WAVE-EQUATION INVERSION USING THE LIPPMANN-SCHWINGER EQUATION

Bjørn Ursin

IPT, NTNU, ROSE meeting 2014

OUTLINE

INTRODUCTION

- The scalar wave equation
- Integral equation solution
- Inverse methods without computing the gradient
- Simplified notation without computational details

The wave equation

$$\mathbf{L}(\mathbf{x})\mathbf{\Psi}_s(\mathbf{x}) = -\mathbf{f}_s(\mathbf{x}), \quad \mathbf{L}(\mathbf{x}) = \nabla^2 - \frac{\omega^2}{c(\mathbf{x})^2} \tag{1}$$

The reference Green's function

$$\left[\nabla^2 - \frac{\omega^2}{c_0(\mathbf{x})^2}\right]\mathbf{G}(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}') \tag{2}$$

THE LIPPMANN-SCHWINGER EQUATION

$$\mathbf{\Psi}_s = \mathbf{G}\mathbf{f}_s + \omega^2 \mathbf{G}\mathbf{V}\mathbf{\Psi}_s \tag{3}$$

with

$$\mathbf{V} = \frac{1}{c_0(\mathbf{x})^2} - \frac{1}{c(\mathbf{x})^2} \tag{4}$$

Data

$$\mathbf{d}_s = \mathbf{R}_s\mathbf{G}\mathbf{f}_s + \omega^2 \mathbf{R}_s\mathbf{G}\mathbf{V}\mathbf{\Psi}_s \tag{5}$$

$\mathbf{R}_s$ = restriction operator

The forward scattering series

$$\mathbf{\Psi}_s = \left(\mathbf{I} - \omega^2 \mathbf{GV}\right)^{-1} \mathbf{Gf}_s = \sum_{k=0}^{\infty} \left(\omega^2 \mathbf{GV}\right)^k \mathbf{Gf}_s = \sum_{k=0}^{\infty} \mathbf{\Psi}_{sk} \tag{6}$$

with

$$\mathbf{\Psi}_{sk} = \omega^2 \mathbf{GV} \mathbf{\Psi}_{sk-1} \;\text{ and }\; \mathbf{\Psi}_{s0} = \mathbf{Gf}_s \tag{7}$$

Single-scattering inversion

For $\mathbf{V}_0 = 0$, $\boldsymbol{\Psi}_s = \mathbf{G}\mathbf{f}_s$, and we compute $\mathbf{V}_1$ from

$$\min_{\mathbf{V}_1} \sum_{s=1}^{N_s} \sum_{\omega} \|\mathbf{d}_s - \mathbf{R}_s\mathbf{G}\mathbf{f}_s - \omega^2\mathbf{R}_s\mathbf{G}\mathbf{V}_1\mathbf{G}\mathbf{f}_s\|^2 \tag{8}$$

Iterative Born inversion

Solve the equation

$$\mathbf{d}_s = \mathbf{R}_s\mathbf{G}\mathbf{V}^k \left(\mathbf{I} - \omega^2\mathbf{G}\mathbf{V}^{k-1}\right)^{-1}\mathbf{G}\mathbf{f}_s, \quad k = 2, 3, ... \tag{9}$$

by least squares, starting with $\mathbf{V}_1$, the single-scattering solution.

Note: this requires the inversion

$$\mathbf{\Psi}_s^{k-1} = \left(\mathbf{I} - \omega^2\mathbf{G}\mathbf{V}^{k-1}\right)^{-1}\mathbf{G}\mathbf{f}_s \approx \mathbf{G}\mathbf{f}_s + \omega^2\mathbf{G}\mathbf{V}^{k-1}\mathbf{G}\mathbf{f}_s \tag{10}$$

THE T-MATRIX

Definition

$$\mathbf{TGf}_s = \mathbf{V\Psi}_s \tag{11}$$

From the Lippmann-Schwinger equation

$$\mathbf{V\Psi}_s = \mathbf{VGf}_s + \omega^2 \mathbf{VGV\Psi}_s \tag{12}$$

we obtain

$$\mathbf{TGf}_s = \mathbf{VGf}_s + \omega^2 \mathbf{VGTGf}_s \tag{13}$$

or

$$\mathbf{T} = \mathbf{V} + \omega^2 \mathbf{VGT}, \; \mathbf{T} = \left(\mathbf{I} - \omega^2 \mathbf{VG}\right)^{-1} \mathbf{V} \tag{14}$$

T-matrix inversion

Compute

$$\mathbf{\Psi}_s^k = \mathbf{G}\mathbf{f}_s + \omega^2 \mathbf{G}\mathbf{T}^k\mathbf{G}\mathbf{f}_s, \quad \mathbf{T}^1 = \mathbf{V}_1 \tag{15}$$

and solve for $\mathbf{V}^{k+1}$

$$\min_{\mathbf{V}^{k+1}} \sum_{s=1}^{N_s} \sum_{\omega} \|\mathbf{d}_s - \mathbf{R}_s\mathbf{G}\mathbf{f}_s - \omega^2 \mathbf{R}_s\mathbf{G}\mathbf{V}^{k+1}\mathbf{\Psi}_s^k\|^2 \tag{16}$$

Update

$$\mathbf{T}^{k+1} = \left(\mathbf{I} - \omega^2\mathbf{V}^{k+1}\mathbf{G}\right)^{-1}\mathbf{V}^{k+1} \approx \left(\mathbf{I} + \omega^2\mathbf{V}^{k+1}\mathbf{G}\right)\mathbf{V}^{k+1} \tag{17}$$

Comparison

Born:

$$\mathbf{d}_s = \mathbf{R_s}\left[\mathbf{I} + \omega^2\mathbf{GV}\left(\mathbf{I} - \omega^2\mathbf{GV}\right)^{-1}\right]\mathbf{Gf}_s \tag{18}$$

T-matrix:

$$\mathbf{d}_s = \mathbf{R_s}\left[\mathbf{I} + \omega^2\mathbf{G}\left(\mathbf{I} - \omega^2\mathbf{VG}\right)^{-1}\mathbf{V}\right]\mathbf{Gf}_s \tag{19}$$

Note:

$$\mathbf{V}\left(\mathbf{I} - \omega^2\mathbf{GV}\right)^{-1} = \left(\mathbf{I} - \omega^2\mathbf{VG}\right)^{-1}\mathbf{V} \tag{20}$$

$$\left(\mathbf{I} - \omega^2\mathbf{VG}\right)\mathbf{V} = \mathbf{V}\left(\mathbf{I} - \omega^2\mathbf{GV}\right) \tag{21}$$

Inverse scattering series

From the Lippmann-Schwinger equation

$$\mathbf{d}_s = \mathbf{R}_s \left[\mathbf{I} + \omega^2 \mathbf{G}\mathbf{V} \sum_{j=0}^{\infty} \left(\omega^2 \mathbf{G}\mathbf{V} \right)^j \right] \mathbf{G}\mathbf{f}_s \tag{22}$$

with

$$\mathbf{V} = \sum_{k=1}^{\infty} \mathbf{V}_k \tag{23}$$

The first-order term gives the single-scattering solution $\mathbf{V}_1$.

Inverse scattering series

Matching higher-order terms gives

$$\begin{aligned} \mathbf{G}\mathbf{V}_2\mathbf{G} &= -\omega^2\mathbf{G}\mathbf{V}_1\mathbf{G}\mathbf{V}_1\mathbf{G} \\ \mathbf{G}\mathbf{V}_3\mathbf{G} &= \omega^4\mathbf{G}\mathbf{V}_1\mathbf{G}\mathbf{V}_1\mathbf{G}\mathbf{V}_1\mathbf{G} \\ &\vdots \\ \mathbf{G}\mathbf{V}_k\mathbf{G} &= \left[-\omega^2\right]^{k-1}\underbrace{\mathbf{G}\mathbf{V}_1\cdot\ldots\cdot\mathbf{G}\mathbf{V}_1}_{\text{k terms}}\mathbf{G} \end{aligned} \tag{24}$$

or

$$\mathbf{G}\mathbf{V}_k\mathbf{G} = -\omega^2\mathbf{G}\mathbf{V}_{k-1}\mathbf{G}\mathbf{V}_1\mathbf{G} \tag{25}$$

This gives, formally,

$$\mathbf{GVG} = \sum_{k=1}^{\infty} \mathbf{GV}_k\mathbf{G} = \sum_{k=0}^{\infty} \left(-\omega^2 \mathbf{GV}_1\right)^k \mathbf{GV}_1\mathbf{G} = \left(\mathbf{I} + \omega^2 \mathbf{GV}_1\right)^{-1} \mathbf{GV}_1\mathbf{G} \tag{26}$$

Contrast-source inversion

The contrast sources $\mathbf{W}_s = \mathbf{V}\boldsymbol{\Psi}_s$.

From the Lippmann-Schwinger equation

$$\mathbf{W}_s = \mathbf{V}\mathbf{G}\mathbf{f}_s + \omega^2 \mathbf{V}\mathbf{G}\mathbf{W}_s \tag{27}$$

and

$$\mathbf{d}_s = \mathbf{R}_s\mathbf{G}\mathbf{f}_s + \omega^2 \mathbf{R}_s\mathbf{G}\mathbf{W}_s \tag{28}$$

For $k = 2, \ldots$ solve, for each s and ω, by least-squares,

$$\begin{cases} \mathbf{d}_s & = & \mathbf{R}_s\mathbf{G}\mathbf{f}_s + \omega^2 \mathbf{R}_s\mathbf{G}\mathbf{W}_s^k \\ \mathbf{W}_s^k & = & \mathbf{V}^{k-1}\mathbf{G}\mathbf{f}_s + \omega^2 \mathbf{V}^{k-1}\mathbf{G}\mathbf{W}_s^k \end{cases}$$

with $\mathbf{V}^1 = \mathbf{V}_1$. Next compute $\mathbf{V}^k$ from

$$\min_{\mathbf{V}_k} \sum_{s=1}^{N_s} \sum_{\omega} \|\mathbf{W}_s^k - \mathbf{V}^k\mathbf{G}\mathbf{f}_s - \omega^2 \mathbf{V}^k\mathbf{G}\mathbf{W}_s^k\|^2 \tag{29}$$

Discussion

- All methods start with the single-scattering solution.
- Need $N_{rs} \cdot N_s \cdot N_\omega > N_V$ to have an overdetermined linear system.
- Born and T-matrix inversions are very similar, both update the model fit to the data.
- Born inversion and T-matrix inversion require one inversion per iteration (in addition to the data fit equation).
- The inverse scattering series depends on the data only through the single scattering solution.
- Contrast-source inversion requires no inversion, but N_s fits to the data in each iteration.
- Convergence and uniqueness issues for all methods.

Selected references

- Jakobsen, M., and Ursin, B., 2014, Full waveform inversion using a Born iterative T-matrix method in the frequency domain. Journal of Geophysics and Engineering, submitted.
- Malcolm, A.E., and de Hoop, M.V., 2005, A method for inverse scattering based on the generalized Bremmer coupling series. Inverse Problems, 21, 1137-1167.
- Van den Berg, P.M., and Kleinmann, R.E., 1997, A contrast source inversion method. Inverse Problems, 13, 1607-1620.
- Weglein, A.B. et al., 2003, Inverse scattering series and seismic exploration. Inverse Problems, 19, R27-R83.

ACKNOWLEDGEMENTS

This research has been supported by the Norwegian Research Council via the ROSE project.

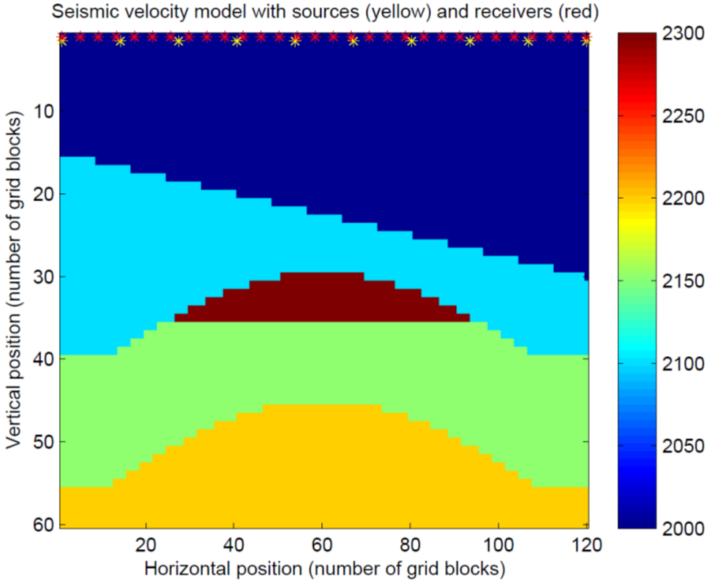
NONLINEAR SEISMIC WAVEFORM INVERSION USING A BORN ITERATIVE T-MATRIX METHOD

Morten Jakobsen[1] Bjørn Ursin[2]

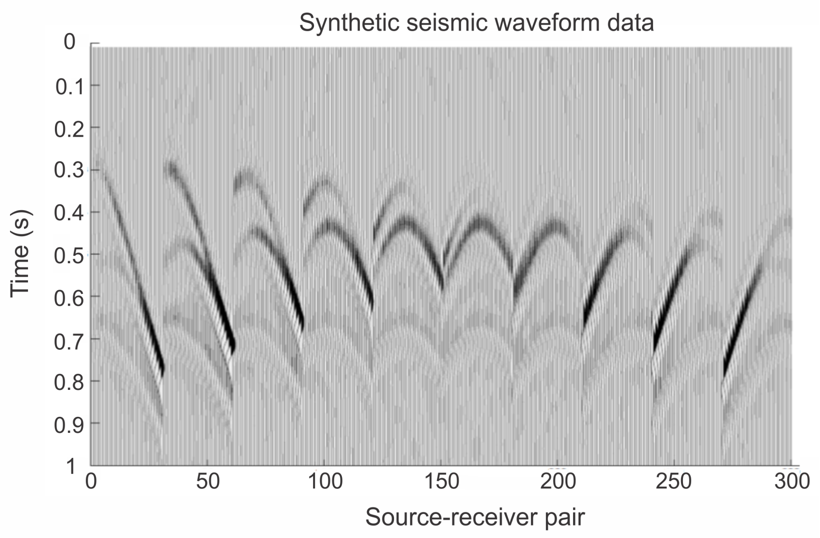
[1]Department of Earth Science
University of Bergen

[2]Department of Applied Geophysics and Petroleum Technology
NTNU

A simple 2D example



Synthetic seismic waveform data for multiple sources



True vs inverted contrasts

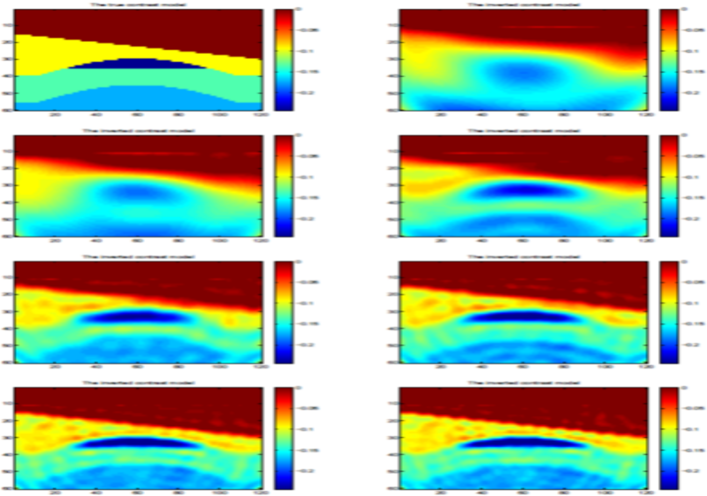


Figure : True versus inverted contrast source models corresponding to different frequencies.