Low-frequency ORT

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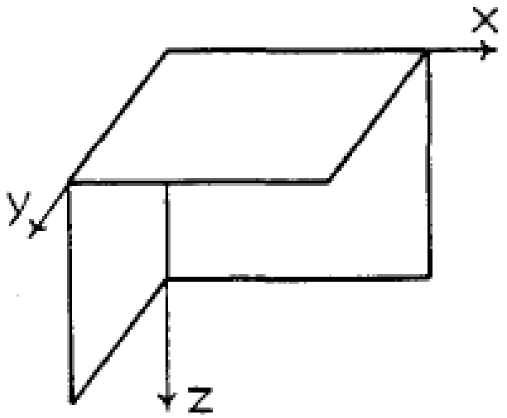
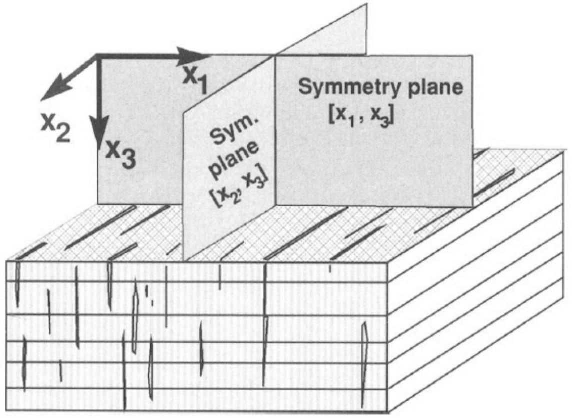


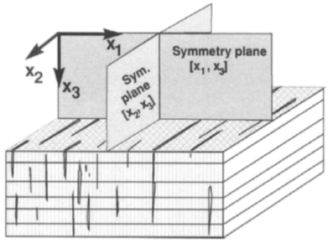


OUTLINE

- Low-frequency properties of layered medium
- ORT medium and parameterization
- BCH series for ORT
- Eigenvalues, multipliers and frequency dependent velocities
- Interpretation of dispersion in terms of ORT parameters
- Conclusions

Low-frequency properties of the medium

- Zero- and infinite-frequency limits
- Given frequency w=w$_0$ (non-physical medium)
- Low-frequency approximation



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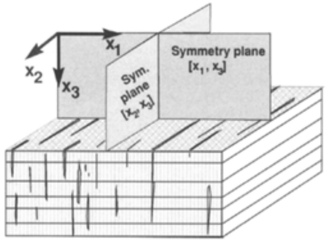

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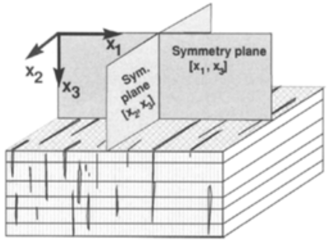


ORT: stiffness coefficient matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & & & \\ & c_{22} & c_{23} & & & \\ & & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{55} & \\ & & & & & c_{66} \end{pmatrix}$$

$$\left(c_{ij}\right) \Leftrightarrow \left(v_{P0}, v_{S0}, \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2\right)$$

Tsvankin, 1997

System matrix for ORT

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{M} \\ \mathbf{N} & 0 \end{pmatrix}$$

$$\mathbf{N} = \begin{pmatrix} -\dfrac{1}{c_{33}} & -p_1 \dfrac{c_{13}}{c_{33}} & -p_2 \dfrac{c_{23}}{c_{33}} \\ -p_1 \dfrac{c_{13}}{c_{33}} & s_{11} & s_{12} \\ -p_2 \dfrac{c_{23}}{c_{33}} & s_{12} & s_{22} \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} -\rho & -p_1 & -p_2 \\ -p_1 & -\dfrac{1}{c_{55}} & 0 \\ -p_2 & 0 & -\dfrac{1}{c_{44}} \end{pmatrix}$$

$$s_{11} = \left(c_{11} - \frac{c_{13}^2}{c_{33}} \right) p_1^2 + c_{66} p_2^2 - \rho, \quad s_{12} = \left(c_{12} + c_{66} - \frac{c_{13} c_{23}}{c_{33}} \right) p_1 p_2, \quad s_{22} = \left(c_{22} - \frac{c_{23}^2}{c_{33}} \right) p_2^2 + c_{66} p_1^2 - \rho.$$

Upscaling (replacement of Schoenberg-Muir)

$$\tilde{\mathbf{A}} = \langle \mathbf{A} \rangle$$

$$B_1 = \left\langle \frac{1}{c_{33}} \right\rangle, \ B_2 = \left\langle \frac{c_{13}}{c_{33}} \right\rangle, \ B_3 = \left\langle \frac{c_{23}}{c_{33}} \right\rangle,$$

$$B_4 = \left\langle \frac{1}{c_{44}} \right\rangle, \ B_5 = \left\langle \frac{1}{c_{55}} \right\rangle, \ B_6 = \langle c_{66} \rangle,$$

$$B_7 = \left\langle c_{11} - \frac{c_{13}^2}{c_{33}} \right\rangle, \ B_8 = \left\langle c_{12} + c_{66} - \frac{c_{13}c_{23}}{c_{33}} \right\rangle, \ B_9 = \left\langle c_{22} - \frac{c_{23}^2}{c_{33}} \right\rangle$$

$$\tilde{c}_{11} = B_7 + \frac{B_2^2}{B_1}, \ \tilde{c}_{12} = B_8 - B_6 + \frac{B_2 B_3}{B_1}, \ \tilde{c}_{13} = \frac{B_2}{B_1},$$

$$\tilde{c}_{22} = B_9 + \frac{B_3^2}{B_1}, \ \tilde{c}_{23} = \frac{B_3}{B_1}, \ \tilde{c}_{33} = \frac{1}{B_1},$$

$$\tilde{c}_{44} = \frac{1}{B_4}, \ \tilde{c}_{55} = \frac{1}{B_5}, \ \tilde{c}_{66} = B_6.$$

Zero-frequency limit

The BCH series

$$\mathbf{A}\left(\omega\right)=\mathbf{F_0}+\left(i\omega H\right)\mathbf{F_1}+\left(i\omega H\right)^2\mathbf{F_2}+\ldots$$

Zero-frequency limit

Roganov and Stovas, 2012

The BCH series

$$\mathbf{F_0} = \alpha \mathbf{A_1} + (1-\alpha)\mathbf{A_2},$$

$$\mathbf{F_1} = \frac{1}{2}\alpha(1-\alpha)[\mathbf{A_2}, \mathbf{A_1}],$$

$$\mathbf{F_2} = \frac{1}{12}\alpha(1-\alpha)\left\{(1-\alpha)\left[\mathbf{A_2},[\mathbf{A_2}, \mathbf{A_1}]\right] + \alpha\left[\mathbf{A_1},[\mathbf{A_1}, \mathbf{A_2}]\right]\right\},$$

[x,y] is a commuting operator

α is a volume fraction

Roganov and Stovas, 2012

The BCH series

$$\left[\mathbf{A_2},\mathbf{A_1}\right]=\begin{pmatrix}\mathbf{M}_2\mathbf{N}_1-\mathbf{M}_1\mathbf{N}_2 & 0\\ 0 & \mathbf{N}_2\mathbf{M}_1-\mathbf{N}_1\mathbf{M}_2\end{pmatrix},$$

$$\left[\mathbf{A_2},\left[\mathbf{A_2},\mathbf{A_1}\right]\right]=2\begin{pmatrix}0 & \mathbf{M}_2\mathbf{M}_1\mathbf{N}_2-\mathbf{M}_2\mathbf{M}_2\mathbf{N}_1\\ \mathbf{N}_2\mathbf{M}_2\mathbf{N}_1-\mathbf{N}_2\mathbf{M}_1\mathbf{N}_2 & 0\end{pmatrix},$$

$$\left[\mathbf{A}_1,\left[\mathbf{A}_1,\mathbf{A}_2\right]\right]=2\begin{pmatrix}0 & \mathbf{M}_1\mathbf{M}_2\mathbf{N}_1-\mathbf{M}_1\mathbf{M}_1\mathbf{N}_2\\ \mathbf{N}_1\mathbf{M}_1\mathbf{N}_2-\mathbf{N}_1\mathbf{M}_2\mathbf{N}_1 & 0\end{pmatrix}.$$

Roganov and Stovas, 2012

Weak contrast

$$\Delta m = 2\frac{m_2 - m_1}{m_2 + m_1},$$
$$\Delta m_a = m_{a2} - m_{a1}$$

Isotropic background
Weak contrast in elastic and anisotropy parameters

$$\mathbf{A}_0 = \begin{pmatrix} 0 & \mathbf{M}_0 \\ \mathbf{N}_0 & 0 \end{pmatrix}$$

$$\mathbf{N}_0 = -\begin{pmatrix} \frac{1}{\rho_0 v_{P0}^2} & \left(1-2\gamma_0^2\right)p_1 & \left(1-2\gamma_0^2\right)p_2 \\ \left(1-2\gamma_0^2\right)p_1 & \rho_0\left(1-4\left(1-2\gamma_0^2\right)p_1^2 v_{S0}^2 - p_2^2 v_{S0}^2\right) & -\rho_0 p_1 p_2 v_{S0}^2\left(3-4\gamma_0^2\right) \\ \left(1-2\gamma_0^2\right)p_2 & -\rho_0 p_1 p_2 v_{S0}^2\left(3-4\gamma_0^2\right) & \rho_0\left(1-p_1^2 v_{S0}^2 - 4\left(1-2\gamma_0^2\right)p_2^2 v_{S0}^2\right) \end{pmatrix}$$

$$\mathbf{M}_0 = -\begin{pmatrix} \rho_0 & p_1 & p_2 \\ p_1 & \frac{1}{\rho_0 v_{S0}^2} & 0 \\ p_2 & 0 & \frac{1}{\rho_0 v_{S0}^2} \end{pmatrix}$$

$$o(2) \equiv o\left(d\rho^2, dv_P^2, dv_S^2, \varepsilon_1^2, \varepsilon_2^2, \delta_1^2, \delta_2^2, \delta_3^2, \gamma_1^2, \gamma_2^2\right)$$

$$\mathbf{A}_1 = \mathbf{A}_0 - \Delta\mathbf{A} + \Delta^2\mathbf{A},$$

Matrix series with respect to contrast

$$\mathbf{A}_2 = \mathbf{A}_0 + \Delta\mathbf{A} + \Delta^2\mathbf{A},$$

Weak contrast

$$\mathbf{F_0} = \mathbf{A}_0 - (1-2\alpha)\boldsymbol{\Delta}\mathbf{A} + \boldsymbol{\Delta}^{\mathbf{2}}\mathbf{A} + o(2)$$

$$\mathbf{F_1} = (1-\alpha)\alpha\left[\boldsymbol{\Delta}\mathbf{A}, \mathbf{A}_0\right],$$

$$\mathbf{F_2} = \frac{1}{6}(1-\alpha)\alpha\left\{-(1-2\alpha)\left[\mathbf{A}_0, \left[\boldsymbol{\Delta}\mathbf{A}, \mathbf{A}_0\right]\right] + \left[\boldsymbol{\Delta}\mathbf{A}, \left[\boldsymbol{\Delta}\mathbf{A}, \mathbf{A}_0\right]\right]\right\},$$

Weak contrast

$$\tilde{\mathbf{A}}(\omega) = \mathbf{R_0} + \mathbf{R}_1 + \mathbf{R}_2$$

Matrix series with respect to contrast

$$\mathbf{R_0} = \mathbf{A}_0,$$

$$\mathbf{R}_1 = -(1-2\alpha)\mathbf{\Delta A} + i\omega H\alpha(1-\alpha)[\mathbf{\Delta A}, \mathbf{A}_0] - (i\omega H)^2 \frac{\alpha(1-\alpha)(1-2\alpha)}{6}[\mathbf{A}_0, [\mathbf{\Delta A}, \mathbf{A}_0]],$$

$$\mathbf{R}_2 = \mathbf{\Delta^2 A} + (i\omega H)^2 \frac{\alpha(1-\alpha)}{6}[\mathbf{\Delta A}, [\mathbf{\Delta A}, \mathbf{A}_0]].$$

No second-order contrasts in dispersion terms!

Characteristic equation (eigenvalues)

$$\det\left[\mathbf{A}(\omega) - q\mathbf{I}\right] = 0$$

$$q^6 + a_4(\omega)q^4 + a_2(\omega)q^2 + a_0(\omega) = 0$$

Characteristic equation (eigenvalues)

$$q_j^2 = q_{j0}^2 + \omega^2 H^2 d_j + o\left(2, \omega^3\right)$$

$$d_j = \frac{2\alpha^2\left(1-\alpha\right)^2}{3} q_j^{(0)} k_j^{-1} \boldsymbol{\psi}_j^{(0)} \boldsymbol{\Delta}\mathbf{A}\left(\mathbf{A}_0 - q_j^{(0)}\mathbf{I}\right)\boldsymbol{\Delta}\mathbf{A}\boldsymbol{\varphi}_j^{(0)}$$

$$k_j = \boldsymbol{\psi}_j^{(0)}\boldsymbol{\varphi}_j^{(0)}$$

Characteristic equation (P-eigenvalues)

$$k_P = \boldsymbol{\psi}_P^{(0)} \boldsymbol{\varphi}_P^{(0)}$$

$$\boldsymbol{\varphi}_P^{(0)} = \left(\boldsymbol{\varphi}_{P1}^{(0)}, \boldsymbol{\varphi}_{P2}^{(0)} \right)^T$$

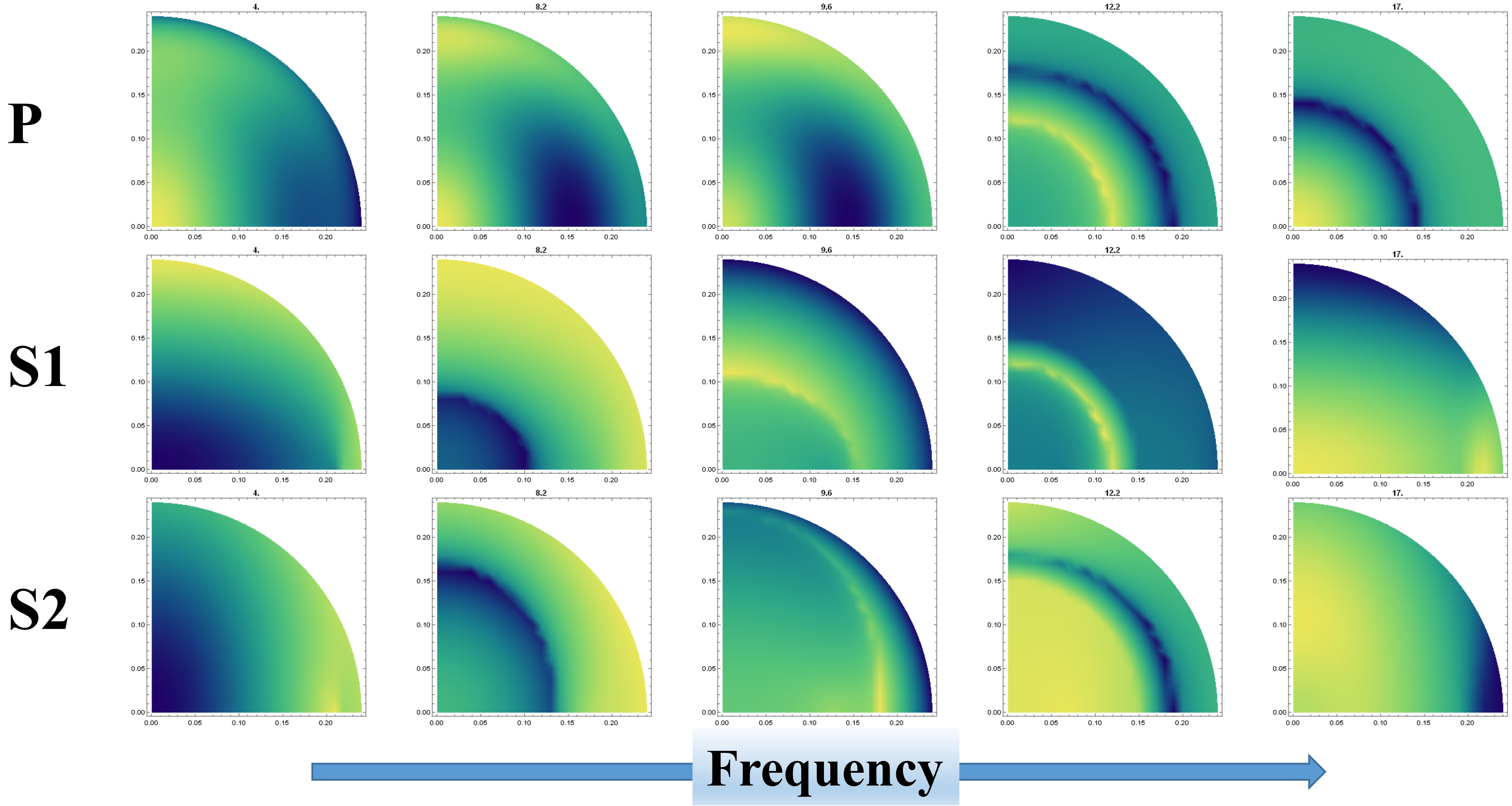
$$\boldsymbol{\psi}_P^{(0)} = \left(\boldsymbol{\varphi}_{P2}^{(0)}, \boldsymbol{\varphi}_{P1}^{(0)} \right)$$

$$\boldsymbol{\varphi}_{P1}^{(0)} = \left(-\rho_0 \left(1 - 2\left(p_1^2 + p_2^2 \right) v_{S0}^2 \right), p_1, p_2 \right)$$

$$\boldsymbol{\varphi}_{P2}^{(0)} = \left(q_P^0, -2 q_P^0 p_1 \rho_0 v_{S0}^2, -2 q_P^0 p_2 \rho_0 v_{S0}^2 \right)$$

$$k_P = -2 \rho_0 q_P^{(0)}$$

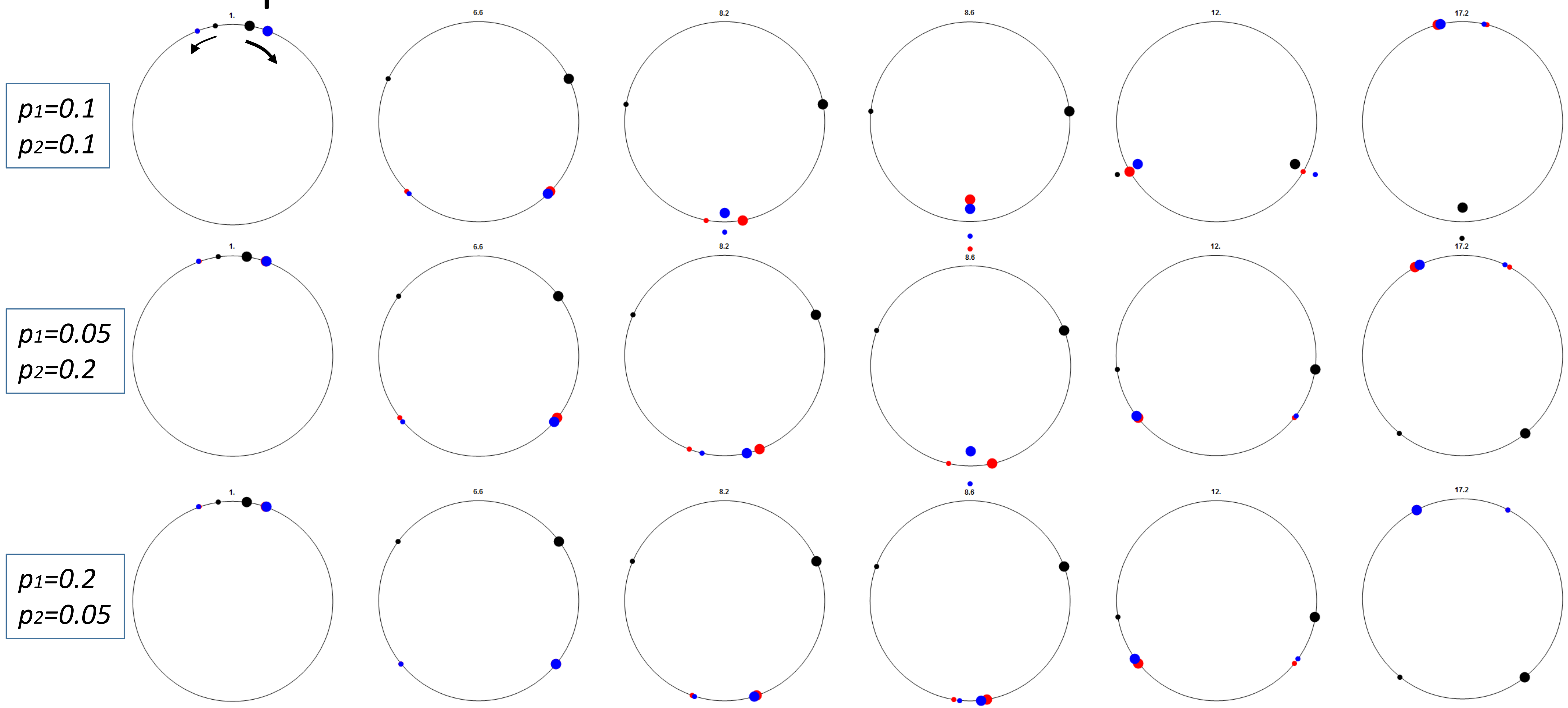
Slowness surface dispersion



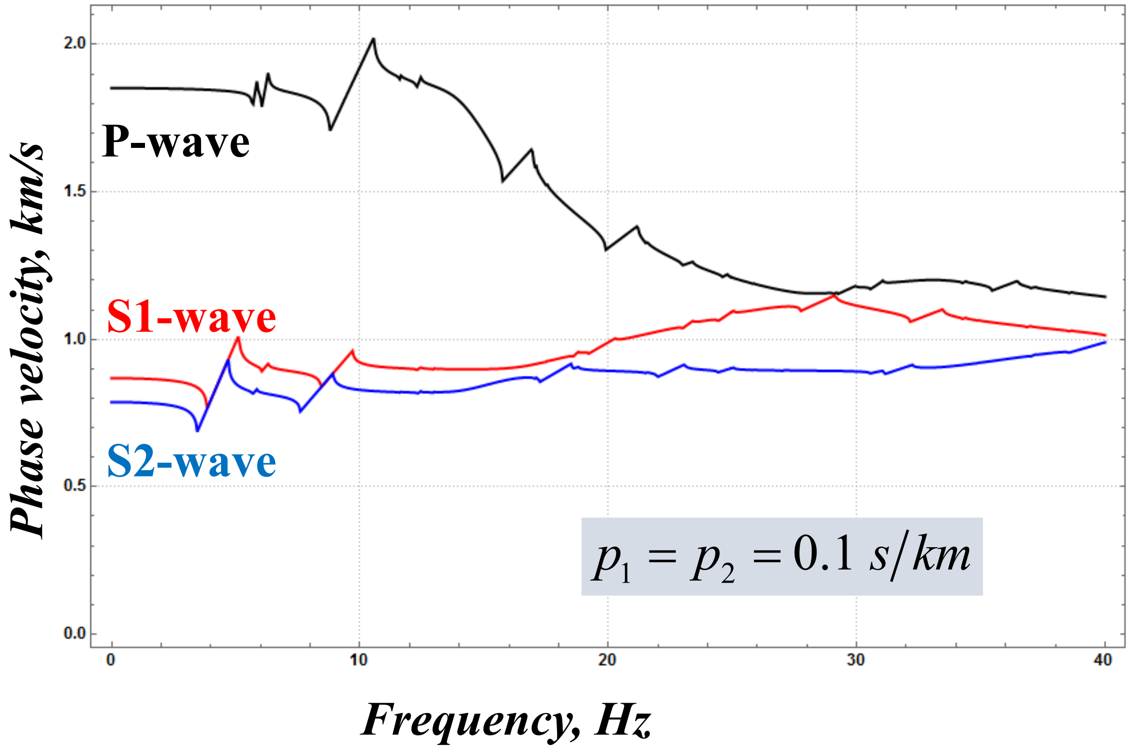
Multipliers

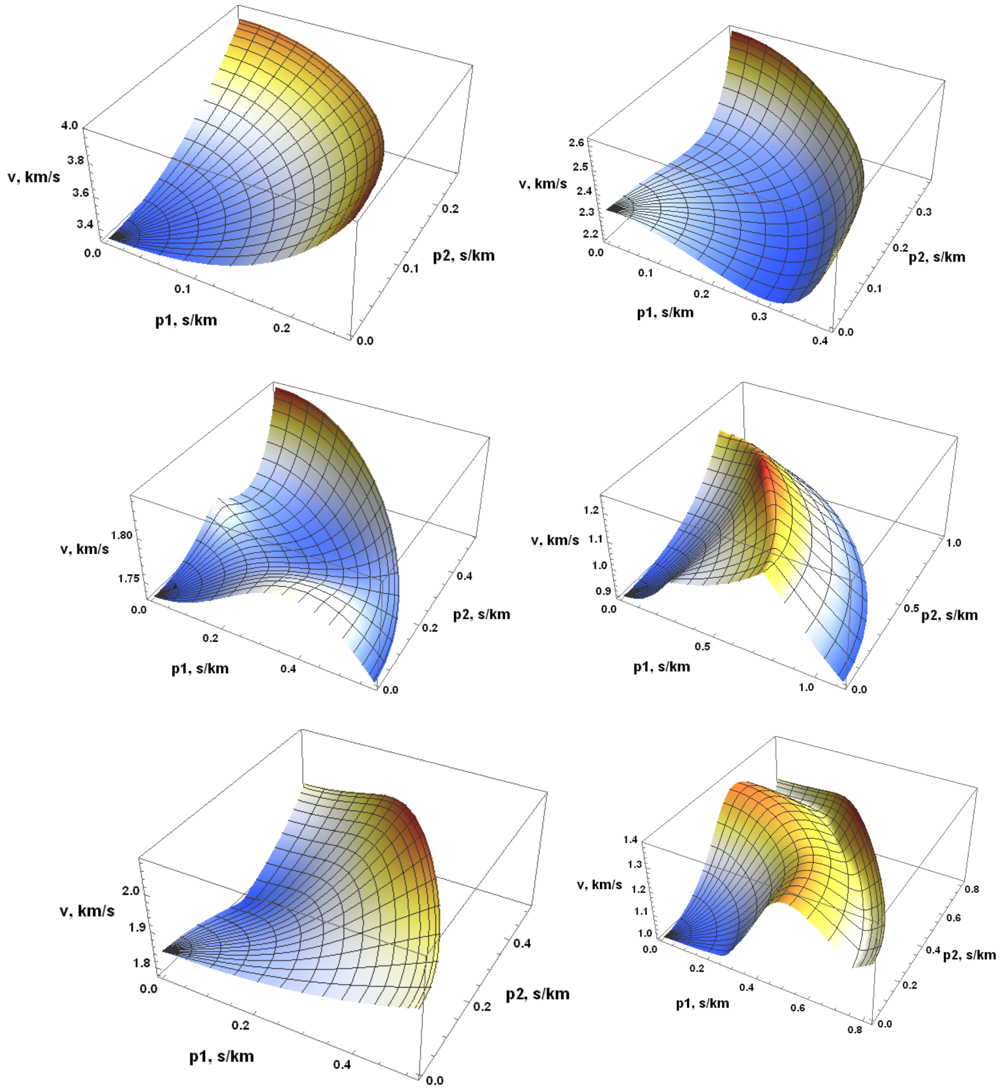
$$e^{i\phi(p_1, p_2, \omega)}$$

$$\phi\left(p_1, p_2, \omega=0\right) = 0$$



Frequency-dependent phase velocity





Wave mode selection

Trial series for dispersion coefficient:

$$d_j = a_{00}^{(j)} + a_{20}^{(j)} p_1^2 + a_{02}^{(j)} p_2^2 + a_{40}^{(j)} p_1^4 + a_{22}^{(j)} p_1^2 p_2^2 + a_{04}^{(j)} p_2^4 + o(2)$$

Three solutions for a_{00} that give the wave mode selection.

Quadratic form

$$a_{00}^{(qP)} = \frac{\alpha^2 (1-\alpha)^2}{3 v_{P0}^4} (\Delta v_P + \Delta \rho)^2 ,$$

$$a_{20}^{(qP)} = \frac{\alpha^2 (1-\alpha)^2}{12 v_{S0}^2} \mathbf{m}_{20} \mathbf{D}_2^{(qP)} \mathbf{m}_{20}^T ,$$

$$a_{02}^{(qP)} = \frac{\alpha^2 (1-\alpha)^2}{12 v_{S0}^2} \mathbf{m}_{02} \mathbf{D}_2^{(qP)} \mathbf{m}_{02}^T ,$$

$$a_{40}^{(qP)} = \frac{\alpha^2 (1-\alpha)^2}{12 \gamma_0^2} \mathbf{m}_{40} \mathbf{D}_4^{(qP)} \mathbf{m}_{40}^T ,$$

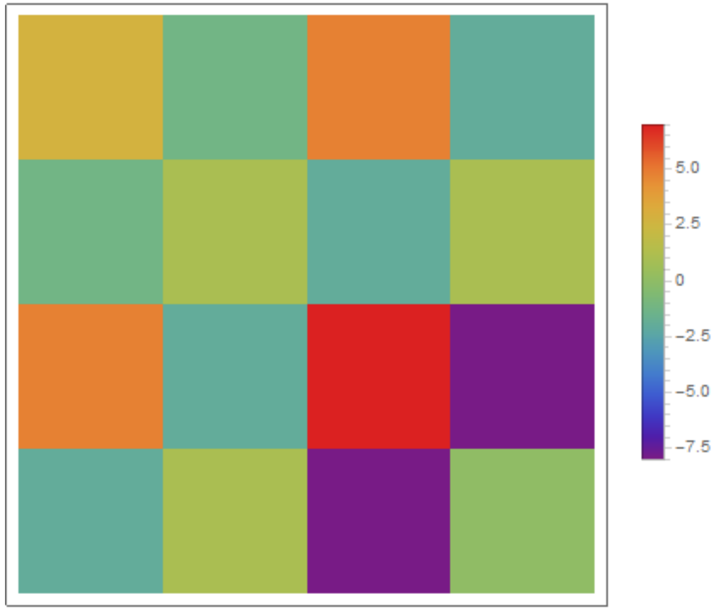
$$a_{04}^{(qP)} = \frac{\alpha^2 (1-\alpha)^2}{12 \gamma_0^2} \mathbf{m}_{04} \mathbf{D}_4^{(qP)} \mathbf{m}_{04}^T ,$$

$$a_{22}^{(qP)} = \frac{\alpha^2 (1-\alpha)^2}{6 \gamma_0^2} \mathbf{m}_{22} \mathbf{D}_{22}^{(qP)} \mathbf{m}_{22}^T ,$$

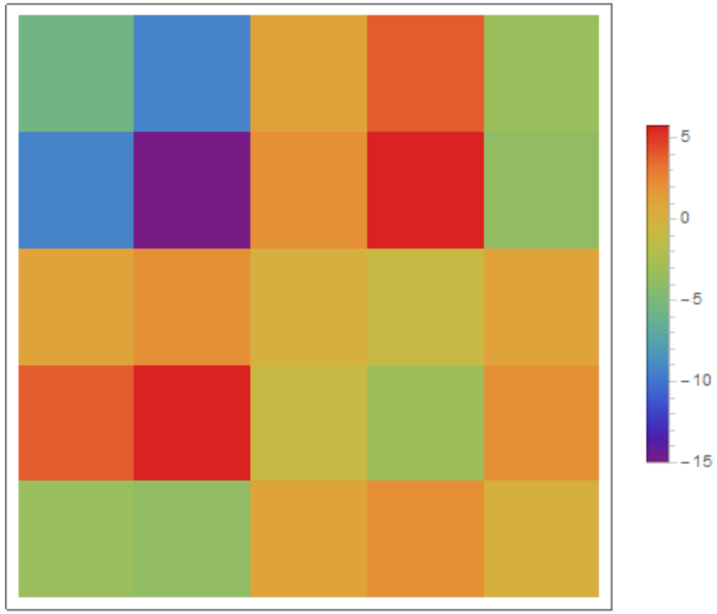
$$\mathbf{m}_{20} = (\Delta\rho, \Delta\delta_2, \Delta v_S, \Delta v_P), \qquad \mathbf{m}_{02} = (\Delta\rho, \Delta\delta_1, \Delta v_S + \Delta\gamma_1 - \Delta\gamma_2, \Delta v_P),$$

$$\mathbf{m}_{40} = (\Delta\rho, \Delta v_S, \Delta v_P, \Delta\delta_2, \Delta\varepsilon_2), \qquad \mathbf{m}_{04} = (\Delta\rho, \Delta v_S + \Delta\gamma_1 - \Delta\gamma_2, \Delta v_P, \Delta\delta_1, \Delta\varepsilon_1),$$

$$\mathbf{m}_{22} = (\Delta\rho, \Delta v_S, \Delta v_P, \Delta\delta_1, \Delta\gamma_1 - \Delta\gamma_2, \Delta\delta_3 + 2\Delta\varepsilon_2).$$



$\mathbf{D}_2$



$\mathbf{D}_4$

Conclusions

- We derive the low frequency approximation for waves propagating in multi-layered orthorhombic model.
- The weak-contrast approximation is introduced.
- We show that the stop-bands are the result of interaction of different wave modes (P, S1 and S2).
- The stop-bands are illustrated by multipliers.
- By defining the low-frequency effective anisotropic parameters, we perform the sensitivity analysis for intrinsic anisotropy parameters.