NTNU
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Computation of the Hessian for Full Waveform Inversion

Vegard Stenhjem Hagen

Iterative methods[1]

- Searching for a model **m** that describes the earth.
- Solve the wave equation

$$\frac{\partial^2}{\partial t^2}\mathbf{u} - \mathbf{c}^2\nabla^2\mathbf{u} = \mathbf{f}, \quad \mathbf{m} = [\mathbf{c}],$$

  where **u** is displacement and the model **m** only consists of velocities **c**, **f** is a source function.
- Compare with true data $\mathbf{u}_0$ using a misfit function

$$\Psi\left(\mathbf{u}(\mathbf{m},\mathbf{x}_r),\mathbf{u}_0\right) = \frac{1}{2}\left(\mathbf{u}_0(\mathbf{x}_r) - \mathbf{u}(\mathbf{m},\mathbf{x}_r)\right)^T\left(\mathbf{u}_0(\mathbf{x}_r) - \mathbf{u}(\mathbf{m},\mathbf{x}_r)\right).$$

- Iterative approach. Find a model update $\delta\mathbf{m}_k$ that decreases the misfit

$$\Psi(\mathbf{m}_{k+1} = \mathbf{m}_k + \delta\mathbf{m}_k) < \Psi(\mathbf{m}_k).$$

[1] Tarantola 1984; Mora 1987; Fichtner et al. 2006; Fichtner 2011.

Iterative methods

— To do this we can calculate the gradient of the misfit

$$\mathbf{J}(\mathbf{m}+\delta\mathbf{m}) = \nabla_m \Psi(\mathbf{m}+\delta\mathbf{m}).$$

— By linearising around the Jacobian we get

$$\mathbf{J}(\mathbf{m}+\delta\mathbf{m}) \simeq \mathbf{J}(\mathbf{m}) + \nabla_m \mathbf{J}(\mathbf{m})\,\delta\mathbf{m} = \mathbf{0}.$$

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— The Hessian is then given by

$$\mathbb{H}(\mathbf{m}) = \nabla_m \mathbf{J}(\mathbf{m}) = \nabla_m \nabla_m \Psi(\mathbf{m}).$$

Newton method[2]

— By solving

$$\mathbb{H}(\mathbf{m})\delta\mathbf{m} = -\mathbf{J}(\mathbf{m})$$

for $\delta\mathbf{m}$ we find the next model update.

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$$\delta\mathbf{m} = -\mathbb{H}^{-1}\mathbf{J}.$$

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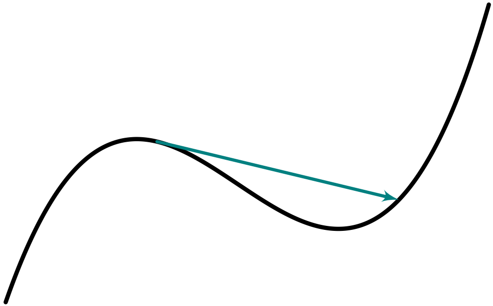
— A common approximation is

$$\delta\mathbf{m} \simeq \alpha\mathbf{J},$$

and a line search for the optimal $\alpha \in \mathbf{R}$.

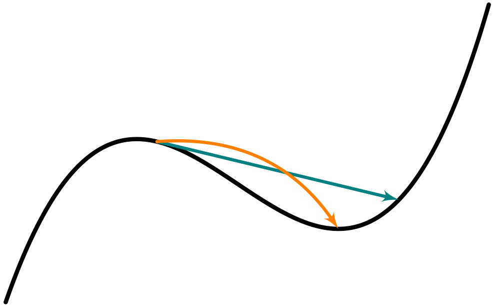
[2]Virieux and Operto 2009; Métivier et al. 2012; Epanomeritakis et al. 2008.

Advantages of using the full Hessian



Linear approximation

Advantages of using the full Hessian



Adjust for slope

Advantages of using the full Hessian[3]

— Fewer iterations are expected.

— Possible to perform resolution analysis.

— Parameter cross-talk analysis.

— Incorporate second order effects.

— Extra transmission-like information.

[3] Pratt et al. 1998; Fichtner and Trampert 2011b; Trampert et al. 2013; Biondi et al. 2015.

Fréchet derivative

— Use the adjoint approach by Tarantola 1984 and further work in Fichtner and Trampert 2011a.

— The Fréchet derivative is defined as

$$\nabla_m \Psi(\mathbf{m})\delta\mathbf{m} = \lim_{\nu \to 0} \frac{1}{\nu}\left[\Psi(\mathbf{m} + \nu\delta\mathbf{m}) - \Psi(\mathbf{m})\right],$$

i.e. the derivative of Ψ with respect to $\mathbf{m}$ in the $\delta\mathbf{m}$ direction.

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— The Jacobian acting on a model perturbation $\delta\mathbf{m}$ we can be written as

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— The Jacobian acting on a model perturbation $\delta\mathbf{m}$ we can be written as

$$\mathbf{J}(\mathbf{m})\delta\mathbf{m} = \nabla_m \Psi(\mathbf{m})\delta\mathbf{m}.$$

— For the Hessian $\mathbb{H}$ we can calculate its action on a model perturbation $\delta\mathbf{m}$ as

$$\mathbb{H}(\mathbf{m})\delta\mathbf{m} = \nabla_m \nabla_m \Psi(\mathbf{m})\delta\mathbf{m}.$$

Fréchet kernel

— The Fréchet kernel is defined as the volumetric densities of the Fréchet derivative

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}V}\nabla_m\Psi$$

giving us the relation

$$\mathbf{J}\delta\mathbf{m} = \nabla_m\Psi\delta\mathbf{m} = \int_G \mathbf{F}\delta\mathbf{m}\mathrm{d}^3\mathbf{x}, \quad G \in \mathbb{R}^3.$$

— By studying **F** we can design more efficient inversion schemes and interpret the results in a meaningful way.

Fréchet kernel

— Introducing the adjoint wavefield $\mathbf{u}^\dagger$ defined as the solution to

$$\frac{\partial^2}{\partial t^2}\mathbf{u}^\dagger - \mathbf{c}^2\nabla^2\mathbf{u}^\dagger = -\big(\mathbf{u}_0(\mathbf{x}_r) - \mathbf{u}(\mathbf{x}_r)\big),$$

which is referred to as backpropagating the residuals.

— The Fréchet kernel can now be calculated as

$$\mathbf{F}(\mathbf{u}^\dagger, \mathbf{u}) = 2\mathbf{c}\int_T \nabla\mathbf{u}^\dagger \nabla\mathbf{u}\,\mathrm{d}t,$$

i.e. by cross correlating the divergence of the forward and adjoint displacement fields and scaling by the background velocity.

Perturbed fields

- In order to more easily calculate the Hessian we need to introduce two more fields.
- The perturbed forward field

$$\begin{aligned} \delta \mathbf{u} &= \nabla_m \mathbf{u} \delta \mathbf{m} \\ &= \lim_{\nu \to 0} \frac{1}{\nu} [\mathbf{u}^\dagger(\mathbf{m} + \nu \delta \mathbf{m}) - \mathbf{u}^\dagger(\mathbf{m})] \end{aligned}$$

- The perturbed adjoint field

$$\begin{aligned} \delta \mathbf{u}^\dagger &= \nabla_m \mathbf{u}^\dagger \delta \mathbf{m} \\ &= \lim_{\nu \to 0} \frac{1}{\nu} [\mathbf{u}(\mathbf{m} + \nu \delta \mathbf{m}) - \mathbf{u}(\mathbf{m})] \end{aligned}$$

Hessian kernels

- The Hessian kernel **H** can be broken down into three parts

$$\mathbf{H} = \mathbf{H}^{\mathbf{u}^\dagger,\delta\mathbf{u}} + \mathbf{H}^{\delta\mathbf{u}^\dagger,\mathbf{u}} + \mathbf{H}^{\mathbf{u}^\dagger,\mathbf{u}}.$$

- Written out these are

$$\mathbf{H}^{\mathbf{u}^\dagger,\delta\mathbf{u}} = 2\mathbf{c}\int_T \nabla(\mathbf{u}^\dagger)\cdot\nabla(\delta\mathbf{u})\mathrm{d}t$$
$$\mathbf{H}^{\delta\mathbf{u}^\dagger,\mathbf{u}} = 2\mathbf{c}\int_T \nabla(\delta\mathbf{u}^\dagger)\cdot\nabla(\mathbf{u})\mathrm{d}t$$
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— Written out these are

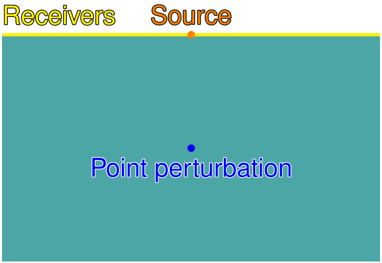
$$\begin{aligned}
\mathbf{H}^{\mathbf{u}^\dagger,\delta\mathbf{u}} &= 2\mathbf{c}\int_T \nabla(\mathbf{u}^\dagger)\cdot\nabla(\delta\mathbf{u})\mathrm{d}t &&= \mathbf{F}(\mathbf{u}^\dagger,\delta\mathbf{u}),\\
\mathbf{H}^{\delta\mathbf{u}^\dagger,\mathbf{u}} &= 2\mathbf{c}\int_T \nabla(\delta\mathbf{u}^\dagger)\cdot\nabla(\mathbf{u})\mathrm{d}t &&= \mathbf{F}(\delta\mathbf{u}^\dagger,\mathbf{u}),\\
\mathbf{H}^{\mathbf{u}^\dagger,\mathbf{u}} &= 2\delta\mathbf{c}\int_T \nabla(\mathbf{u}^\dagger)\cdot\nabla(\mathbf{u})\mathrm{d}t &&= \frac{\delta\mathbf{c}}{\mathbf{c}}\mathbf{F}(\mathbf{u}^\dagger,\mathbf{u}).
\end{aligned}$$

Fields

— We now have four fields we need to calculate.

$\mathbf{u}$ — Forward field.
$\mathbf{u}^{\dagger}$ — Adjoint field.
$\delta\mathbf{u}$ — Perturbed forward field.
$\delta\mathbf{u}^{\dagger}$ — Perturbed adjoint field.

Model



- Acoustic 2-D
- Background 2.0 km/s
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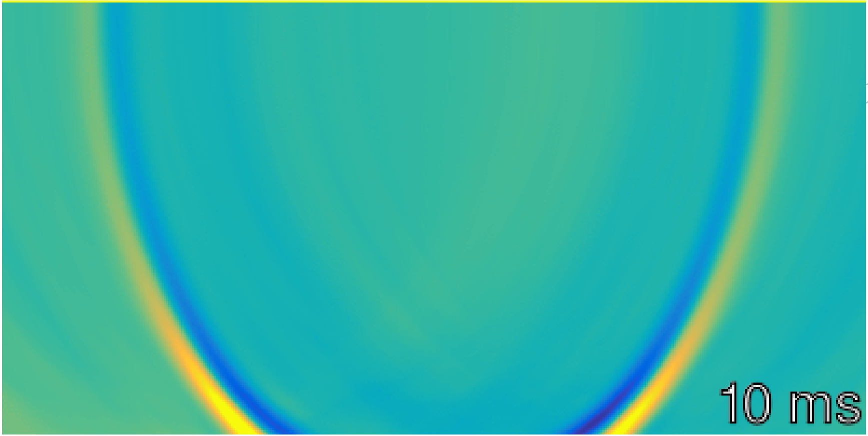
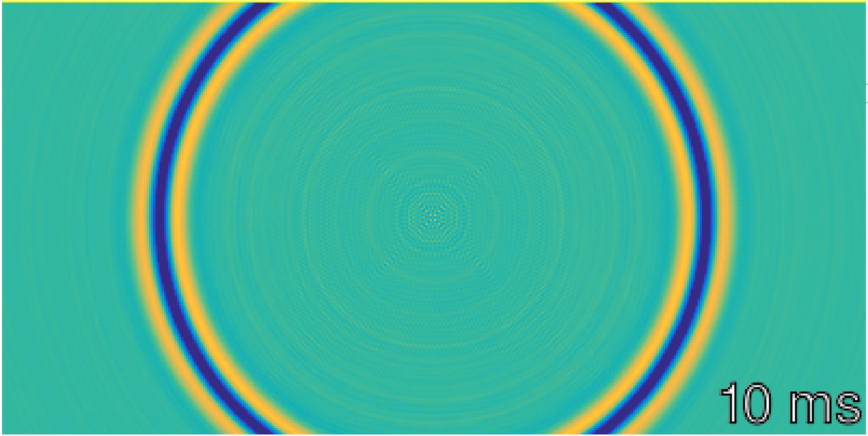
Forward field

Adjoint field



Perturbed forward field

Perturbed adjoint field

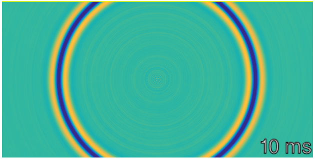


Fields

Forward field

Perturbed forward field

Adjoint field

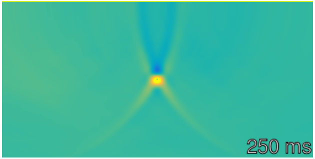


Perturbed adjoint field

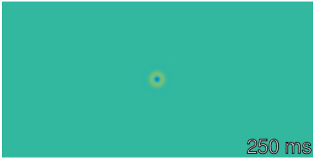
Fields

Forward field

Perturbed forward field



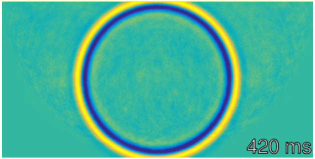
Adjoint field



Perturbed adjoint field

Fields

Forward field



Perturbed forward field

Adjoint field

Perturbed adjoint field

Reflection - Fréchet kernel — $\mathbf{F}(\mathbf{u}^{\dagger}, \mathbf{u})$

Divergence Correlation

Background field

Adjoint field

Reflection — $\mathbf{H}^{\mathbf{u}^\dagger,\delta\mathbf{u}} = \mathbf{F}(\mathbf{u}^\dagger,\delta\mathbf{u})$

Divergence Correlation

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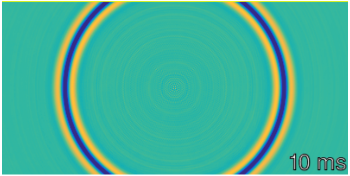
Divergence Correlation

Background field



Perturbed adjoint field

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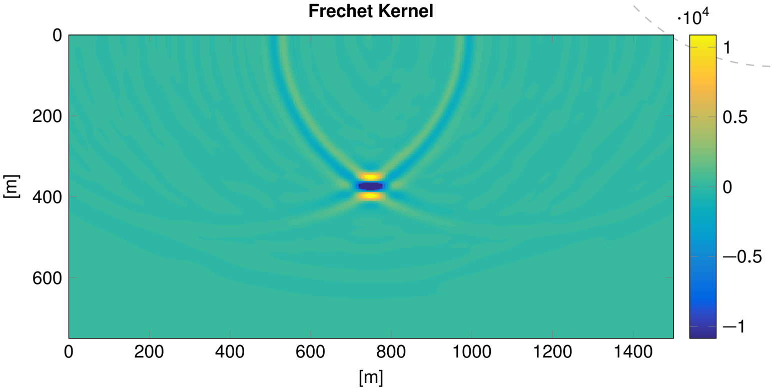
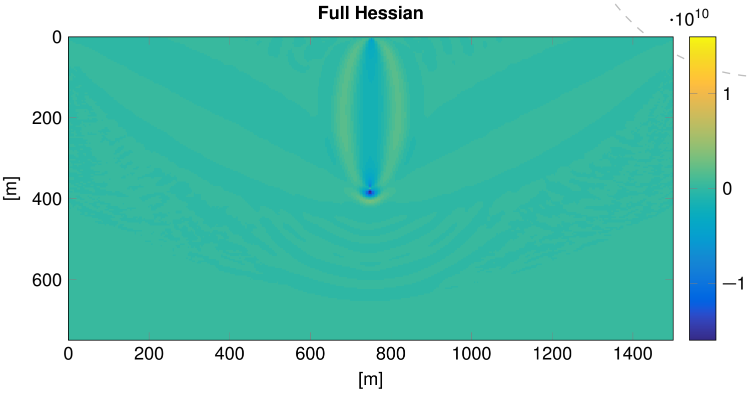
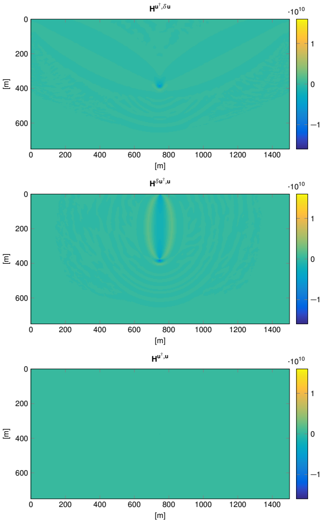
Reflection - Fréchet kernel — $F(u^{\dagger}, u)$



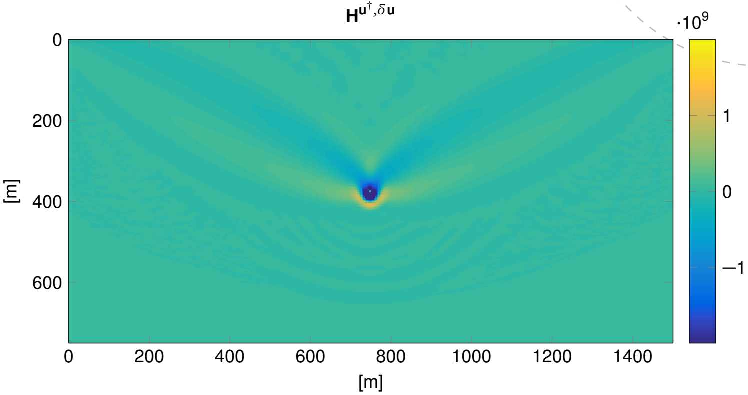
Reflection - Full Hessian Kernel — H



Reflection


$\mathbf{H}^{\mathbf{u}^\dagger,\delta\mathbf{u}}$
$\mathbf{H}^{\delta\mathbf{u}^\dagger,\mathbf{u}}$
$\mathbf{H}^{\mathbf{u}^\dagger,\mathbf{u}}$
$\cdot 10^{10}$
[m]
0
200
400
600
800
1000
1200
1400
1
0
−1

Reflection — $\mathbf{H}^{\mathbf{u}^\dagger, \delta\mathbf{u}} = \mathbf{F}(\mathbf{u}^\dagger, \delta\mathbf{u})$



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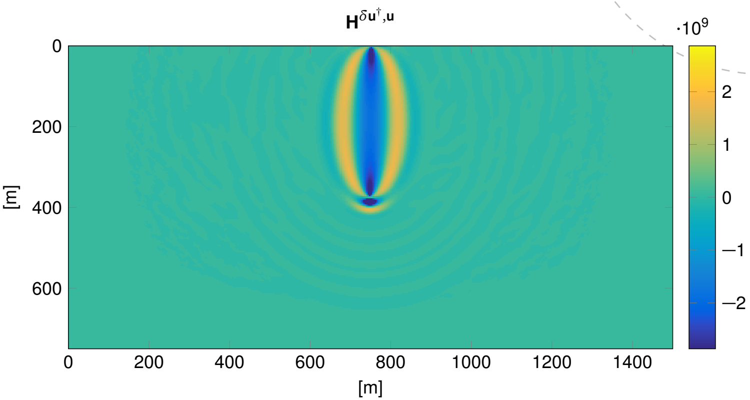

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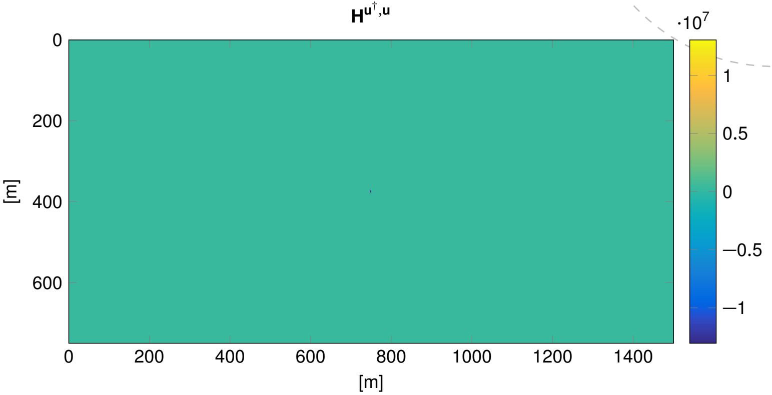

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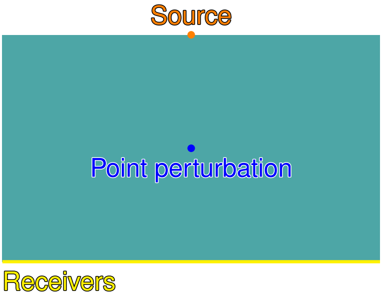

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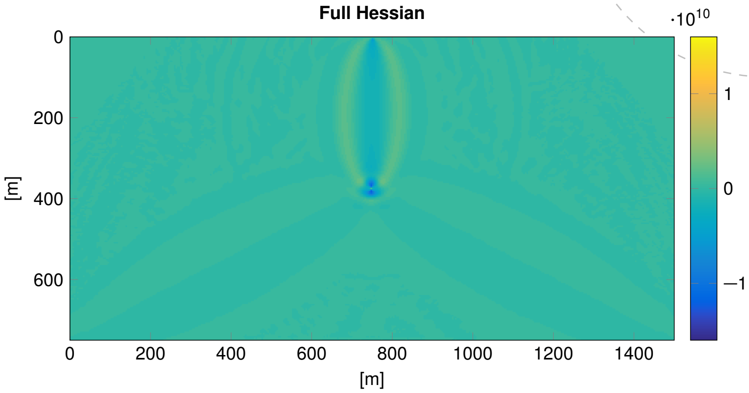
Transmission - Full Hessian Kernel — H



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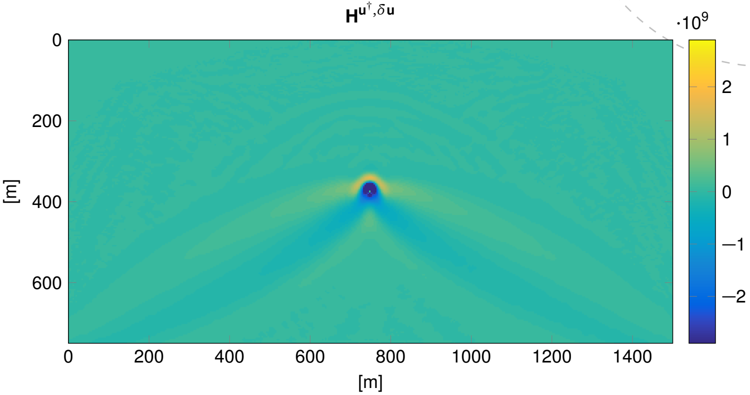

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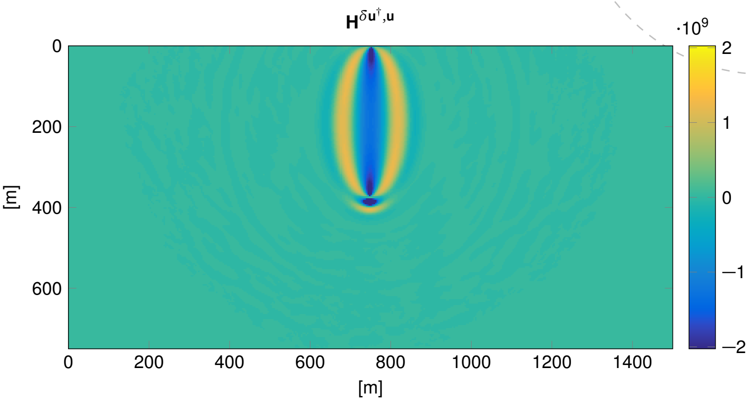

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- Possible to perform accuracy and resolution analysis on the results.
- Parameter cross-talk analysis.

Future work

- Implement in 3-D for Madagascar.
- Calculate the Hessian for an elastic medium.
- Write a full Newton inversion algorithm.
- Investigate cross-talk in different parametrisations.

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Acknowledgements

ROSE Consortium

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