Anelliptic approximation for phase and group velocities of P-waves in orthorhombic media

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Outline

Research Background

Phase and group velocity approximations

- ❑ GMA-type approximation (elastic : 9 pars; acoustic : 6 pars)
- ❑ Extended Fomel approximation (elastic : 9 pars; acoustic : 6 pars)
- ❑ Other approximations

Numerical examples

- ❑ An elastic orthorhombic model
- ❑ An acoustic orthorhombic model
- ❑ Maximum relative error tables

Summary

Research Background

- Tsvankin (1997) notation

$$\left.\begin{aligned} \upsilon_{p0} &= \sqrt{c_{33}} \\ \upsilon_{s0} &= \sqrt{c_{66}} \end{aligned}\right\} \text{ on z-axis}$$

$$\left.\begin{aligned} \varepsilon_1 &\equiv \frac{c_{22} - c_{33}}{2c_{33}} \\ \delta_1 &\equiv \frac{(c_{23} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})} \\ \gamma_1 &\equiv \frac{c_{66} - c_{55}}{2c_{55}} \end{aligned}\right\} \text{ in [y, z] plane}$$

$$\left.\begin{aligned} \varepsilon_2 &\equiv \frac{c_{11} - c_{33}}{2c_{33}} \\ \delta_2 &\equiv \frac{(c_{13} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}(c_{33} - c_{55})} \\ \gamma_2 &\equiv \frac{c_{66} - c_{44}}{2c_{44}} \end{aligned}\right\} \text{ in [x, z] plane}$$

$$\left.\delta_3 \equiv \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})}\right\} \text{ in [x, y] plane}$$

Density normalized stiffness matrix:

$$C^{ORT} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{55} & \\ & & & & & c_{66} \end{pmatrix}$$

Acoustic orthorhombic media $(\upsilon_{s0} = 0)$:

$$\upsilon_{p0},\ \eta_1,\ \delta_1,\ \eta_2,\ \delta_2,\ \delta_3$$

$$\eta_1 \equiv \frac{\varepsilon_1 - \delta_1}{1 + 2\delta_1} \qquad \eta_2 \equiv \frac{\varepsilon_2 - \delta_2}{1 + 2\delta_2}$$

- Examples of approximation (Fomel, 2004; Fomel and Stovas, 2010; Stovas, 2010):

$$\begin{aligned}(\text{Traveltime})^2 &\approx \left(\text{Hyperbolic function}\right)\left(1-\text{Weight}\right)\\ &+ \text{Weight (Nonhyperbolic function)}\end{aligned}$$

$$\begin{aligned}\left(\text{phase velocity}\right)^2 &\approx \left(\text{Elliptic function}\right)\left(1-\text{Weight}\right)\\ &+ \text{Weight (Anelliptic function)}\end{aligned}$$

$$\begin{aligned}\frac{1}{\left(\text{group velocity}\right)^2} &\approx \left(\text{Elliptic function}\right)\left(1-\text{Weight}\right)\\ &+\text{Weight (Anelliptic function)}\end{aligned}$$

❑ From GMA-type phase velocity approximation to Fomel (2004) approximation

GMA-type phase velocity approximation (5 pars)

$$\upsilon_p^2 = (1-\xi)\left(a\cos^2\theta + b\sin^2\theta\right) + \xi\sqrt{a^2\cos^4\theta + 2da\cos^2\theta\sin^2\theta + e\sin^4\theta}$$

Setting $\quad \xi = s, \quad b = c, \quad d = ac + \dfrac{\alpha}{s}, \quad e = c,$

Fomel (2004) approximation (4 pars)

$$\upsilon_p^2 = (1-s)(a\cos^2\theta + c\sin^2\theta) + s\sqrt{(a\cos^2\theta + c\sin^2\theta)^2 + 2\frac{\alpha}{s}\cos^2\theta\sin^2\theta}$$

Phase and group velocity approximations

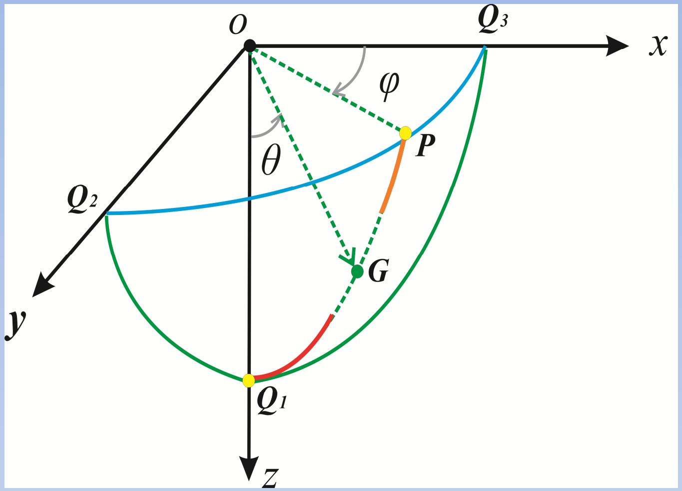
- GMA-type approximation

- phase velocity:

$$\upsilon_p^2(\theta,\varphi) = (1-\xi)\left(a\cos^2\theta + b\sin^2\theta\right) + \xi\sqrt{a^2\cos^4\theta + 2da\cos^2\theta\sin^2\theta + e^2\sin^4\theta}$$

$$\xi = \xi(\varphi) \qquad a = \upsilon_p^2(\theta = \pi/2, \varphi) \qquad b = b(\varphi)$$

$$d = d(\varphi) \qquad e = e(\varphi)$$



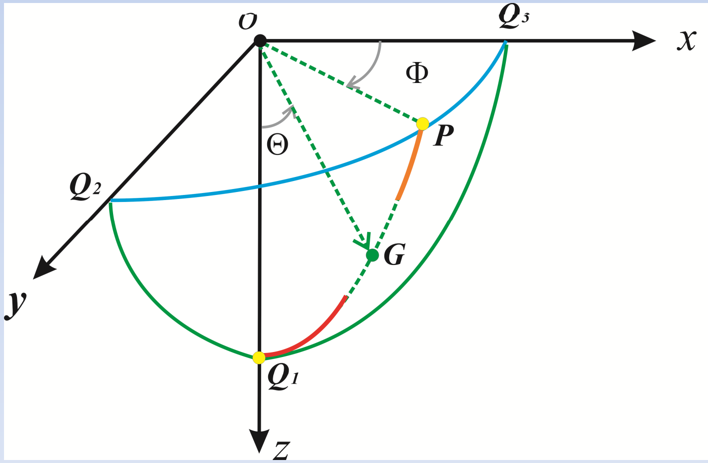
P-wave phase velocity surface

- group velocity:

$$\frac{1}{V_p^2(\Theta,\Phi)} = (1-\Xi)\left(A\cos^2\Theta + B\sin^2\Theta\right) + \Xi\sqrt{A^2\cos^4\Theta + 2DA\cos^2\Theta\sin^2\Theta + E^2\sin^4\Theta}$$

$$\Xi = \Xi(\Phi) \qquad A = \frac{1}{V_p^2(\Theta = 0, \Phi = 0)} \qquad B = B(\Phi)$$

$$D = D(\Phi) \qquad E = E(\Phi)$$



P-wave group velocity surface

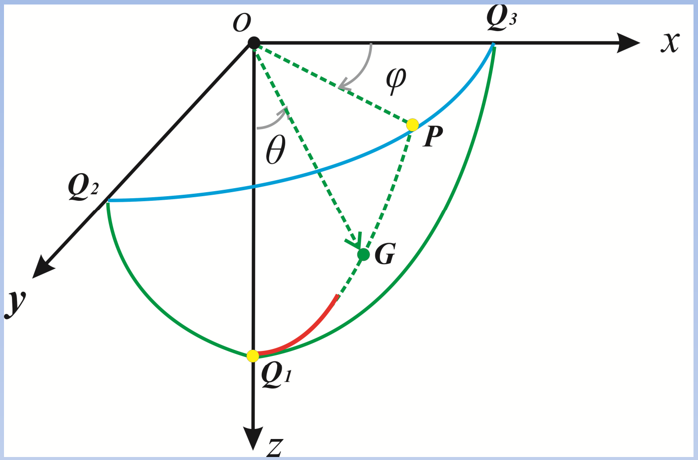
❑ Extended Fomel approximation

- phase velocity:

$$\upsilon_p^2 = (1-s)(a\cos^2\theta + c\sin^2\theta) + s\sqrt{(a\cos^2\theta + c\sin^2\theta)^2 + 2\frac{f}{s}\cos^2\theta\sin^2\theta}$$

$$s = s(\varphi) \quad a = \upsilon_p^2(\theta = \pi/2, \varphi) \quad c = c(\varphi) \quad f = f(\varphi)$$

Further simplification $s(\varphi) \approx 1/2$ for acoustic models.

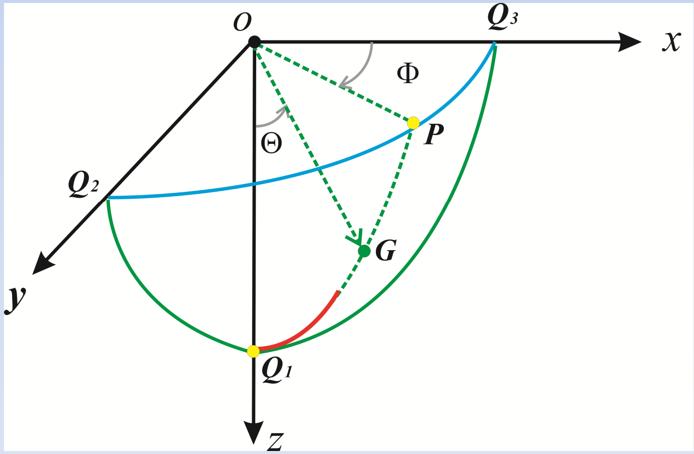


P-wave phase velocity surface

- group velocity:

$$\frac{1}{V_p^2} = (1-S)(A\cos^2\Theta + C\sin^2\Theta) + S\sqrt{(A\cos^2\Theta + C\sin^2\Theta)^2 + 2\frac{F}{S}\cos^2\Theta\sin^2\Theta}$$

$$S = S(\varphi) \quad A = \frac{1}{V_p^2(\Theta = 0, \Phi = 0)} \quad C = C(\varphi) \quad F = F(\varphi)$$



P-wave group velocity surface

❑ Orther approximations

Phase velocity:

- Tsvankin (1997) approximation
- Linearized approximation (Song et al. 2001; Daley and Krebes, 2004a, 2004b)
- Farra (2001) second-order approximation.
- Sripanich and Fomel (2014) approximation

Group velocity:

- Xu et al. (2005) and Vasconcelos and Tsvankin (2006) approximation
- Linearized approximation (Song and Every, 2000; Daley and Krebes, 2004b)
- Sripanich and Fomel (2014) approximation

Numerical examples

- An elastic ORT model (Schoenberg and Helbig,1997)

$$c_{11} = 9.0,\ c_{12} = 3.6,\ c_{13} = 2.25,\ c_{22} = 9.84,\ c_{33} = 5.9375,\ c_{44} = 2.0,\ c_{55} = 1.6,\ c_{66} = 2.182$$

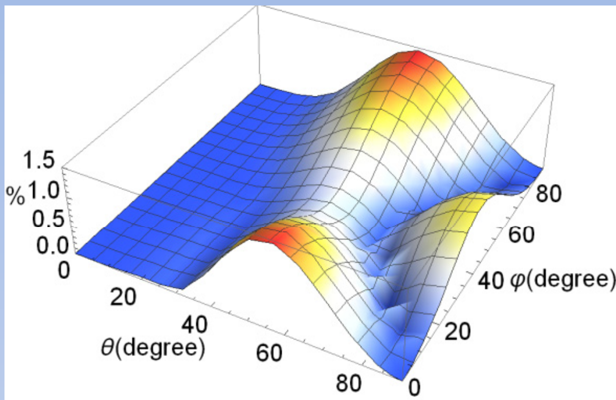
The C_{ij} have the units of $(km/s)^2$.

$$\upsilon_{p0}=2.44\text{km/s},\ \upsilon_{s0}=1.26\text{km/s},\ \varepsilon_1=0.33,\ \delta_1=0.08,\ \gamma_1=0.18,\ \varepsilon_2=0.26,\ \delta_2=\text{-}0.08,$$

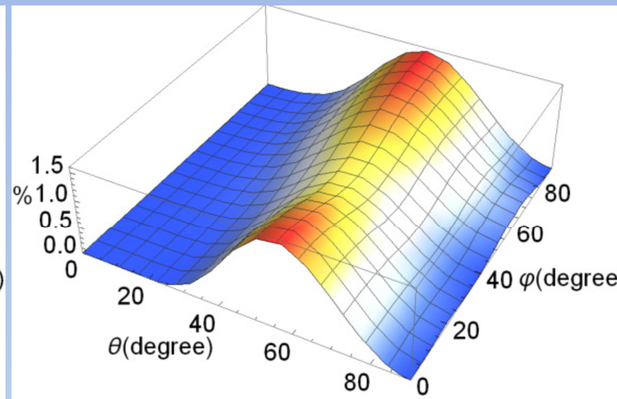
$$\gamma_2 = 0.05, \delta_3 = -0.11$$

$$\eta_1 = \frac{\varepsilon_1 - \delta_1}{1+2\delta_1} = 0.21 \qquad \eta_2 = \frac{\varepsilon_2 - \delta_2}{1+2\delta_2} = 0.40 \qquad \eta_3 = \frac{\varepsilon_1 - \varepsilon_2 - \delta_3(1+2\varepsilon_2)}{(1+2\delta_3)(1+2\varepsilon_2)} = 0.19$$

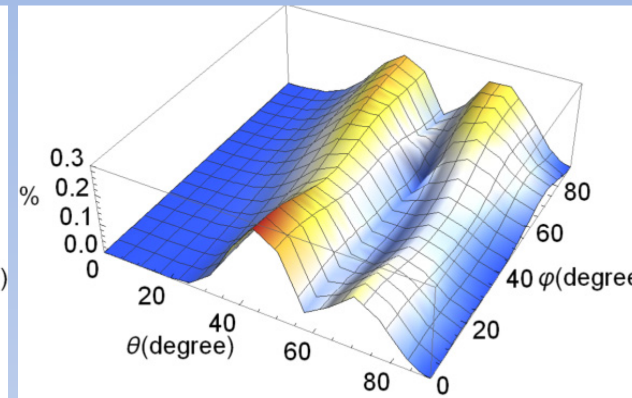
Relative error of phase-velocity approximations in elastic orthorhombic media



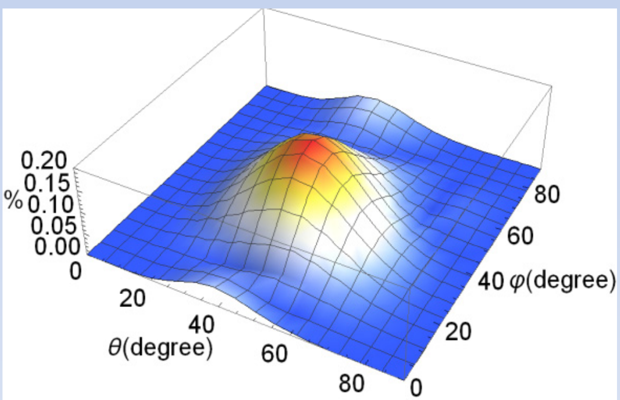
Tsvankin (1997) approximation



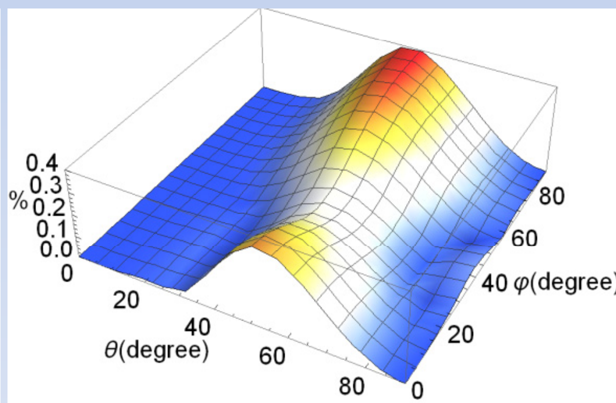
Linearized approximation (Daley and Krebes, 2004a,b)



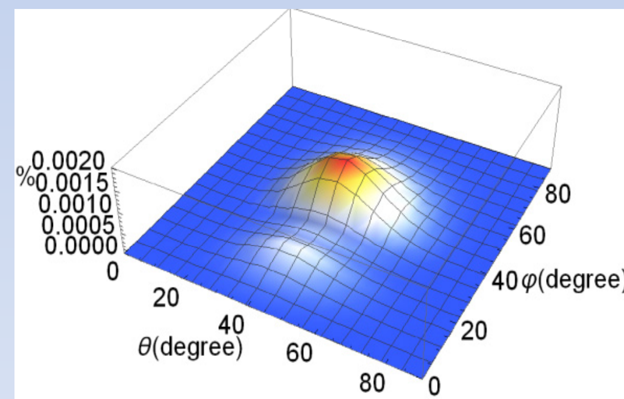
Farra (2001) second-order approximation



Sripanich and Fomel (2014) approximation

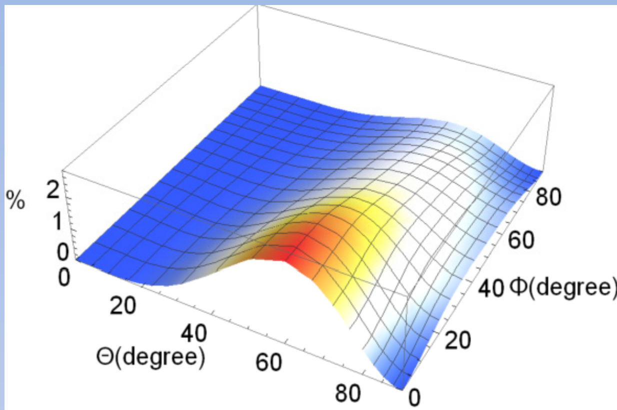


Extended Fomel approximation

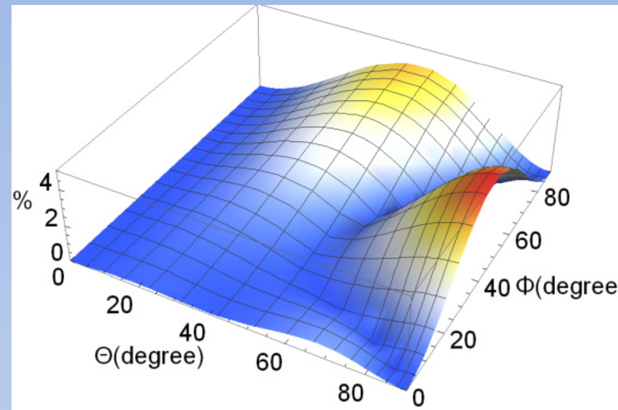


GMA-type approximation

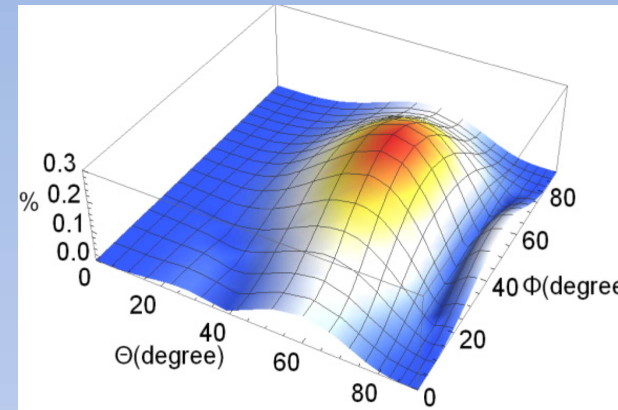
Relative error of group-velocity approximations for elastic orthorhombic media



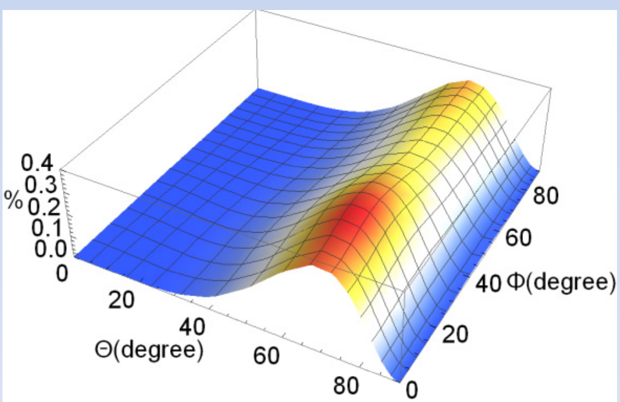
Xu et al. (2005) and Vasconcelos and Tsvankin (2006) approximation



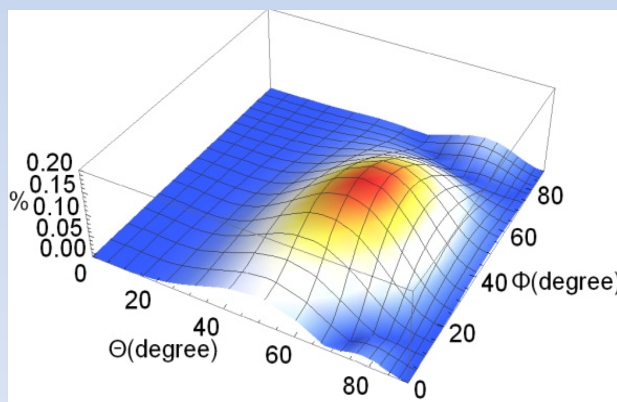
Linearized (Song and Every, 2000; Daley and Krebes, 2004b) approximation.



Sripanich and Fomel (2014) approximation



Extended Fomel approximation



GMA-type approximation

❑ An acoustic ORT model

(Schoenberg and Helbig,1997)

$$c_{11}=9.0,\ c_{12}=3.6,\ c_{13}=2.25,\ c_{22}=9.84,\ c_{33}=5.9375,\ c_{44}=2.0,\ c_{55}=1.6,\ c_{66}=2.182$$

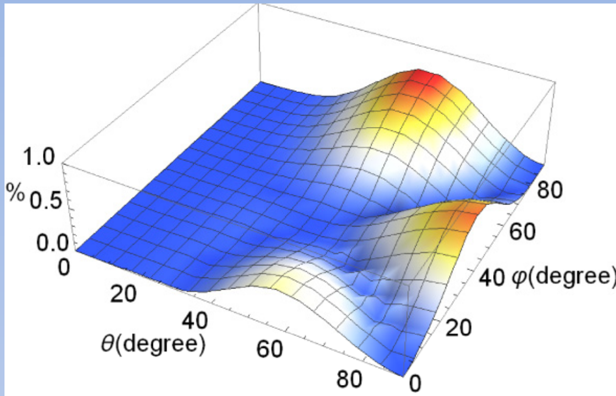
The C_{ij} have the units of $(km/s)^2$.

↓

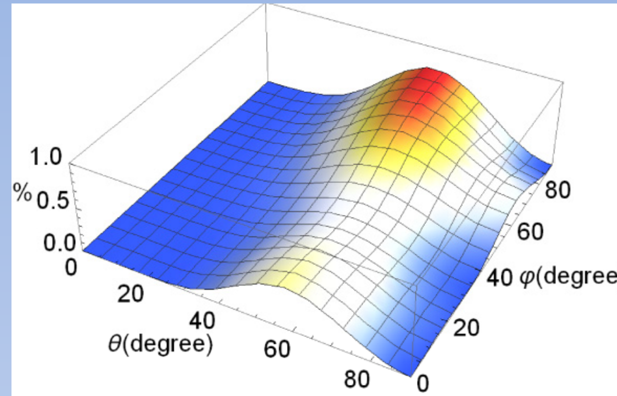
$$\upsilon_{p0}=2.44\text{km/s},\ \varepsilon_1=0.33,\ \delta_1=0.08,\ \varepsilon_2=0.26,\ \delta_2=-0.08,\ \delta_3=-0.11$$

$$\eta_1=\frac{\varepsilon_1-\delta_1}{1+2\delta_1}=0.21 \qquad \eta_2=\frac{\varepsilon_2-\delta_2}{1+2\delta_2}=0.40 \qquad \eta_3=\frac{\varepsilon_1-\varepsilon_2-\delta_3(1+2\varepsilon_2)}{(1+2\delta_3)(1+2\varepsilon_2)}=0.19$$

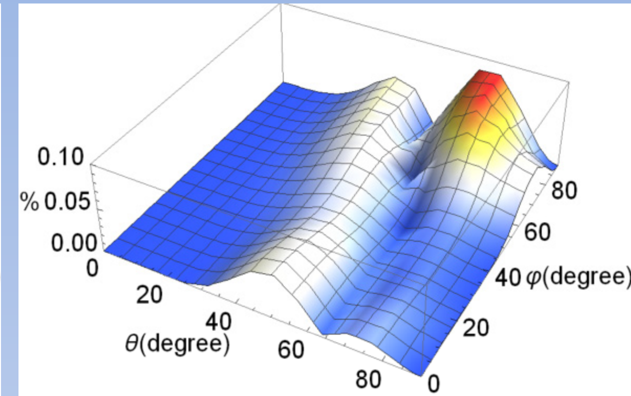
Relative error of phase-velocity approximations in acoustic orthorhombic media



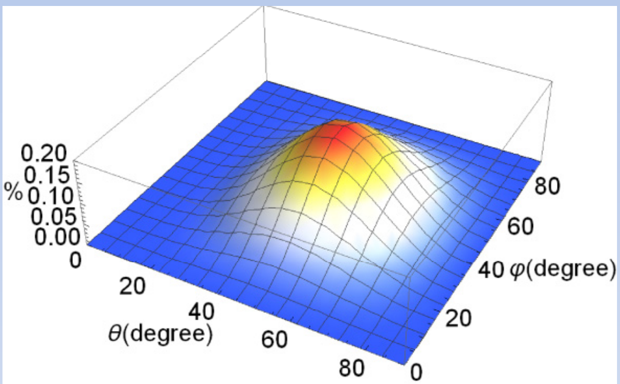
Tsvankin (1997) approximation



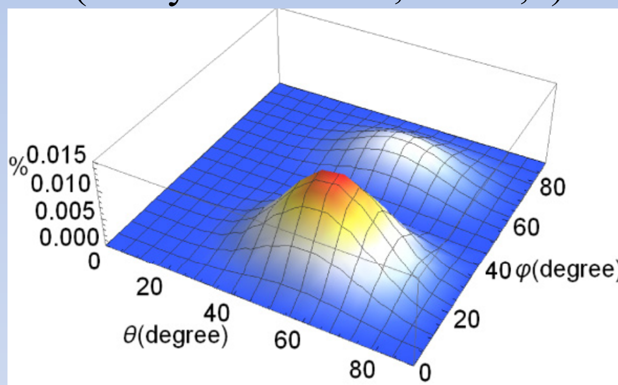
Linearized approximation (Daley and Krebes, 2004a,b)



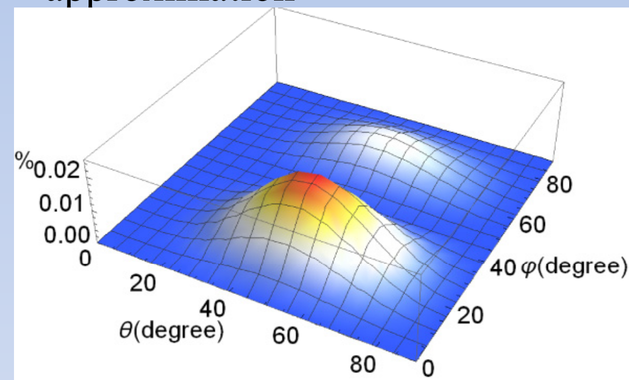
Farra (2001) second-order approximation



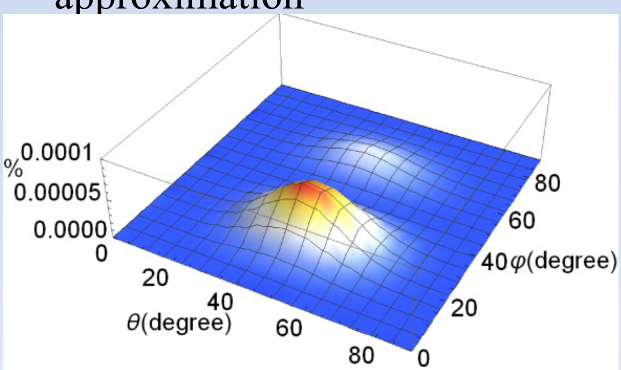
Sripanich and Fomel (2014) approximation



Extended Fomel approximation

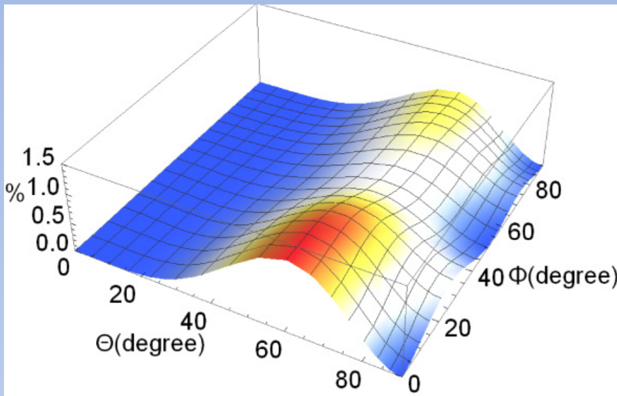


Simplified extended Fomel approximation

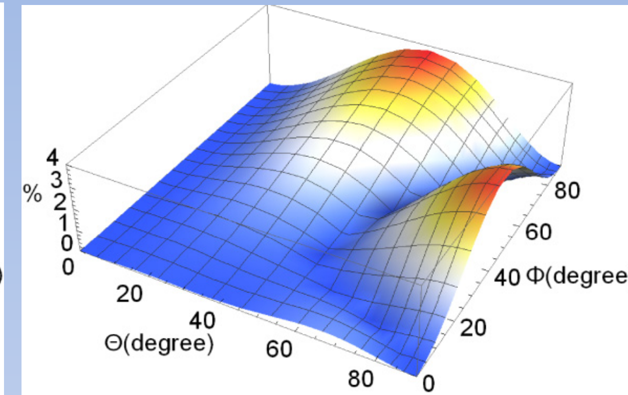


GMA-type approximation

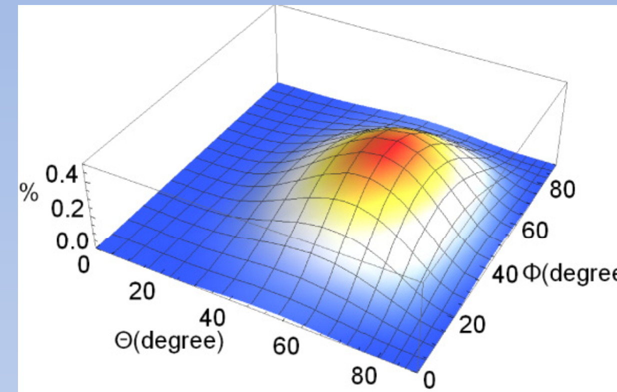
Relative error of group-velocity approximations for acoustic orthorhombic media



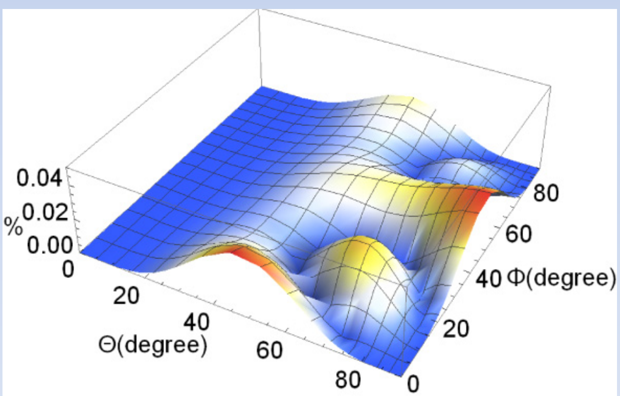
Xu et al. (2005) and Vasconcelos and Tsvankin (2006) approximation



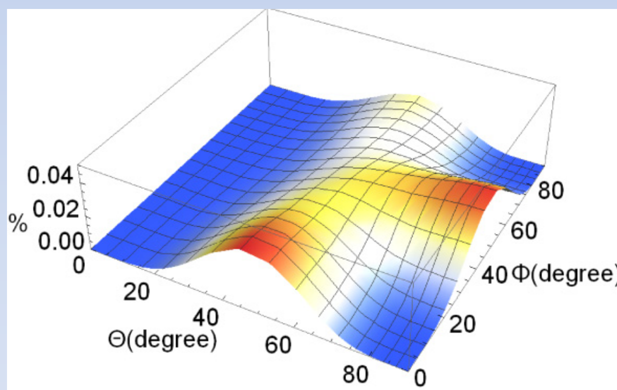
Linearized (Song and Every, 2000; Daley and Krebes, 2004b) approximation.



Sripanich and Fomel (2014) approximation



Extended Fomel approximation



GMA-type approximation

❑ Maximum relative error tables

- Accuracy test for elastic orthorhombic models

Table 1. Density-normalized stiffness parameters (unit: km^2/s^2) for orthorhombic models. Models 1-4 are taken from Mah and Schmitt (2003), Mahmoudian et al. (2014), Sano et al. (1992) and Miller and Spencer (1994), respectively.

Model	c_{11}	c_{22}	c_{33}	c_{44}	c_{55}	c_{66}	c_{12}	c_{13}	c_{23}
1	15.9	15.5	11.1	3.4	3.0	3.8	7.0	6.8	6.9
2	8.70	13.25	12.25	2.89	2.34	2.28	4.68	5.07	5.13
3	13.75	18.49	21.39	8.55	7.57	7.38	2.30	2.77	2.02
4	6.30	6.871	5.411	1.00	0.80	1.50	2.70	2.25	2.393

Table 2. Maximum relative error in phase-velocity for different approximations for models listed in Table 1.
"S & F" : Sripanich and Fomel (2014) approximation
"EF" : Our extended Fomel approximation
"GMA-type : GMA-type approximation.
"Tsvankin" : Tsvankin (1997) approximation.
"Linearized" : Linearized approximation (Song et al., 2001; Daley and Krebe, 2004a, b).
"Farra" : Farra second-order approximation (Farra, 2001).

Model	Tsvankin (%)	Linearized (%)	Farra (%)	S & F (%)	EF (%)	GMA-type (%)
1	0.721	0.544	0.065	0.078	0.059	3.0×10^{-3}
2	0.507	0.995	0.121	0.052	0.069	2.1×10^{-4}
3	4.198	1.090	0.203	0.207	0.209	5.3×10^{-3}
4	1.282	0.932	0.148	0.724	0.186	7.0×10^{-4}

Table 3. Maximum relative error in group-velocity for different approximations for models listed in Table 1.
"Tsvankin" : Xu et al. (2005) and Vasconcelos and Tsvankin (2006) approximation.
"Linearized" : linearized approximation (Song and Every, 2000; Daley and Krebe, 2004b). "S & F" : Sripanich and Fomel (2014) approximation.
"EF" : Our extended Fomel approximation.
"GMA-type" : Our GMA-type approximation.

Model	Tsvankin (%)	Linearized (%)	S & F (%)	EF (%)	GMA-type (%)
1	0.202	1.218	0.037	0.060	0.368
2	1.344	3.846	0.126	0.057	0.107
3	0.840	1.295	0.079	0.218	0.109
4	4.958	7.767	0.572	0.156	0.167

- Accuracy test for acoustic orthorhombic models

Table 4. Acoustic orthorhombic models converted from the elastic ones shown in Table 1.

Model	υ_{p0} (km/s)	$\varepsilon^{(1)}$	$\delta^{(1)}$	$\varepsilon^{(2)}$	$\delta^{(2)}$	$\delta^{(3)}$
1	3.332	0.198	0.274	0.216	0.169	-0.077
2	3.500	0.041	-0.102	-0.145	-0.178	0.065
3	4.625	-0.068	-0.097	-0.179	-0.142	0.303
4	2.326	0.135	-0.166	0.082	-0.240	-0.089

Table 4. Maximum relative error in phase-velocity for different approximations for models listed in Table 1.

"S & F" : Sripanich and Fomel (2014) approximation

"Tsvankin" : Tsvankin (1997) approximation.

"Linearized" : Linearized approximation (Song et al., 2001; Daley and Krebe, 2004a, b).

"Farra" : Farra second-order approximation (Farra, 2001).

"EF" : Our extended Fomel approximation

"Simplified" : Our simplified extended Fomel approximation.

"GMA-type : Our GMA-type approximation.

Model	Tsvankin (%)	Linearized (%)	Farra (%)	S & F (%)	EF (%)	Simplified (%)	GMA-type (%)
1	0.681	0.423	4.25×10^{-2}	2.81×10^{-2}	1.52×10^{-2}	2.10×10^{-2}	9.5×10^{-4}
2	0.367	0.789	7.61×10^{-2}	5.74×10^{-2}	3.12×10^{-2}	4.45×10^{-2}	6.3×10^{-5}
3	3.889	0.624	7.28×10^{-2}	2.17×10^{-2}	1.73×10^{-2}	2.39×10^{-2}	9.3×10^{-4}
4	1.037	0.754	9.90×10^{-2}	0.458	1.86×10^{-2}	3.04×10^{-2}	2.4×10^{-4}

Table 5. Maximum relative error in group-velocity for different approximations for models listed in Table 1.

"Tsvankin" : Xu et al. (2005) and Vasconcelos and Tsvankin (2006) approximation.

"Linearized" : linearized approximation (Song and Every, 2000; Daley and Krebe, 2004b). "S & F" : Sripanich and Fomel (2014) approximation.

"EF" : Our extended Fomel approximation.

"GMA-type" : Our GMA-type approximation.

Model	Tsvankin (%)	Linearized (%)	S & F (%)	EF (%)	GMA-type (%)
1	0.202	1.312	0.082	0.072	0.167
2	1.546	3.792	0.131	0.076	0.021
3	0.705	0.964	0.070	0.083	0.384
4	4.785	7.651	0.754	0.311	0.311

Summary

- We propose a method of using anelliptic functions to analytically calculate phase and group velocities of P-waves for elastic and acoustic orthorhombic media.
- The proposed approximations (including the GMA-type and extended Fomel approximations for both phase and group velocities and the simplified version of the extended Fomel approximation for phase velocity) are stable and accurate for both elastic and acoustic orthorhombic media with strong anisotropy.

	Phase velocity	Group velocity
Extended Fomel	×	×
GMA-type	×	×
Simplified Extended Fomel	×	

Acknowledgements

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