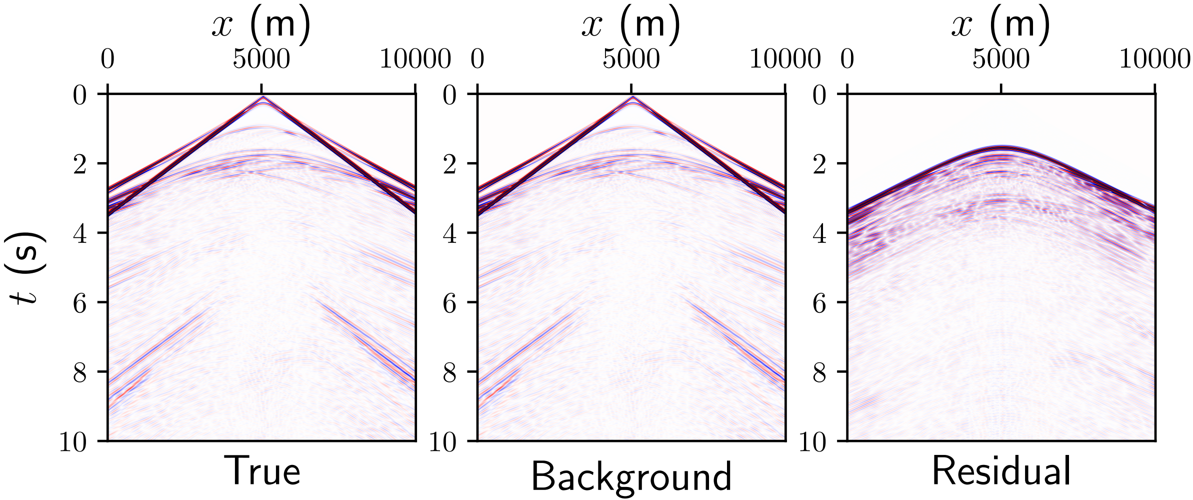
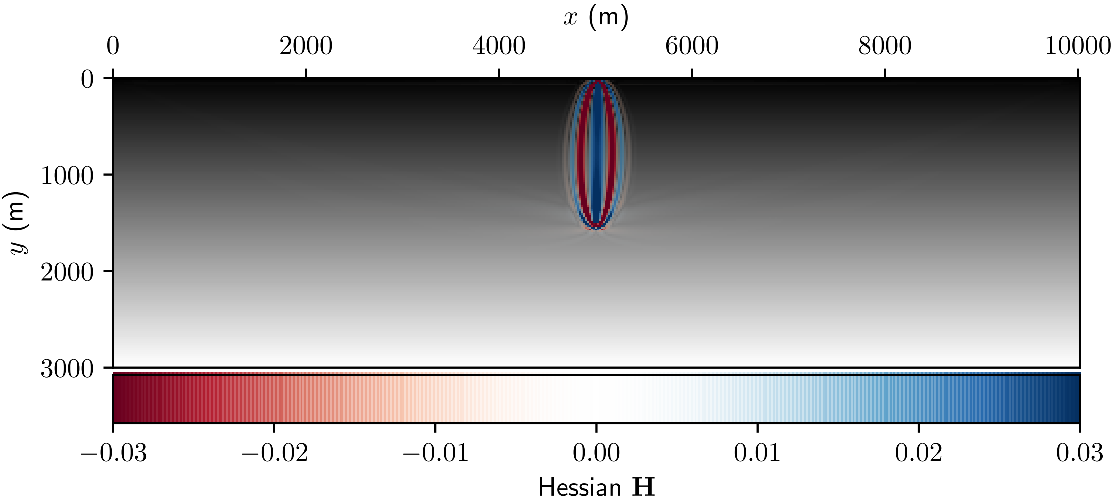
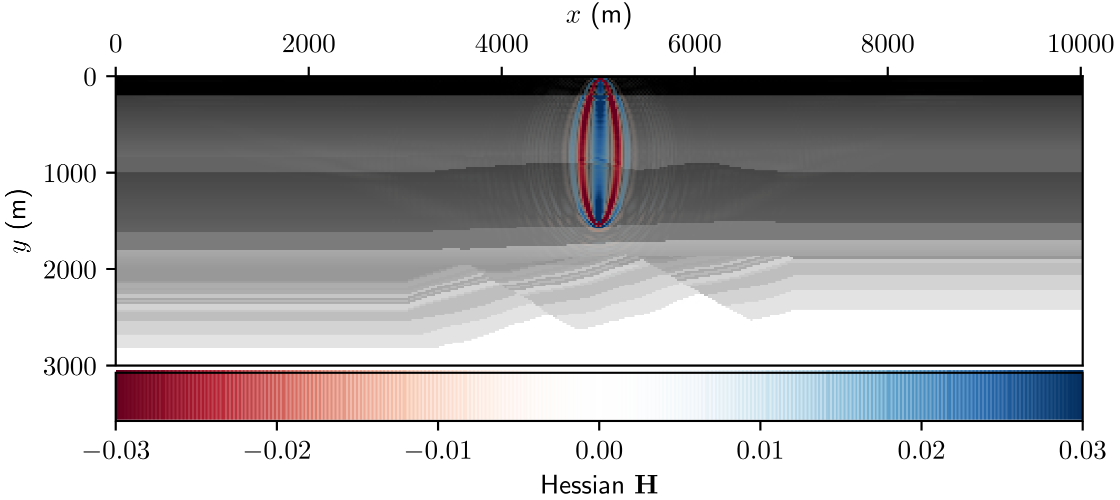


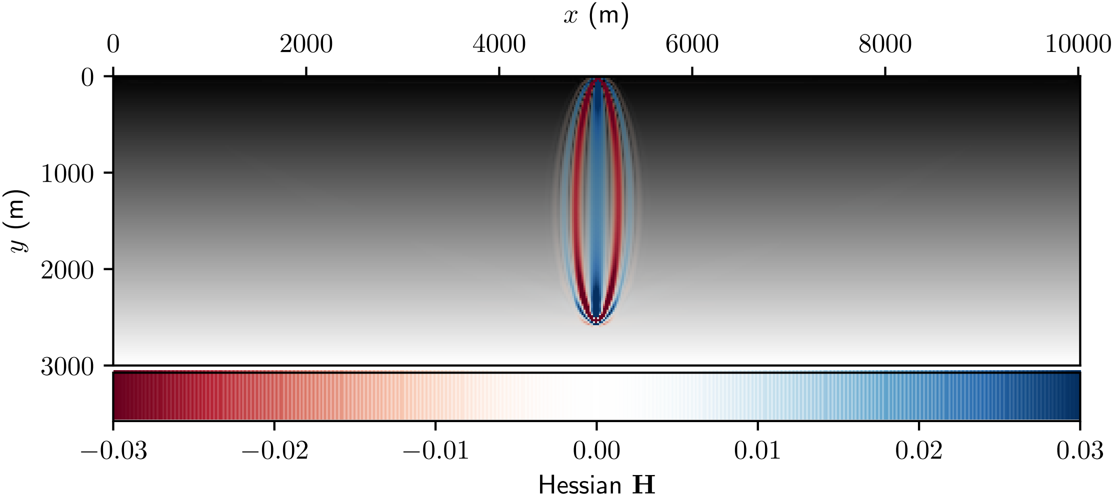
Gradient Hessian shot 1, perturbation at 1500 m



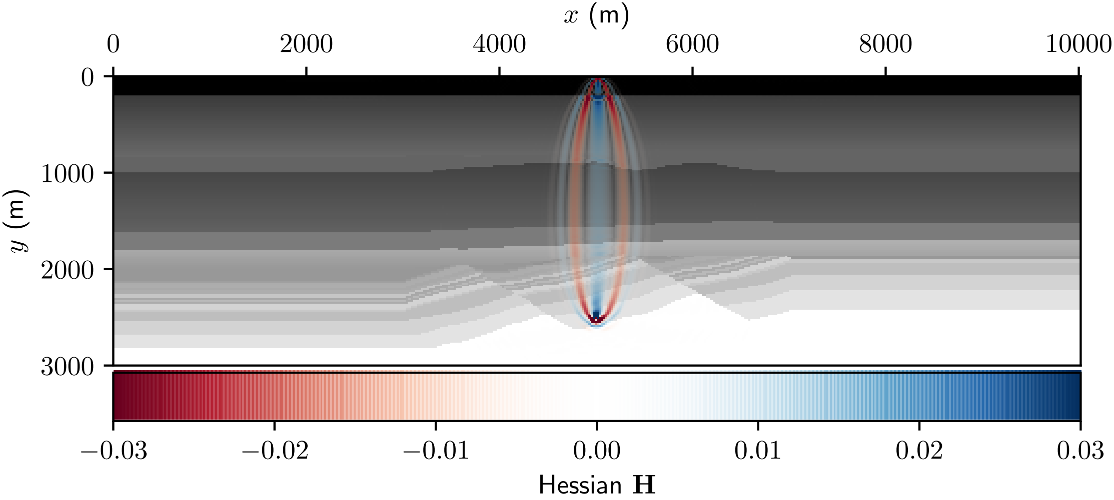
Gullfaks Hessian shot 1, perturbation at 1500 m



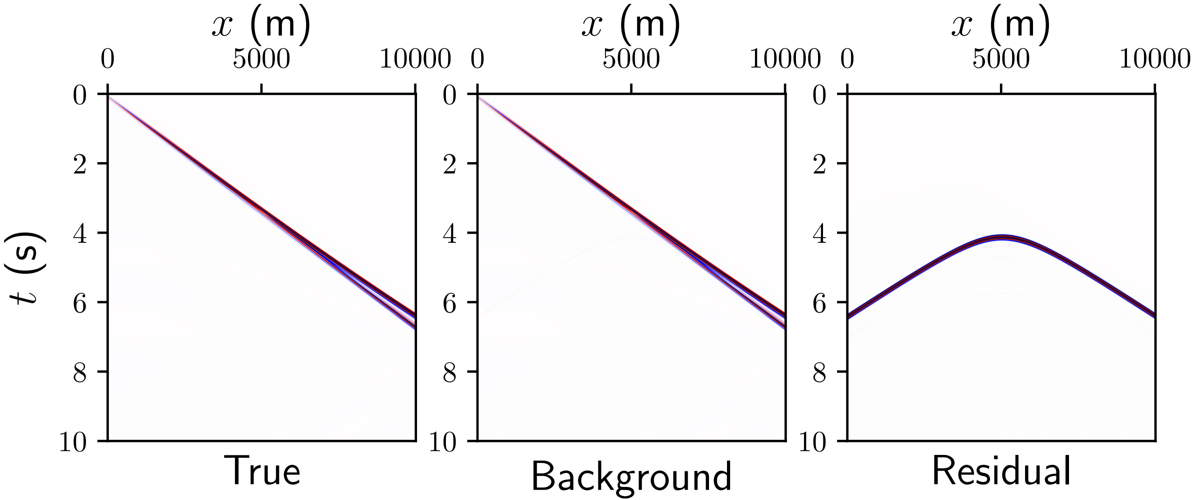
Gradient Hessian shot 1, perturbation at 2500 m



Gullfaks Hessian shot 1, perturbation at 2500 m



Gradient recording shot 2, perturbation at 1500 m



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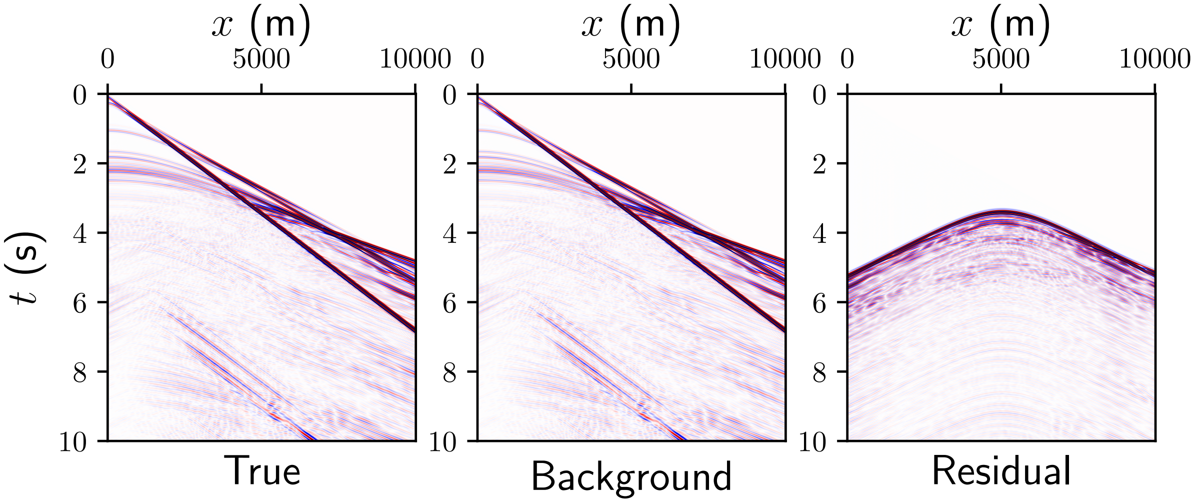
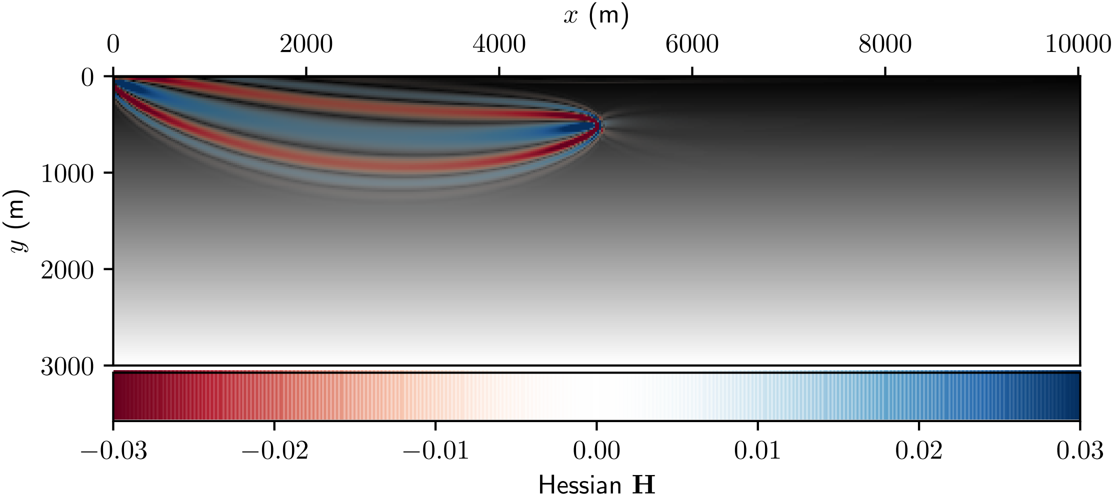
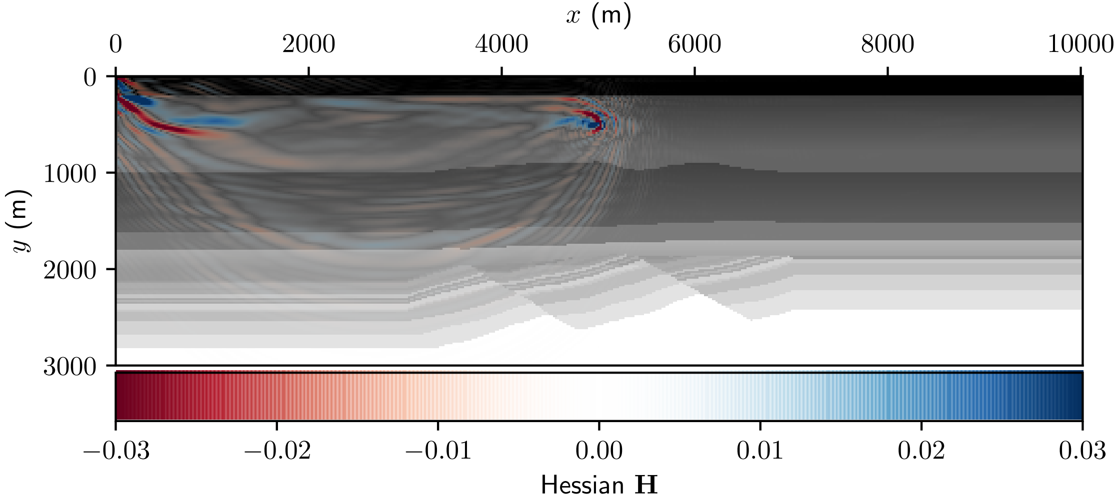


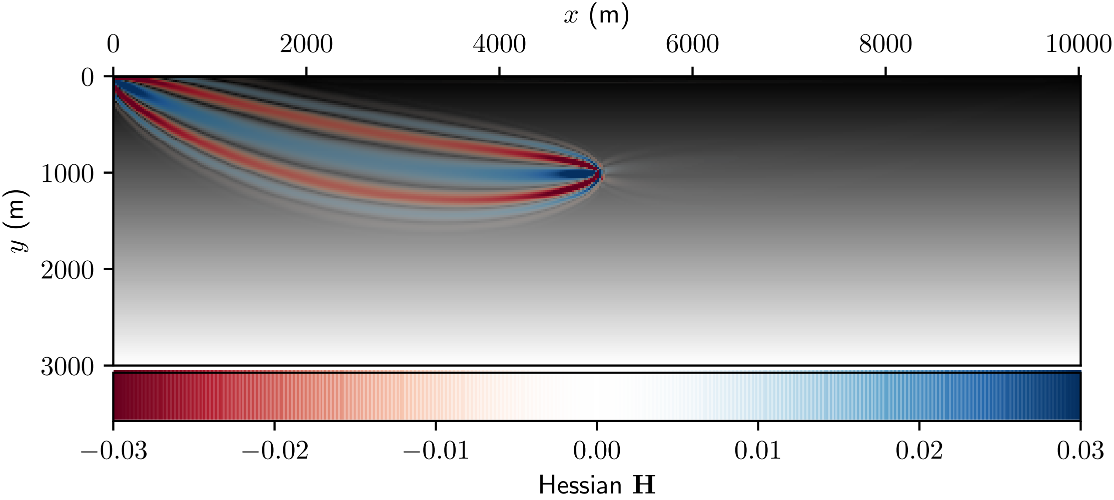
Gradient Hessian shot 2, perturbation at 500 m



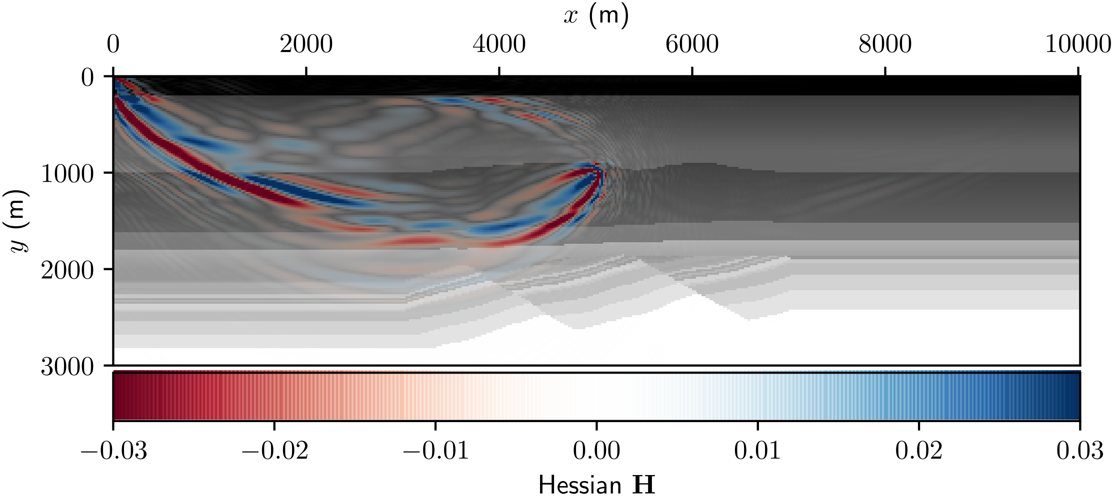
Gullfaks Hessian shot 2, perturbation at 500 m



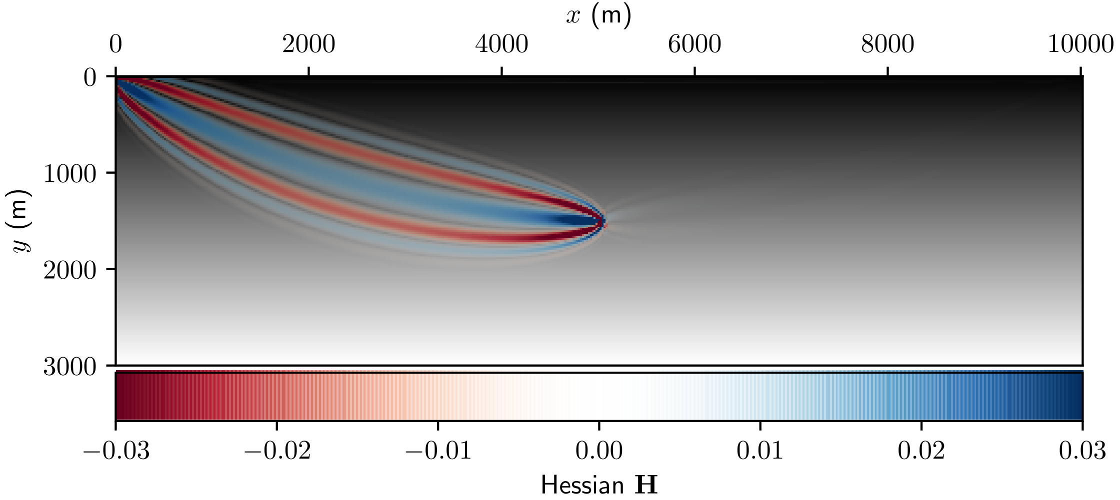
Gradient Hessian shot 2, perturbation at 1000 m



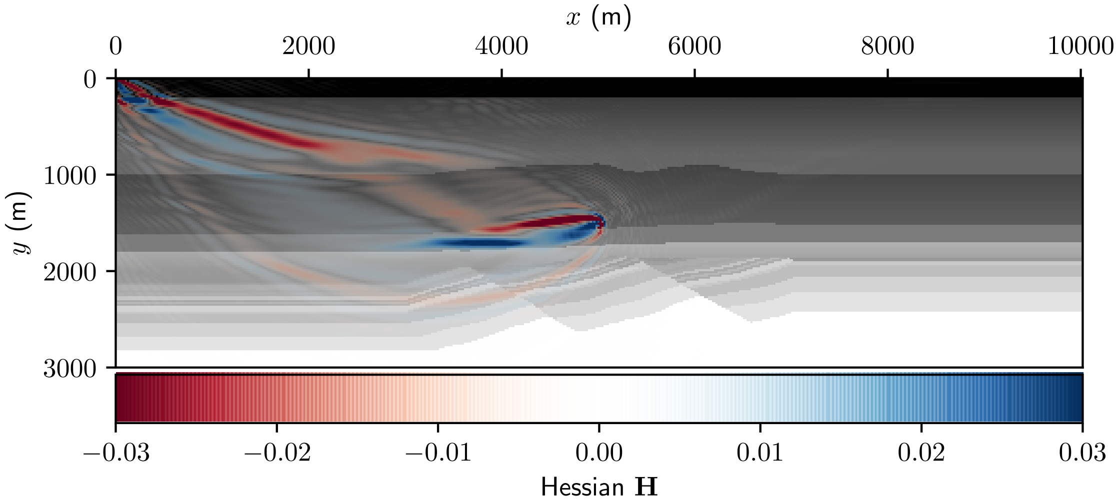
Gullfaks Hessian shot 2, perturbation at 1000 m



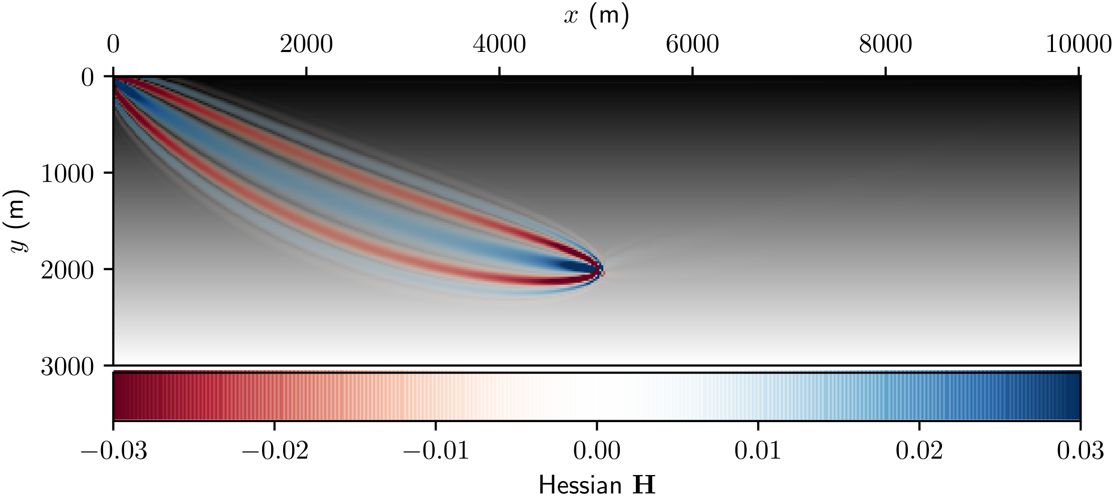
Gradient Hessian shot 2, perturbation at 1500 m



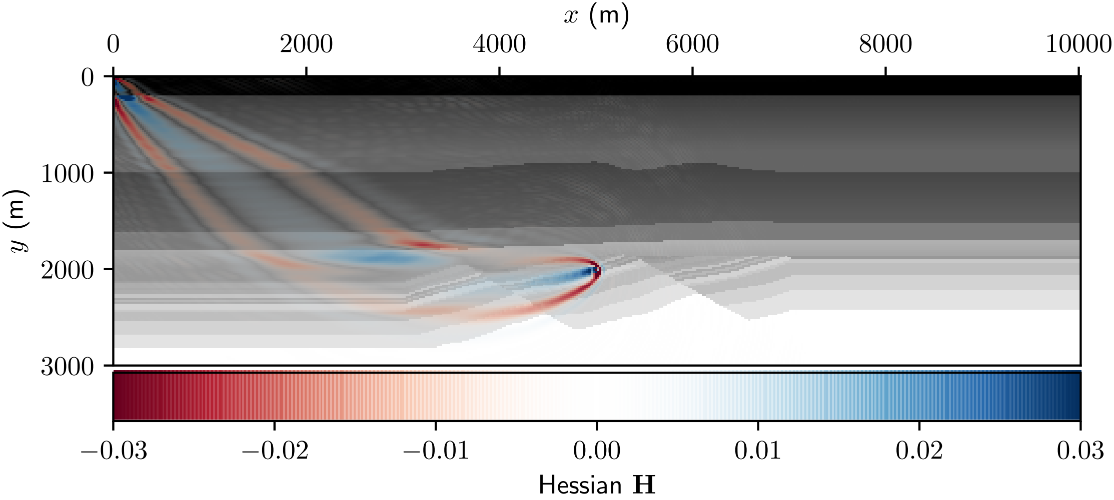
Gullfaks Hessian shot 2, perturbation at 1500 m



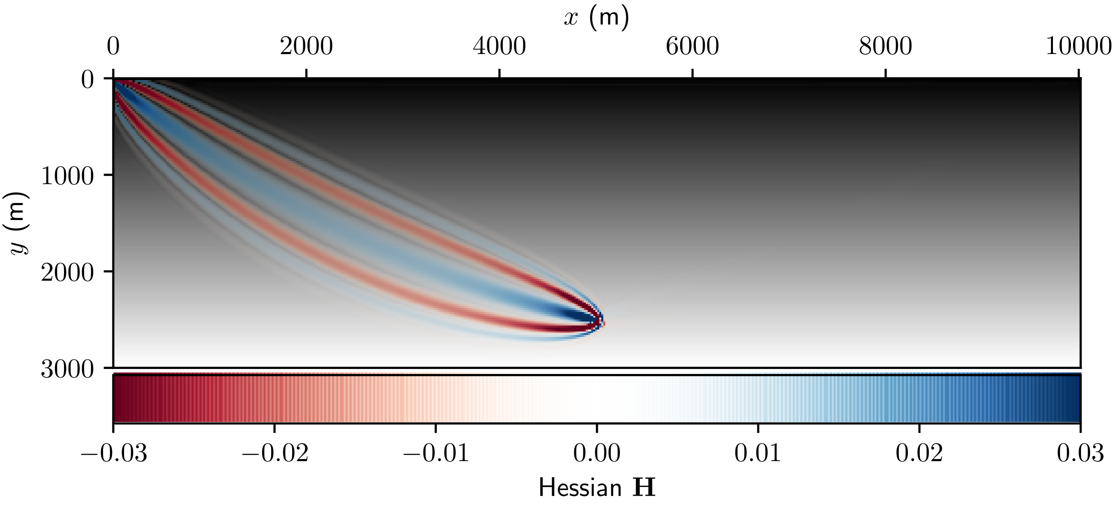
Gradient Hessian shot 2, perturbation at 2000 m



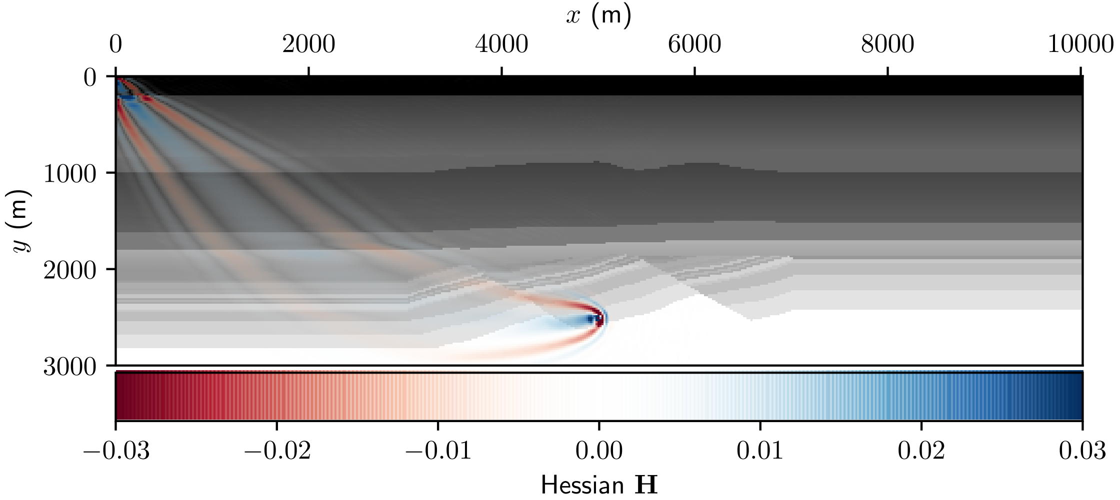
Gullfaks Hessian shot 2, perturbation at 2000 m



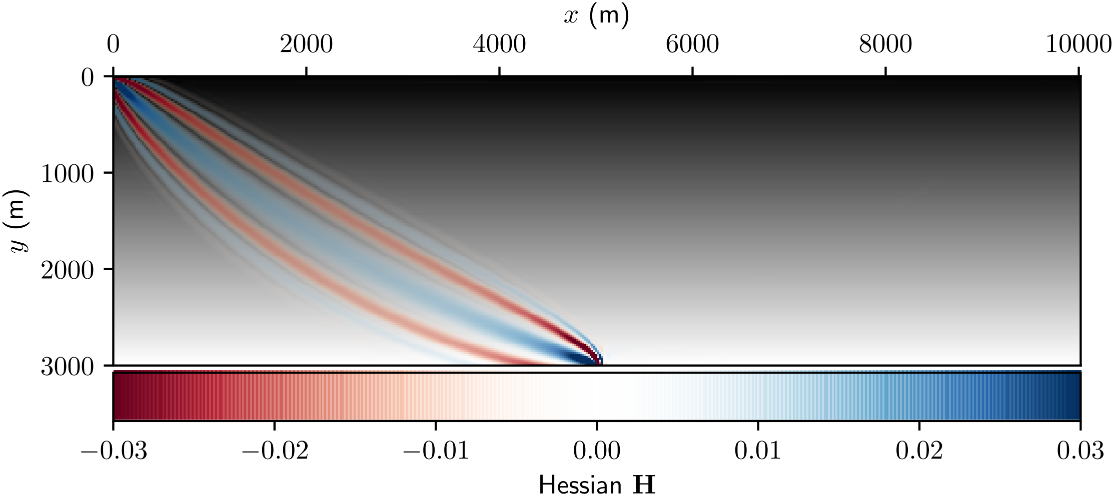
Gradient Hessian shot 2, perturbation at 2500 m



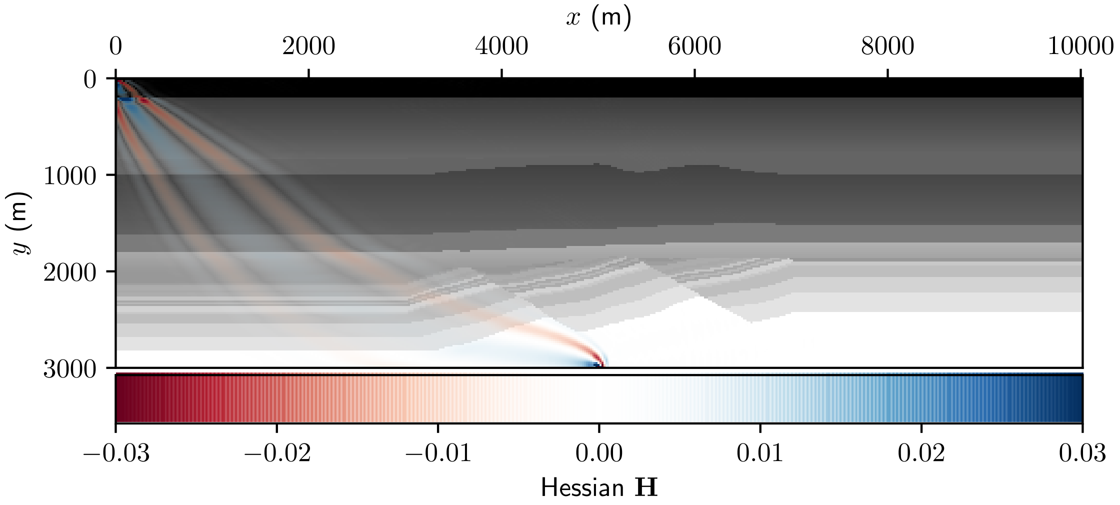
Gullfaks Hessian shot 2, perturbation at 2500 m



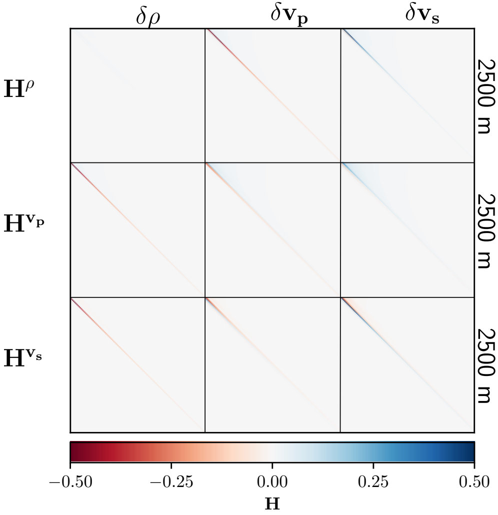
Gradient Hessian shot 2, perturbation at 3000 m



Gullfaks Hessian shot 2, perturbation at 3000 m



Gradient model Hessian constructed from shot 1



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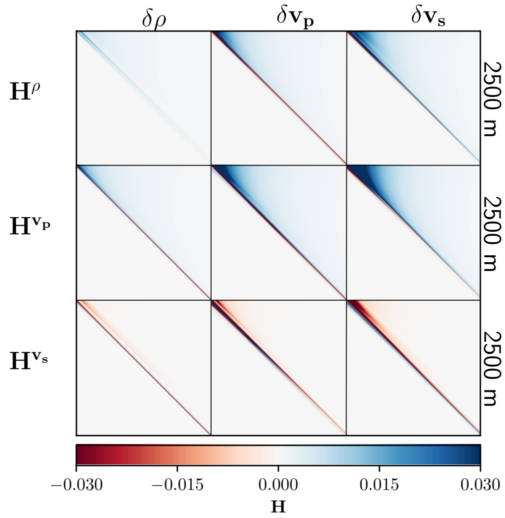
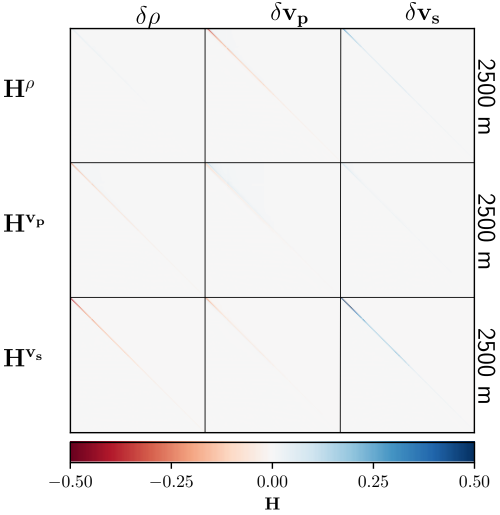
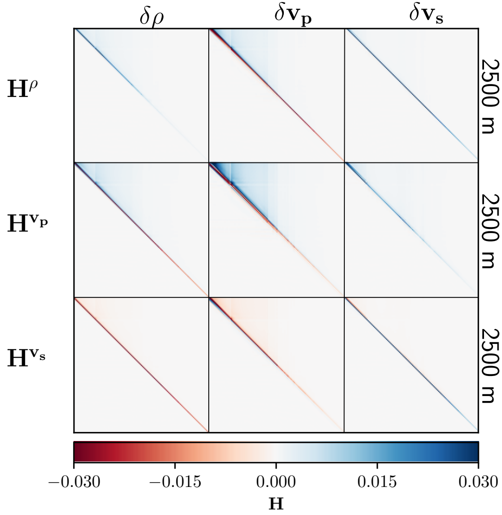


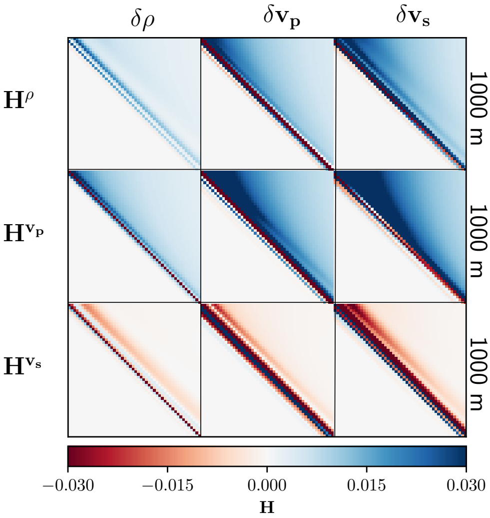
Gullfaks model Hessian constructed from shot 1



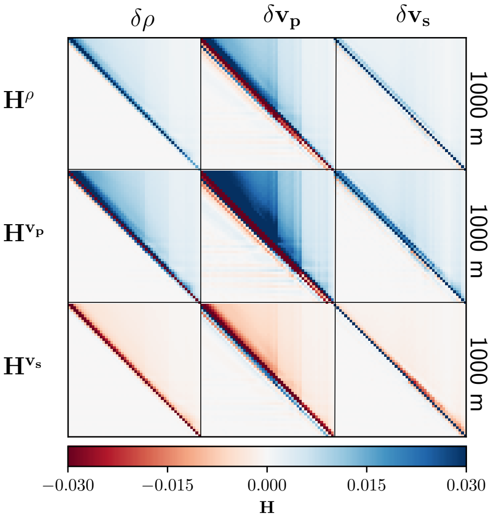
Gullfaks model Hessian constructed from shot 1



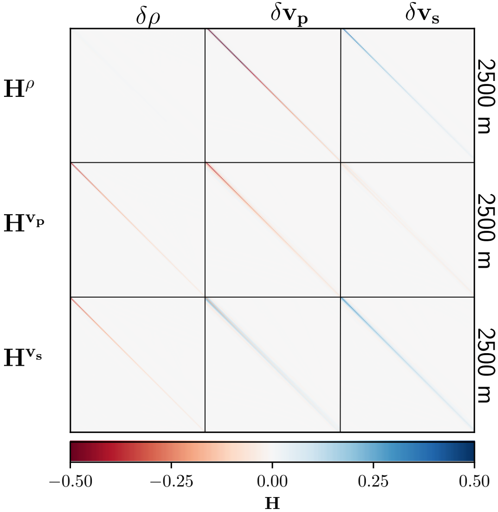
Gradient model Hessian constructed from shot 1 – 1km-2km zoom



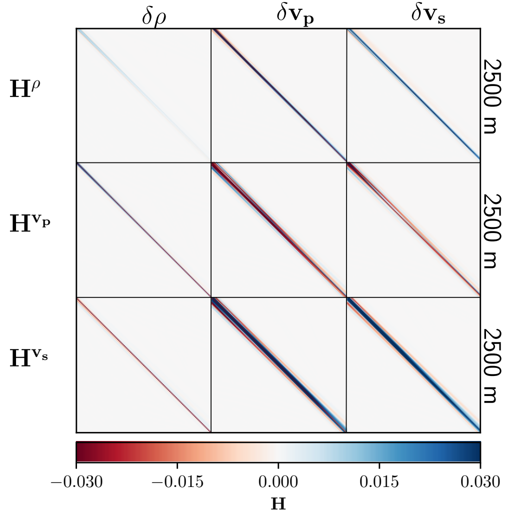
Gullfaks model Hessian constructed from shot 1 – 1km-2km zoom



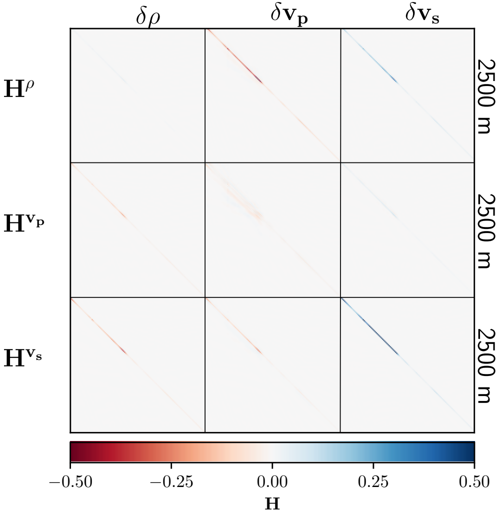
Gradient model Hessian constructed from shot 2



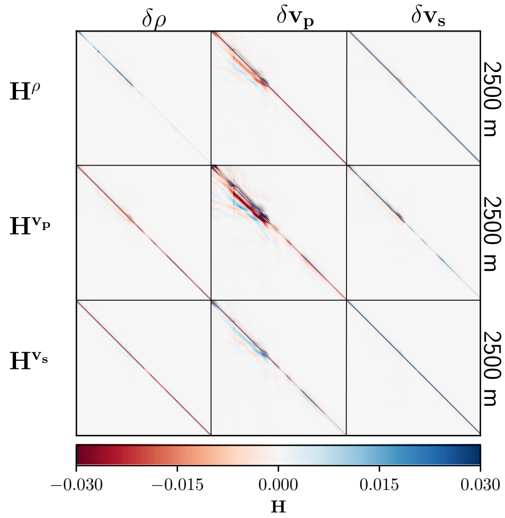
Gradient model Hessian constructed from shot 2



Gullfaks model Hessian constructed from shot 2



Gullfaks model Hessian constructed from shot 2



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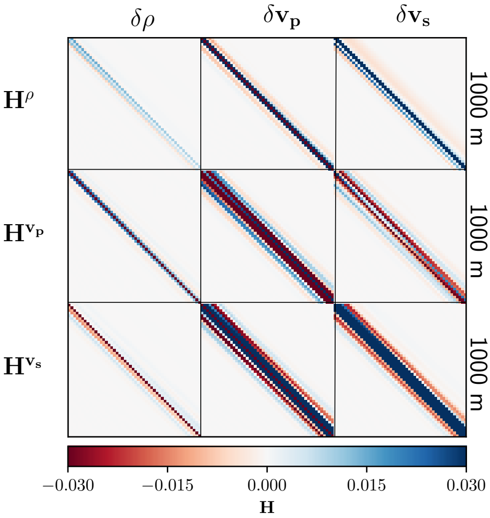
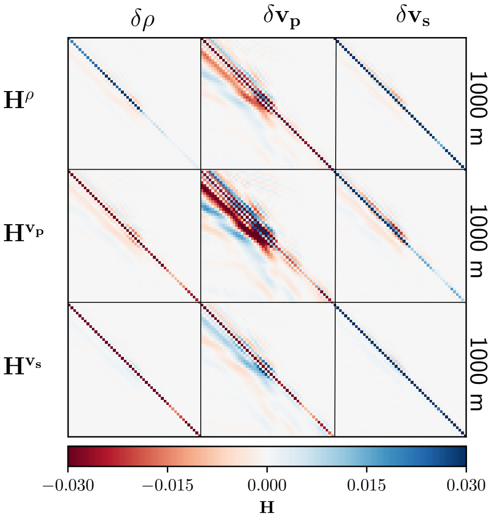


Gullfaks model Hessian constructed from shot 2 – 1km-2km zoom



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- Low recovery of density in the given geometries.

Future work

- Explore different shot-receiver geometries.
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References

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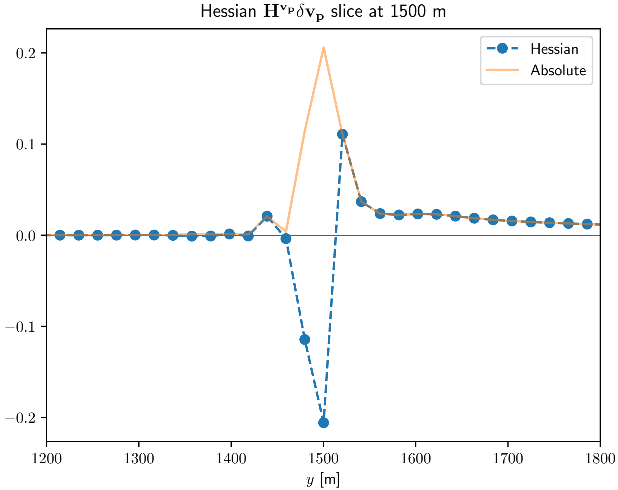
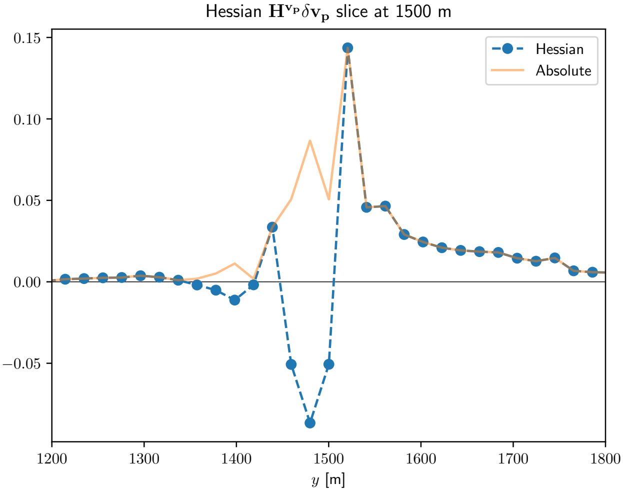
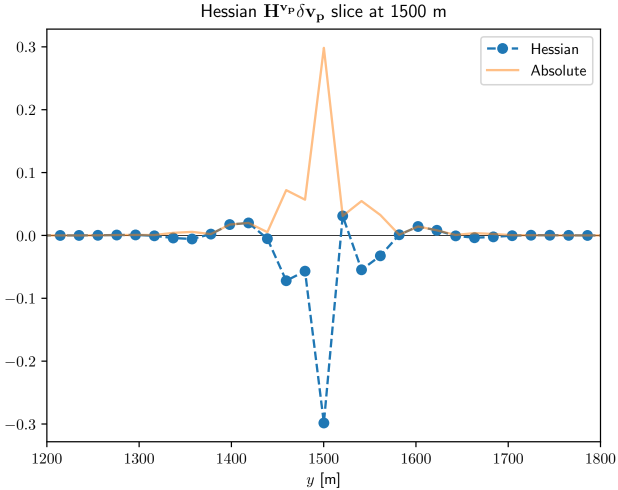
Gradient model Hessian slice shot 1 – 1500m



Gullfaks model Hessian slice shot 1 – 1500m



Gradient model Hessian slice shot 2 – 1500m



Gullfaks model Hessian slice shot 2 – 1500m