Elimination of time dispersion from wave equation modelling in elastic and anelastic media

Lasse Amundsen & Ørjan Pedersen

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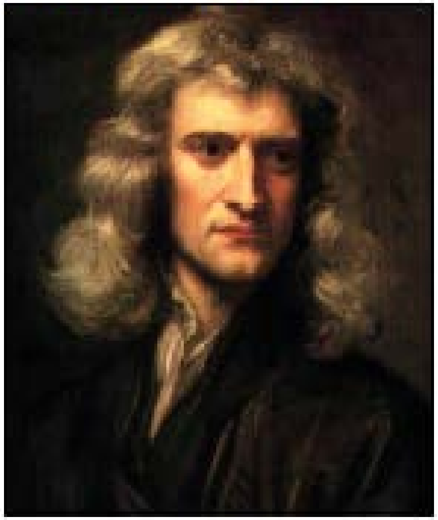
Finite Difference Modeling: Part I
Become Expert in Five Minutes

I recoil with dismay and horror at this lamentable plague of functions which do not have derivatives.

Letter to Dutch mathematician Thomas Joannes Stieltjes (1856–1894) from French mathematician Charles Hermite (1822–1901).

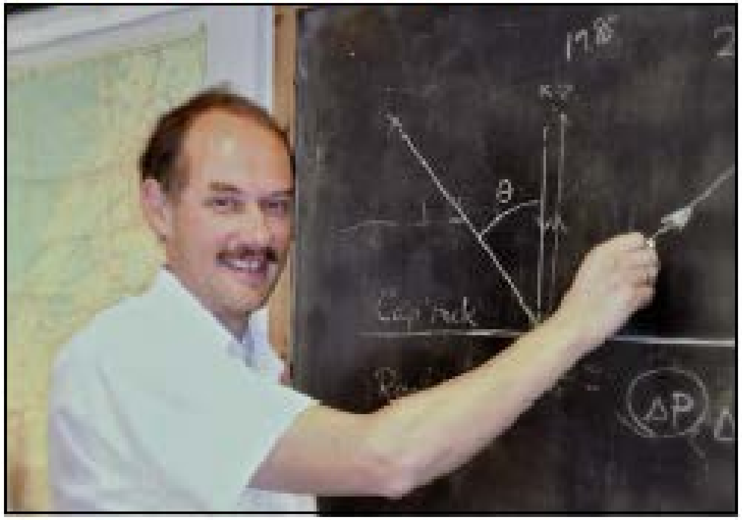
LASSE AMUNDSEN, ØRJAN PEDERSEN, and MARTIN LANDRØ

Newton, 1666:
"I've invented calculus!"

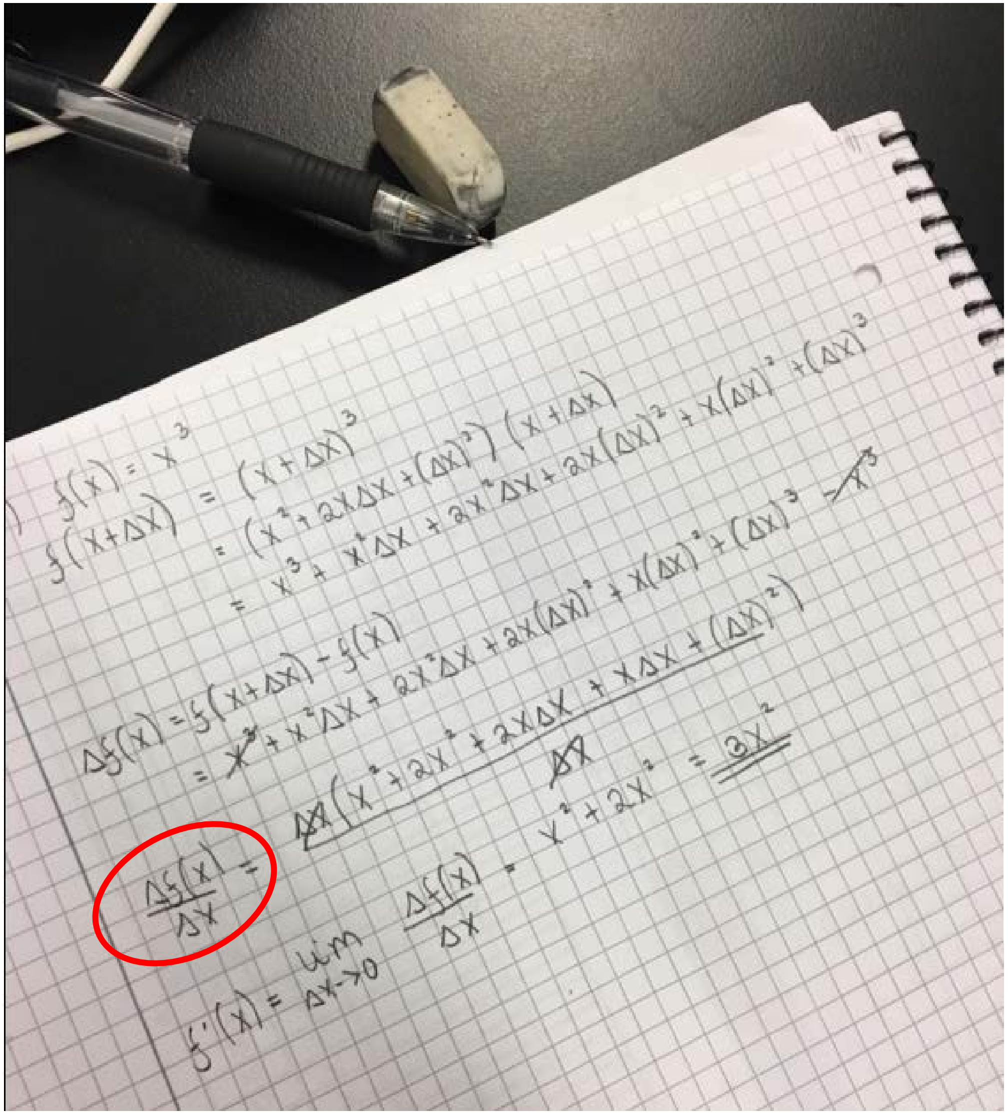


Leibniz, 1674:
"I've invented calculus!"

Landrø, 2014:
"Really? Sounds a little bit… derivative."



www.geoexpro.com or www.bivrostgeo.no (to appear)



High-school students are FD experts,
and are cheap to hire for our industry

$$\frac{\partial f}{\partial t} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} \quad ; \quad \frac{\Delta f}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

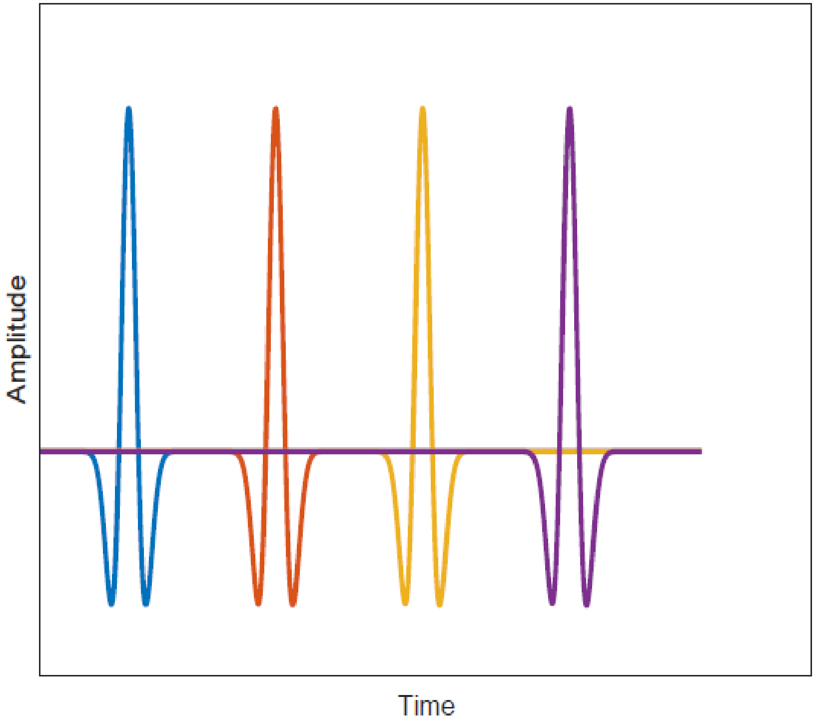
$$f = t^3 \; ; \; \frac{\Delta f}{\Delta t} = 3t^2 + 3t\Delta t + (\Delta t)^2$$

Error

$$\frac{\partial^2 f}{\partial t^2} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{(\Delta t)^2}$$

In a homogeneous 1-D medium, the FD solution of the wave equation (for small Δt and Δx) produces a wave that moves outwards from the source without change, in agreement with the solution which d'Alembert discovered in 1746.

$$\frac{\partial^2 u(t,x)}{\partial t^2} = c^2(x)\frac{\partial^2 u(t,x)}{\partial x^2} + s(t,x)$$



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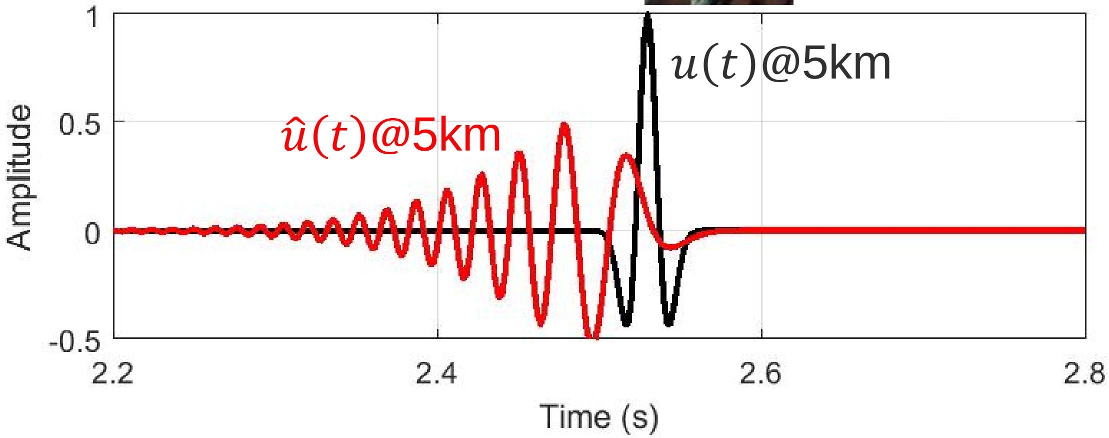
However, too large Δt gives Time dispersion

$$\frac{\partial^2 u(x,t)}{\partial t^2} = L(x)u(x,t) + a(t)\delta(x)$$

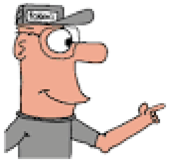


$c = 2000$ m/s

$\Delta t = 2$ ms

$$\frac{D^2\hat{u}(x,t)}{Dt^2} = L(x)\hat{u}(x,t) + a(t)\delta(x)$$

$$\frac{D^2\hat{u}(\boldsymbol{x},t)}{Dt^2} = \frac{\hat{u}(\boldsymbol{x},t+\Delta t) - 2\hat{u}(\boldsymbol{x},t) + \hat{u}(\boldsymbol{x},t-\Delta t)}{(\Delta t)^2}$$



How to clean dispersion

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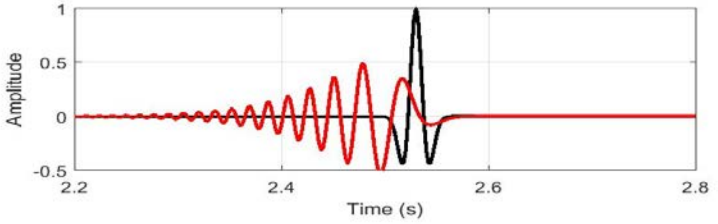
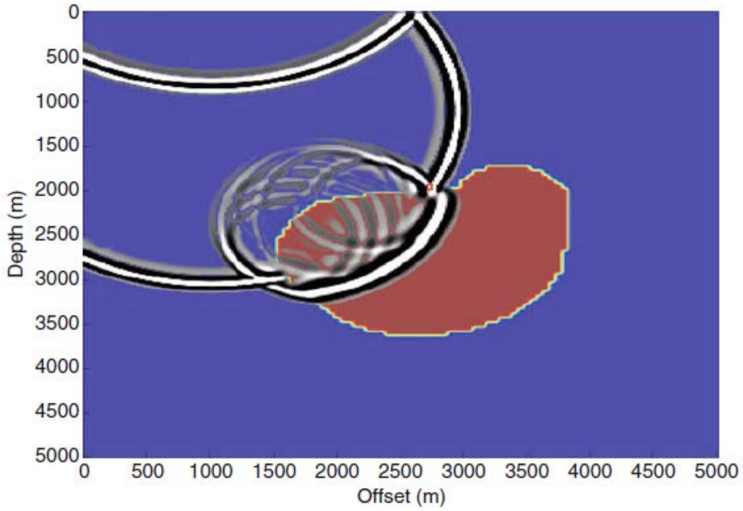
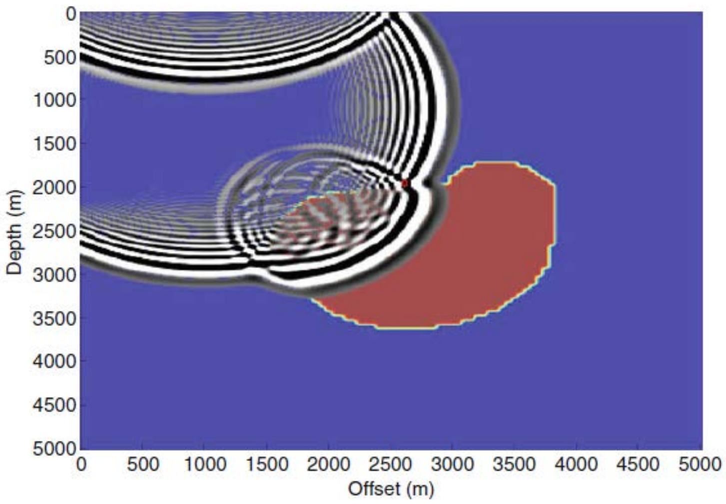
Numerical dispersion

Numerical dispersion is the separation of different Fourier components of an FD approximation into a train of oscillations that travel with different speeds.
It occurs whenever the dispersion relation for the difference approximation is nonlinear.

- Temporal dispersion leads the signal …
- Spatial dispersion follows the signal …
- Both can be corrected …







Robertsson and Blanch, 2011

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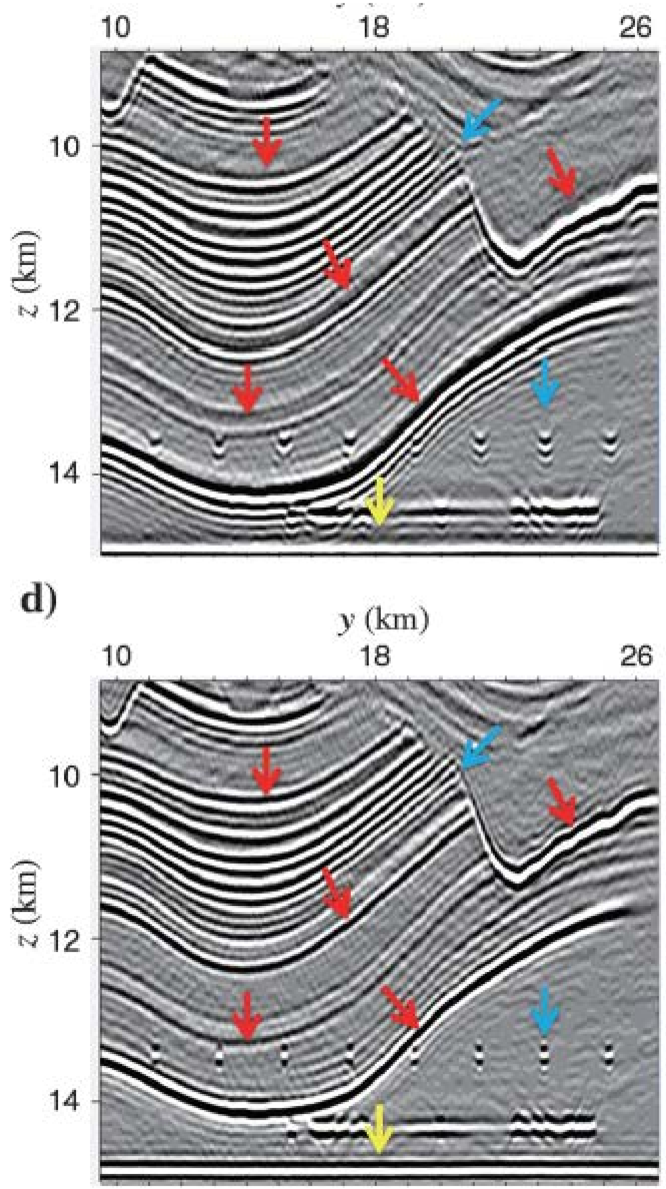
Numerical temporal dispersion is independent of the wave's propagation path – and is cleaned post-modeling

Zhang et al (2011, 2013) – Chevron (patent)
Stork (2013)
Dai et al. (2014)
Wang and Xu (2015) - Statoil Gulf Services
Anderson et al. (2015) - Exxon
Li et al. (2016)
Qin et al. (2017)
Xu et al. (2017) - WesternGeco
Mittet (2017) - EMGS
Koene et al. (2017) - ETH
Amundsen and Pedersen (2018) (absorption)



Wang and Xu (2015)

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Our approach for elastic media

The discrete FD equations are not identical to the original differential equations …

$$\frac{\partial^2 u(x,t)}{\partial t^2} = L(x)u(x,t) + s(x,t)$$

$$\frac{D^2 \hat{u}(x,t)}{Dt^2} = L(x)\hat{u}(x,t) + \hat{s}(x,t,\Delta t) \qquad \frac{D^2 \hat{u}(\boldsymbol{x},t)}{Dt^2} = \frac{\hat{u}(\boldsymbol{x},t+\Delta t) - 2\hat{u}(\boldsymbol{x},t) + \hat{u}(\boldsymbol{x},t-\Delta t)}{(\Delta t)^2}$$

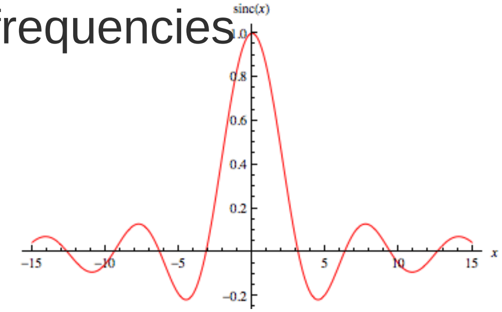
When we use the Fourier transform to relate the two differential equations and their solutions …

$$F(\omega) = \int_{-\infty}^{\infty} dt \exp(i\omega t) f(t) \qquad |\omega| \le \frac{\pi}{\Delta t}$$

we find that the exact and FD wave equations then have the frequencies

$$\omega$$

$$\widehat{\omega} = \widehat{\omega}(\omega, \Delta t) = \frac{2}{\Delta t}\sin\left(\frac{\omega \Delta t}{2}\right) = \omega \operatorname{sinc}\left(\frac{\omega \Delta t}{2}\right)$$



Fourier transforms yield

$$[\widehat{\omega}^2 + L(x)]u(x,\widehat{\omega}) = -s(x,\widehat{\omega}).$$

$$F(\widehat{\omega}) = \int_{-\infty}^{\infty} dt \exp(i\widehat{\omega}t) f(t)$$

$$[\widehat{\omega}^2 + L(x)]\hat{u}(x,\omega) = -\hat{s}(x,\omega,\Delta t)$$

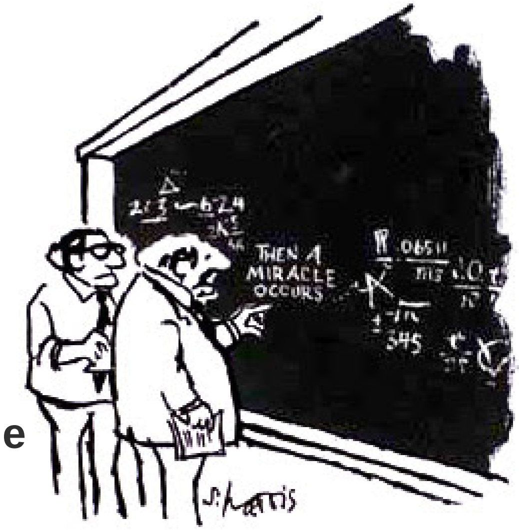
$$F(\omega) = \int_{-\infty}^{\infty} dt \exp(i\omega t) f(t)$$

To keep our courage up, choose

$$\hat{s}(x,\omega,\Delta t) = s(x,\widehat{\omega})$$

THEN A MIRACLE OCCURS

$$\hat{u}(x,\omega) = u(x,\widehat{\omega})$$

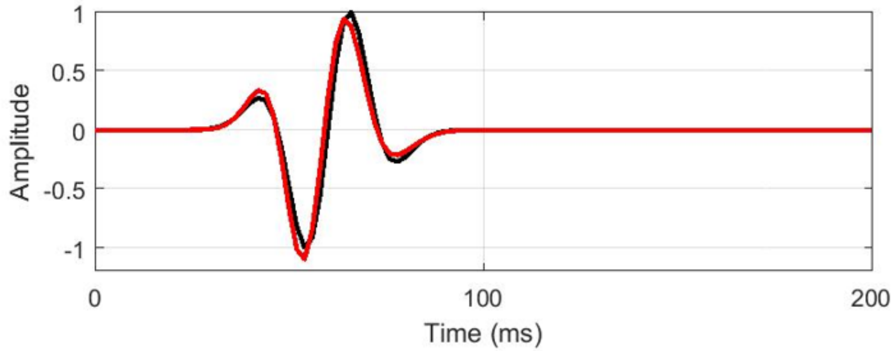


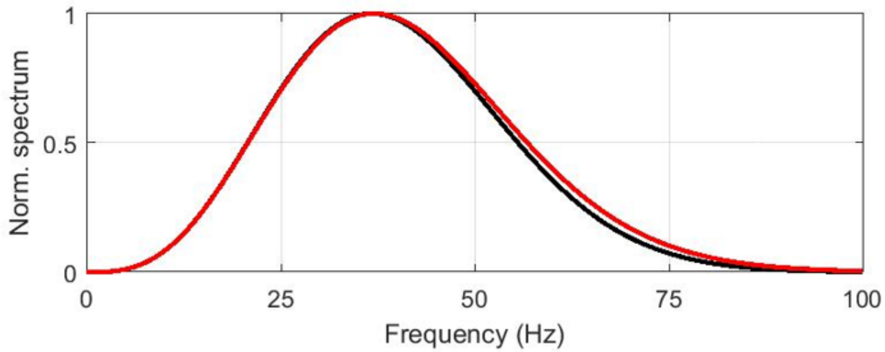
"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Given a solution to one of the wave equations, we can derive the solution to the other wave equation by employing the Fourier transformation.

$$u(x,t) = \frac{1}{2\pi} \int_{-\frac{\pi}{\Delta t}}^{\frac{\pi}{\Delta t}} d\omega \exp(-i\widehat{\omega}t) \cos\left(\frac{\omega\Delta t}{2}\right) \int_{0}^{\infty} dt' \exp(i\omega t')\, \hat{u}(x,t')$$

Elimination of time dispersion for elastic media: Change of signatures



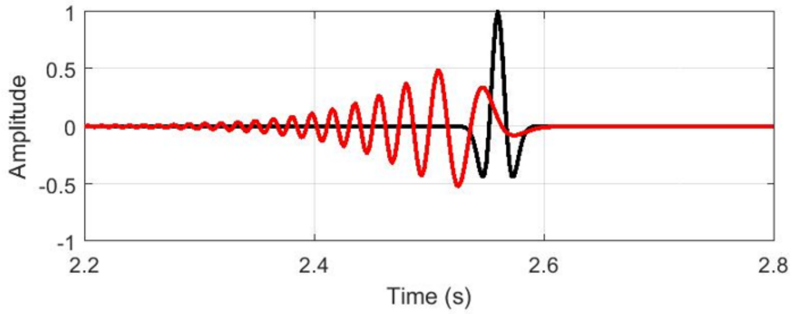


Time derivative of Ricker signature.

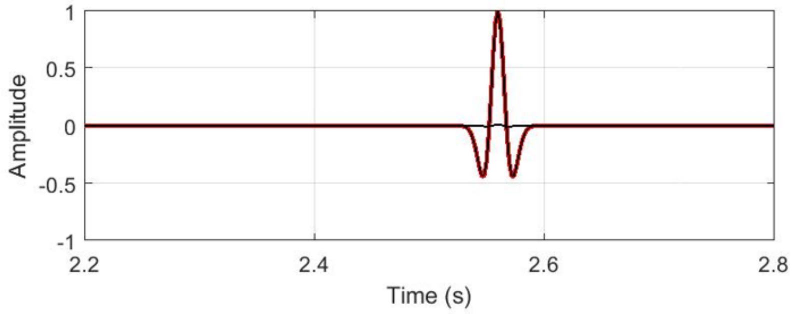
Black: Signature for original differential equation.

Red: Signature for FD equation.

Elimination of time dispersion for elastic media



Reference, exact solution (black line), and FD modeled trace contaminated with numerical dispersion (red line)



Dispersion corrected solution (red line) and error (thin black line)

Our approach for anelastic media

The discrete FD equations are not identical to the original differential equations …

$$[\widehat{\omega}^2 + L(x,\widehat{\omega})]u(x,\widehat{\omega}) = -a(\widehat{\omega})\delta(x)$$

$$[\widehat{\omega}^2 + L_m(x,\omega)]\hat{u}_m(x,\omega) = -\hat{a}(\omega)\delta(x)$$

$\hat{u}_m$ is modelled in time-space, not with the system operator (medium) of interest $L(x,t)$, but with the frequency-modified system operator $L_m(x,t)$.

Choose

$$\hat{a}(\omega) = a(\widehat{\omega})$$

Then

$$\hat{u}_m(x,\omega) = u(x,\widehat{\omega})$$

Our approach for anelastic media

Finite-difference experts tend to implement realistic attenuation in the time-domain methods by using a generalized Maxwell body or generalized Zener body. Since the rheology of the two models is one and the same, we choose the modulus

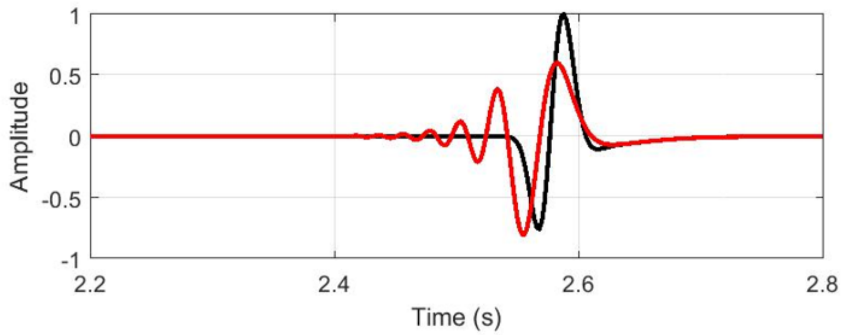
$$M(x,t) = M_u(x)\delta(t) - \Delta M(x) \sum_{j=1}^{N} a_j \omega_j \exp(-\omega_j t) H(t)$$

For FD modelling we should use the modulus

$$M_m(x,t) = M_u'(x)\delta(t) - \Delta M \sum_{j=1}^{N} a_j' \omega_j' \exp(-\omega_j' t) H(t)$$

$$M_u' = M_u - \Delta M \sum_{j=1}^{N} a_j \frac{\omega_j \Delta t}{2} \left[\frac{\frac{\omega_j \Delta t}{2}}{2\left(1+\left(\frac{\omega_j \Delta t}{2}\right)^2\right)} - \frac{\left(2-\left(\frac{\omega_j \Delta t}{2}\right)^2\right)\operatorname{arcsinh}\left(\frac{\omega_j \Delta t}{2}\right)}{12\left(1+\left(\frac{\omega_j \Delta t}{2}\right)^2\right)^{\frac{3}{2}}} + O\left(\operatorname{arcsinh}\left(\frac{\omega_j \Delta t}{2}\right)\right)^2 \right], \qquad \omega_j' = \frac{2}{\Delta t}\operatorname{arcsinh}\left(\frac{\omega_j \Delta t}{2}\right)$$

Elimination of time dispersion for elastic media



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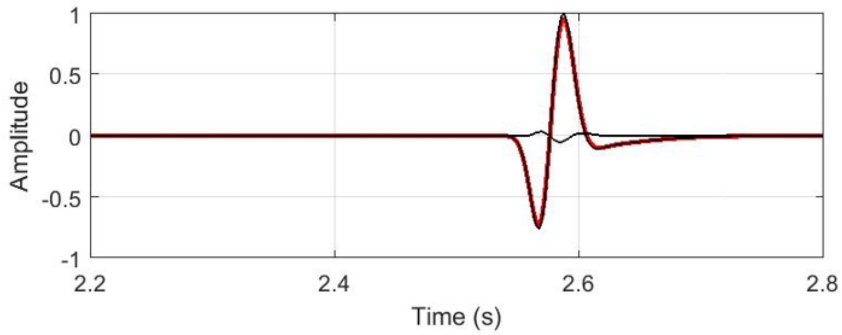
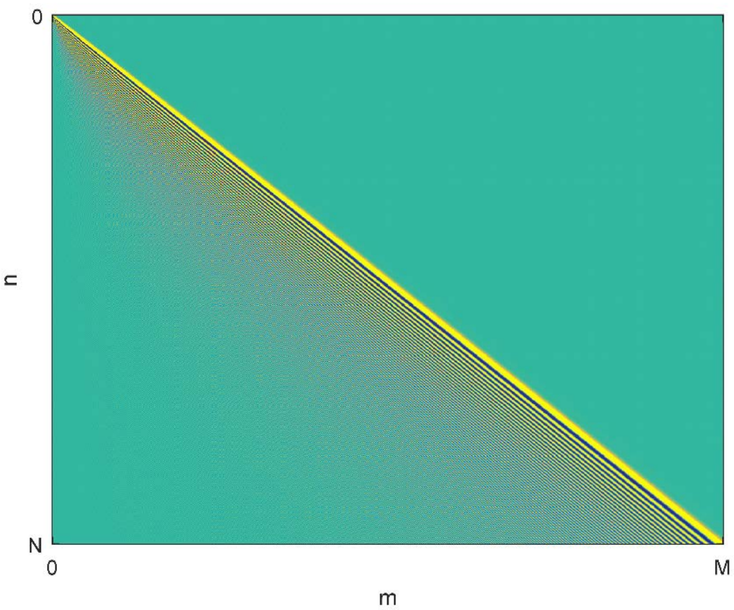
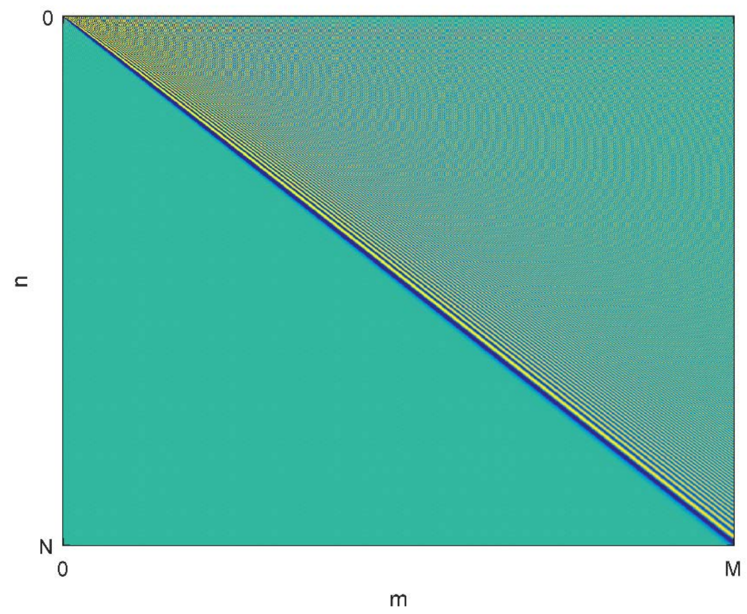


Dispersion corrected solution (red line) and error (thin black line)

Digital filters for eliminating and adding dispersion are – independent of sampling interval! (Mittet, pers.comm.)





$$u(n) = \sum_{m} F(n,m)\hat{u}(m)$$

$$F(n,m) = \sum_{k=0}^{m} (-1)^k f_c(n,m,k) j_{m-k}(2n) + \sum_{k=0}^{m-1} (-1)^k f_s(n,m,k)\, J_{1+m-k}(2n)$$

$$\hat{u}(n) = \sum_{m} G(n,m)\, u(m)$$

$$G(n,m) = J_{2n}(2m) + \sum_{k=0}^{n-1} (-1)^k g_s(n,m,k)\, j_{n-k}(2m)$$

Conclusions

- High-school students early on learn FD, and are FD experts on par with the industry experts (read: me and Ørjan)

- Students: Patents are a source of information where you get early information about news and trends in petroleum seismics