Theoretical Background

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Deep Vision Ltd and Sharp Reflections

AVO

Elastic plane wave incident on a planar interface:

- Stresses are continuous across the boundary.
- Particle displacements are continuous across the boundary.

→ Zoeppritz equations for transmission and reflection coeffcients

→ Horizontal slowness is preserved (Snell's law)

Zoeppritz equations

Simplifications of Zoeppritz equations

Aki & Richards

$$r_{PP}(\vartheta) \approx \frac{1}{2}\left(1 - p^2 V_S^2\right)\frac{\Delta\rho}{\rho} + \frac{1}{2\cos^2\vartheta}\frac{\Delta V_P}{V_P} - 4p^2 V_S^2 \frac{\Delta V_S}{V_S}$$

Shuey

$$r_{PP}(\vartheta) \approx \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right) + \left[\frac{1}{2}\frac{\Delta V_P}{V_P} - 2\frac{V_S^2}{V_P^2}\left(2\frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}\right)\right]\sin^2\vartheta + \frac{1}{2}\frac{\Delta V_P}{V_P}\left(\tan^2\vartheta - \sin^2\vartheta\right)$$

These assume that the contrast in properties across the boundaries is small (say $< 10\%$). They are NOT small angle approximations.

Fatti, Bortfeld, Hilterman ... Mostly make assumptions, small angles, Gardner for density, Vp/Vs is some known value.

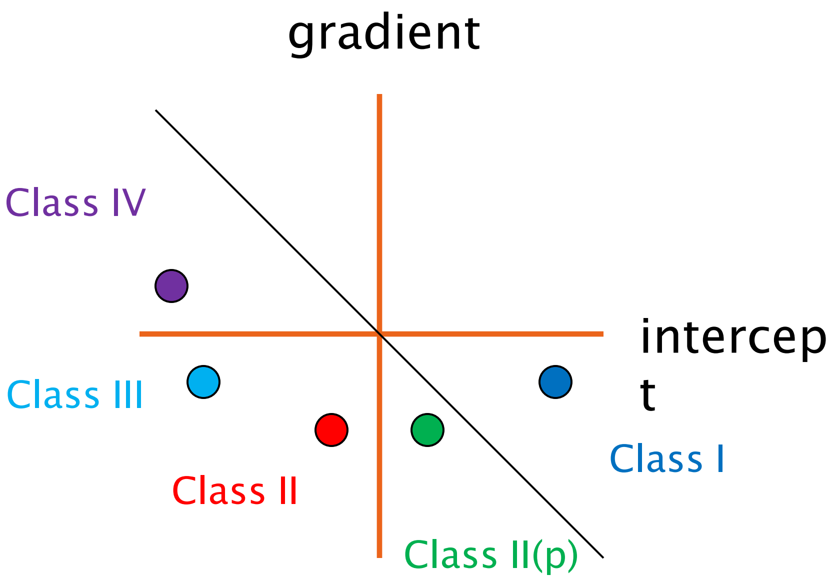
Simplifications of Zoeppritz equations

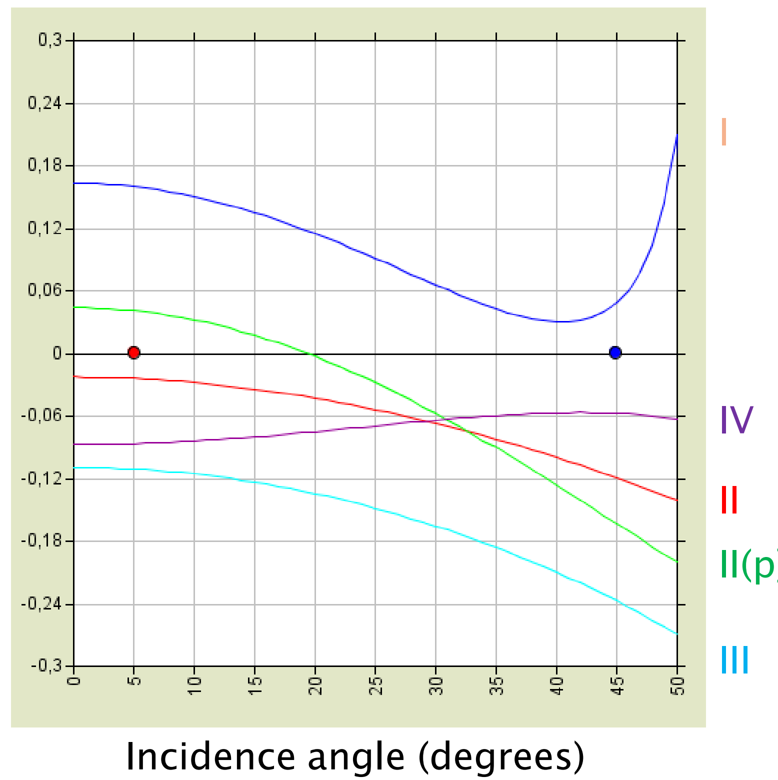
Two-term Shuey

$$r_{PP}(\vartheta) \approx \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right) + \left[\frac{1}{2}\frac{\Delta V_P}{V_P} - 2\frac{V_S^2}{V_P^2}\left(2\frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}\right)\right]\sin^2\vartheta$$

OK'ish for incidence angles up to 30°

AVO Classes

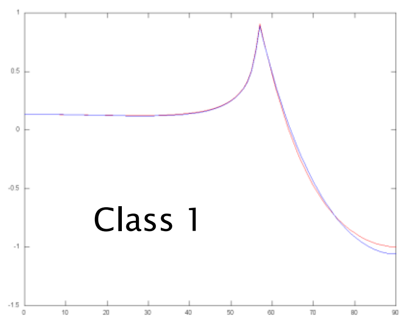


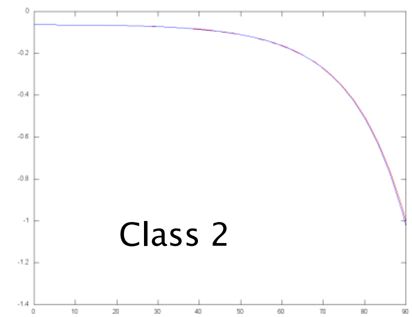


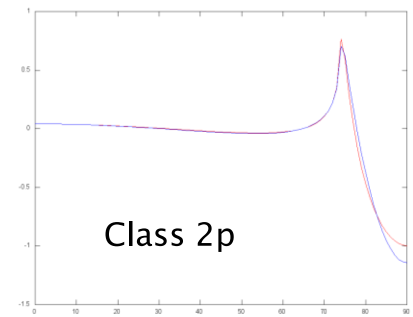
Rutherford and Williams

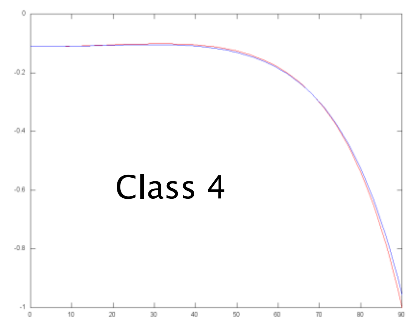
Simplifications of Zoeppritz equations

Aki & Richards v exact equations









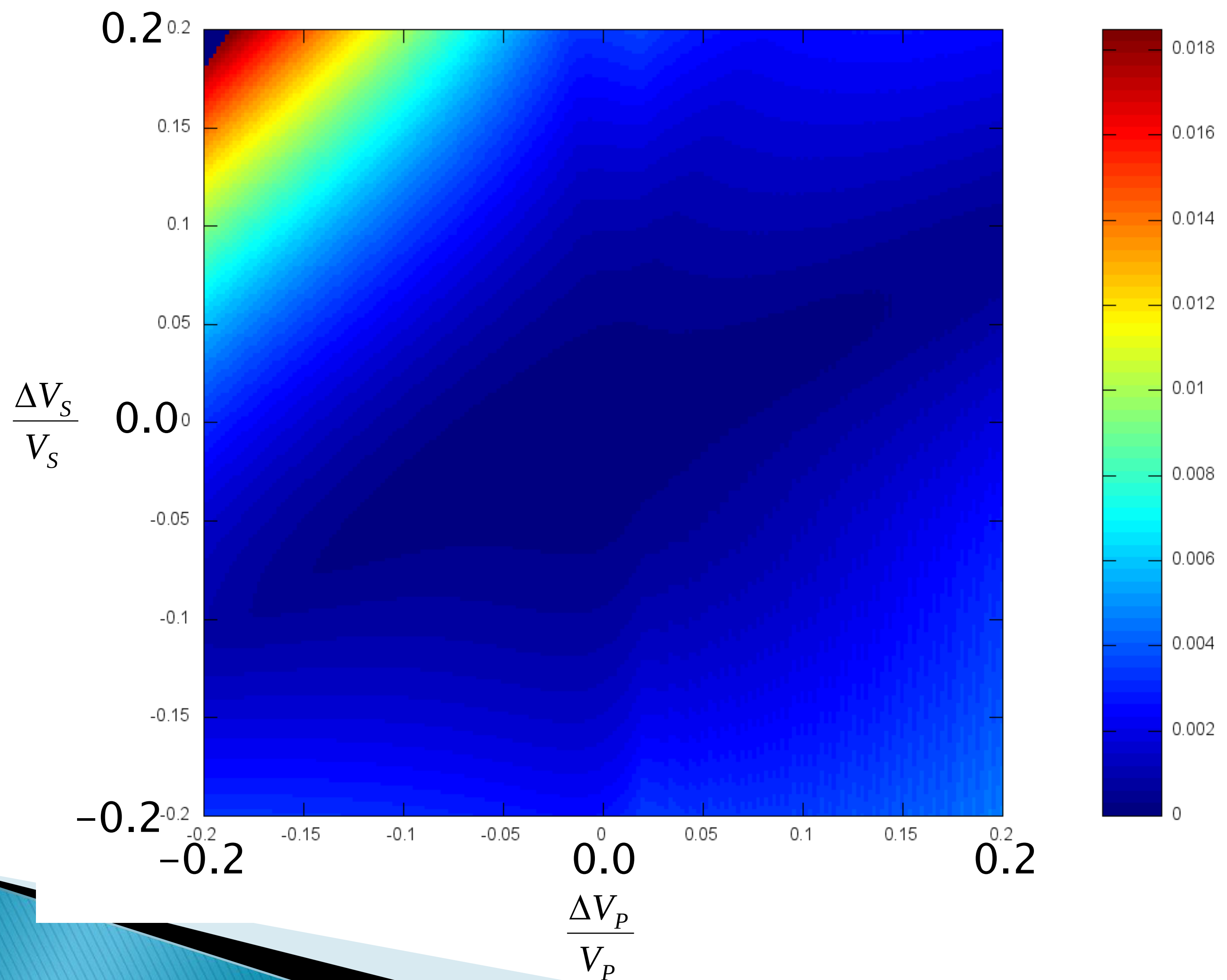
Real parts plotted,
Aki & Richards red,
Exact solution blue

Aki and Richards 3 term equation is a small *contrast* approximation, not a small *angle* approximation.
The two term equations assume small angles *and* small contrasts.

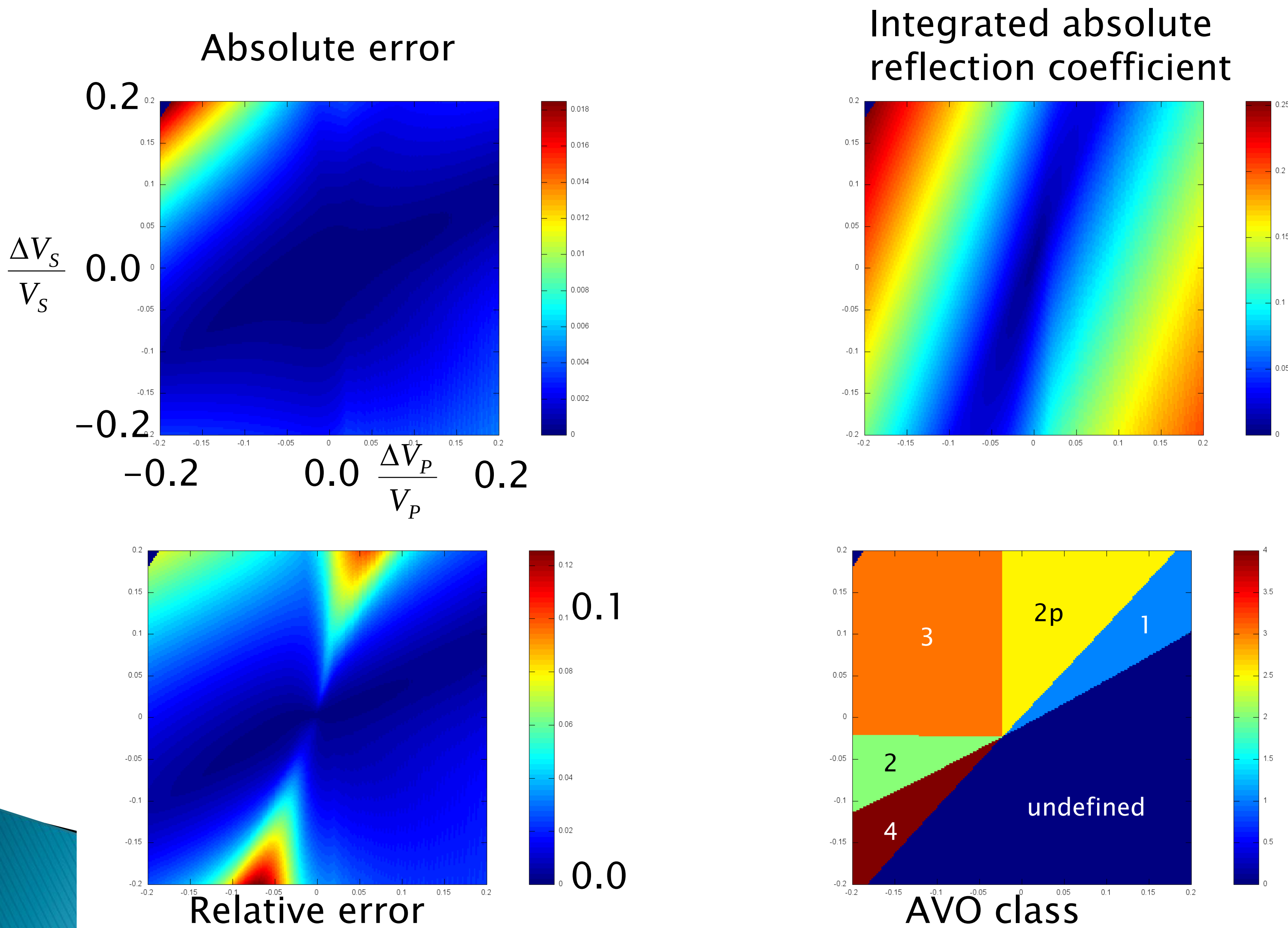
Simplifications of Zoeppritz equations

Aki & Richards v exact equations

Absolute error integrated over range 0 – 0.75 critical angle (or 72°) for fixed density contrast.



Simplifications of Zoeppritz equations



Simplifications of Zoeppritz equations

R_1	R_2	R_3	C_1	C_2	C_3
Vp	Vs	ρ	$\frac{1}{2}\left(1+\tan^2\vartheta\right)$	$-4\gamma^2\sin^2\vartheta$	$\frac{1}{2}\left(1-4\gamma^2\sin^2\vartheta\right)$
Zp	Zs	ρ	$\frac{1}{2}\left(1+\tan^2\vartheta\right)$	$-4\gamma^2\sin^2\vartheta$	$\frac{1}{2}\left(4\gamma^2\sin^2\vartheta-\tan^2\vartheta\right)$
Zp	υ	ρ	$\frac{1}{2}\left(1+\tan^2\vartheta\right)-4\gamma^2\sin^2\vartheta$	$2\left(1-2\gamma^2\right)\left(1-\gamma^2\right)\sin^2\vartheta$	$\frac{1}{2}\left(4\gamma^2\sin^2\vartheta-\tan^2\vartheta\right)$
Zp	Vp/Vs	ρ	$\frac{1}{2}\left(1+\tan^2\vartheta\right)-4\gamma^2\sin^2\vartheta$	$4\gamma^2\sin^2\vartheta$	$\frac{1}{2}\left(4\gamma^2\sin^2\vartheta-\tan^2\vartheta\right)$
K	μ	ρ	$\left(\frac{1}{4}-\frac{1}{3}\gamma^2\right)\left(1+\tan^2\vartheta\right)$	$\gamma^2\left(\frac{1}{3}\left(1+\tan^2\vartheta\right)-2\sin^2\vartheta\right)$	$\frac{1}{4}\left(1-\tan^2\vartheta\right)$

$$r_{PP}(\vartheta)\approx C_1\frac{\Delta R_1}{R_1}+C_2\frac{\Delta R_2}{R_2}+C_3\frac{\Delta R_3}{R_3}$$

$$\gamma^2=\frac{V_s^2}{V_p^2}$$

Simplifications of Zoeppritz equations

R_1	R_2	R_3	C_1	C_2	C_3
Vp	Vs	ρ	$\frac{1}{2}\left(1+\tan^2\vartheta\right)$	$-4\gamma^2\sin^2\vartheta$	$\frac{1}{2}\left(1-4\gamma^2\sin^2\vartheta\right)$
		ρ			$\frac{1}{2}\left(4\gamma^2\sin^2\vartheta-\tan^2\vartheta\right)$
Zp	υ	ρ	$\frac{1}{2}\left(1+\tan^2\vartheta\right)-4\gamma^2\sin^2\vartheta$	$2\left(1-2\gamma^2\right)\left(1-\gamma^2\right)\sin^2\vartheta$	$\frac{1}{2}\left(4\gamma^2\sin^2\vartheta-\tan^2\vartheta\right)$
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
K	μ	ρ	$\left(\frac{1}{4}-\frac{1}{3}\gamma^2\right)\left(1+\tan^2\vartheta\right)$	$\gamma^2\left(\frac{1}{3}\left(1+\tan^2\vartheta\right)-2\sin^2\vartheta\right)$	$\frac{1}{4}\left(1-\tan^2\vartheta\right)$

Suppose that there are velocity contrasts across the boundary, but no impedance contrasts (the density compensates).
Then reflectivity is zero – small at low to moderate incidence angles

$$r_{PP}(\vartheta)\approx C_1\frac{\Delta R_1}{R_1}+C_2\frac{\Delta R_2}{R_2}+C_3\frac{\Delta R_3}{R_3}$$

$$\gamma^2=\frac{V_s^2}{V_p^2}$$

Simplifications of Zoeppritz equations

R_1	R_2	R_3	C_1	C_2	C_3
		ρ			$\frac{1}{2}\left(1-4\gamma^2\sin^2\vartheta\right)$
Zp	Zs	ρ	$\frac{1}{2}\left(1+\tan^2\vartheta\right)$	$-4\gamma^2\sin^2\vartheta$	$\frac{1}{2}\left(4\gamma^2\sin^2\vartheta-\tan^2\vartheta\right)$
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
K	μ	ρ	$\left(\frac{1}{4}-\frac{1}{3}\gamma^2\right)\left(1+\tan^2\vartheta\right)$	$\gamma^2\left(\frac{1}{3}\left(1+\tan^2\vartheta\right)-2\sin^2\vartheta\right)$	$\frac{1}{4}\left(1-\tan^2\vartheta\right)$

Now suppose that there are impedance contrasts across the boundary, but no velocity contrasts (the density compensates).
Then reflectivity is non-zero at low to moderate incidence angles

Near-mid angle reflections are caused by impedance contrasts, not velocity contrasts.
Should invert for impedances, not for velocities.

$$r_{PP}(\vartheta) \approx C_1\frac{\Delta R_1}{R_1} + C_2\frac{\Delta R_2}{R_2} + C_3\frac{\Delta R_3}{R_3}$$

$$\gamma^2 = \frac{V_s^2}{V_p^2}$$

Scattering

Impedance contrasts cause back-scattering (reflections) whereas *velocity* contrasts cause forward scattering.

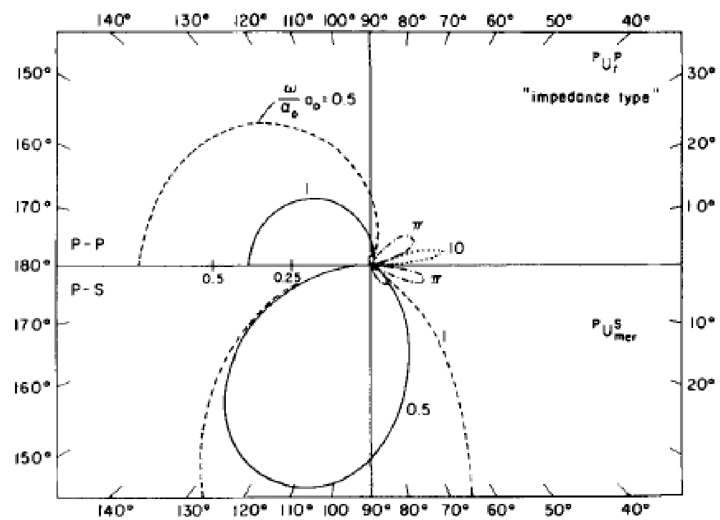


FIG. 28. Scattering patterns of a Gaussian heterogeneity of impedance type for different frequencies. The upper half-plane is for *P-P*, the lower half-plane is for *P-S*.

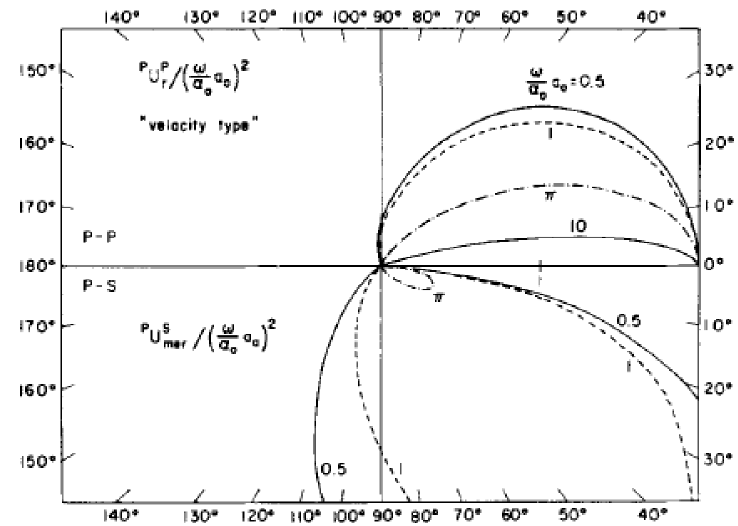


FIG. 29. Same as Figure 28, for velocity type heterogeneity.

Wu & Aki, 1985, Geophysics

Simplifications of Zoeppritz equations

The information in the data is the same, but you won’t get the same answers if you use different equations for the fitting then calculate the same physical quantities.

Curve fitting

Fit a straight line to the amplitudes v $\sin^2\theta$:

$$\begin{pmatrix} 1 & \sin^2\vartheta_1 \\ \vdots & \vdots \\ 1 & \sin^2\vartheta_n \end{pmatrix}\begin{pmatrix} r_0 \\ g \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \qquad Am = d \qquad \hat{m} = \left(A^T A\right)^{-1} A^T d$$

Assume we are equally spaced in $\sin^2\theta$ and there are no negative offsets, then

$$\mathrm{cov}\{\hat{m}\} = \sigma^2 \begin{pmatrix} \frac{1}{n} + \frac{12x_m^2(n-1)}{x_r^2 n(n+1)} & -\frac{12(n-1)x_m}{n(n+1)x_r^2} \\ -\frac{12(n-1)x_m}{n(n+1)x_r^2} & \frac{12(n-1)}{n(n+1)x_r^2} \end{pmatrix}$$

n is fold
x_r is $\sin^2\theta_{max} - \sin^2\theta_{min}$
x_m is $(\sin^2\theta_{max} + \sin^2\theta_{min})/2$
σ^2 is the data noise variance

If the near angle is zero (or close)

$$\mathrm{cov}\{\hat{m}\} = \sigma^2 \begin{pmatrix} \frac{1}{n} + \frac{12x_m^2(n-1)}{x_r^2 n(n+1)} & -\frac{12(n-1)x_m}{n(n+1)x_r^2} \\ -\frac{12(n-1)x_m}{n(n+1)x_r^2} & \frac{12(n-1)}{n(n+1)x_r^2} \end{pmatrix}$$

Curve fitting

If the near angle is zero (or close)

$$\mathrm{cov}\{\hat{m}\} = \sigma^2 \begin{pmatrix} \dfrac{2(2n-1)}{n(n+1)} & -\dfrac{6(n-1)}{n(n+1)x_r} \\ -\dfrac{6(n-1)}{n(n+1)x_r} & \dfrac{12(n-1)}{n(n+1)x_r^2} \end{pmatrix}$$

For reasonably high fold,

$$std\,dev\{\hat{r}_0\} \approx \frac{2\sigma}{\sqrt{n}}$$

Twice the std dev of the stack

$$std\,dev\{\hat{g}\} \approx \frac{2\sqrt{3}}{\sin^2 \vartheta_{\max}} \frac{\sigma}{\sqrt{n}}$$

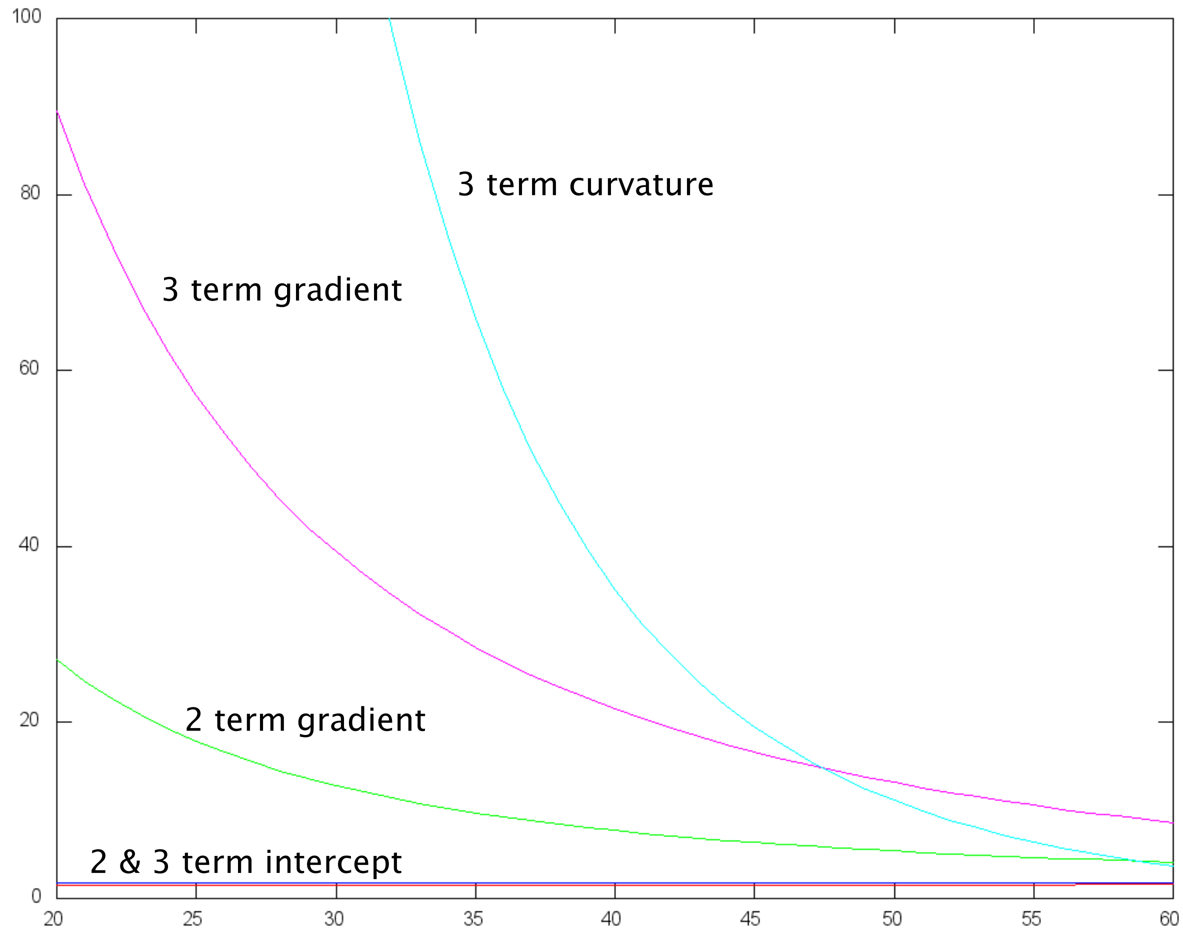
14 times the std dev of the stack for $\theta_{max} = 30°$

$$corr\{\hat{r}_0, \hat{g}\} \approx -\frac{\sqrt{3}}{2}$$

Curve fitting: 2 term v 3 term

Similar analysis is intractable for the 3 term fit. However we still expect the maximum incidence angle to be the key controlling parameter.

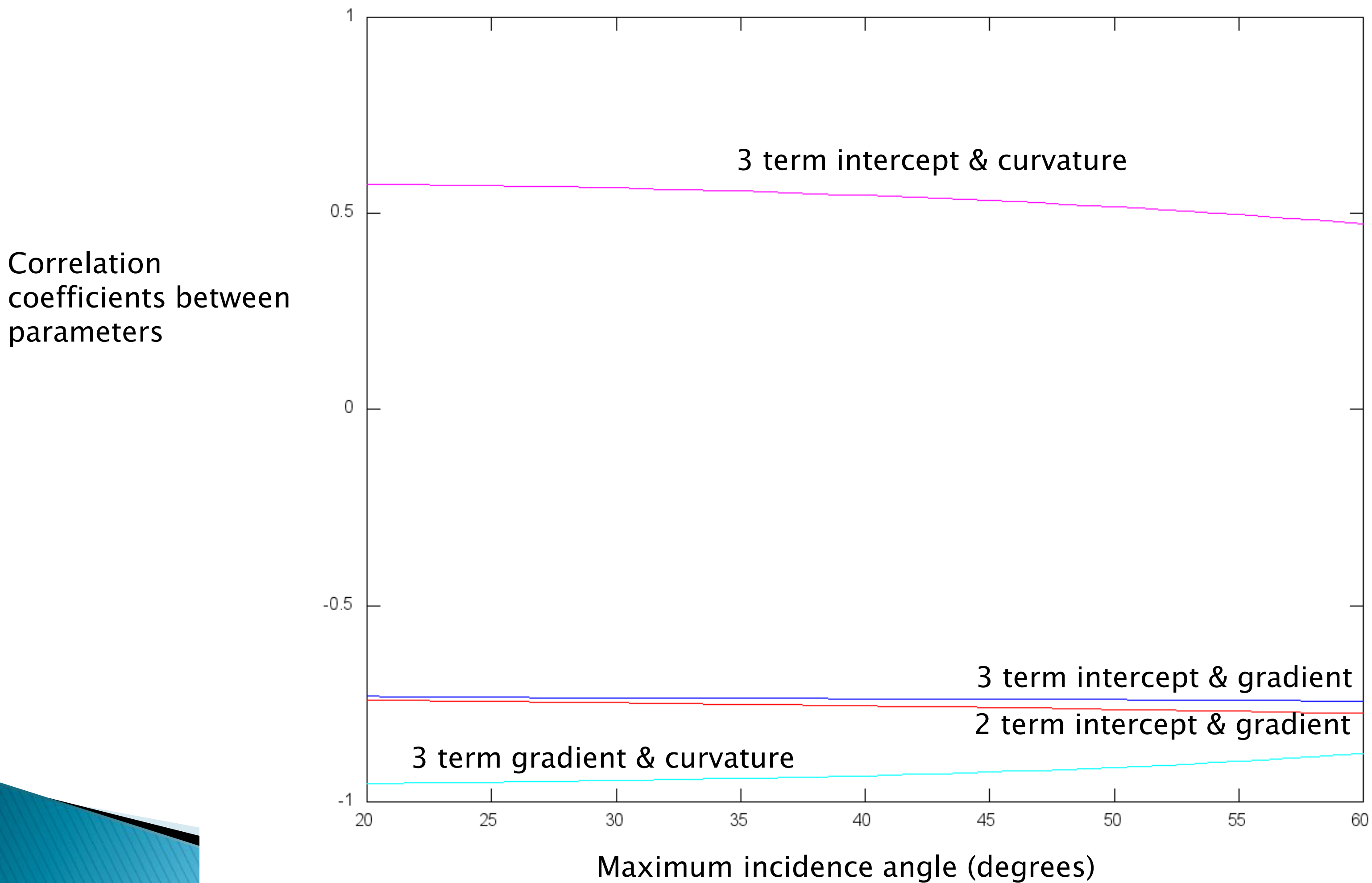
Standard deviations of the parameter estimates expressed as a multiplier of the stack standard deviation



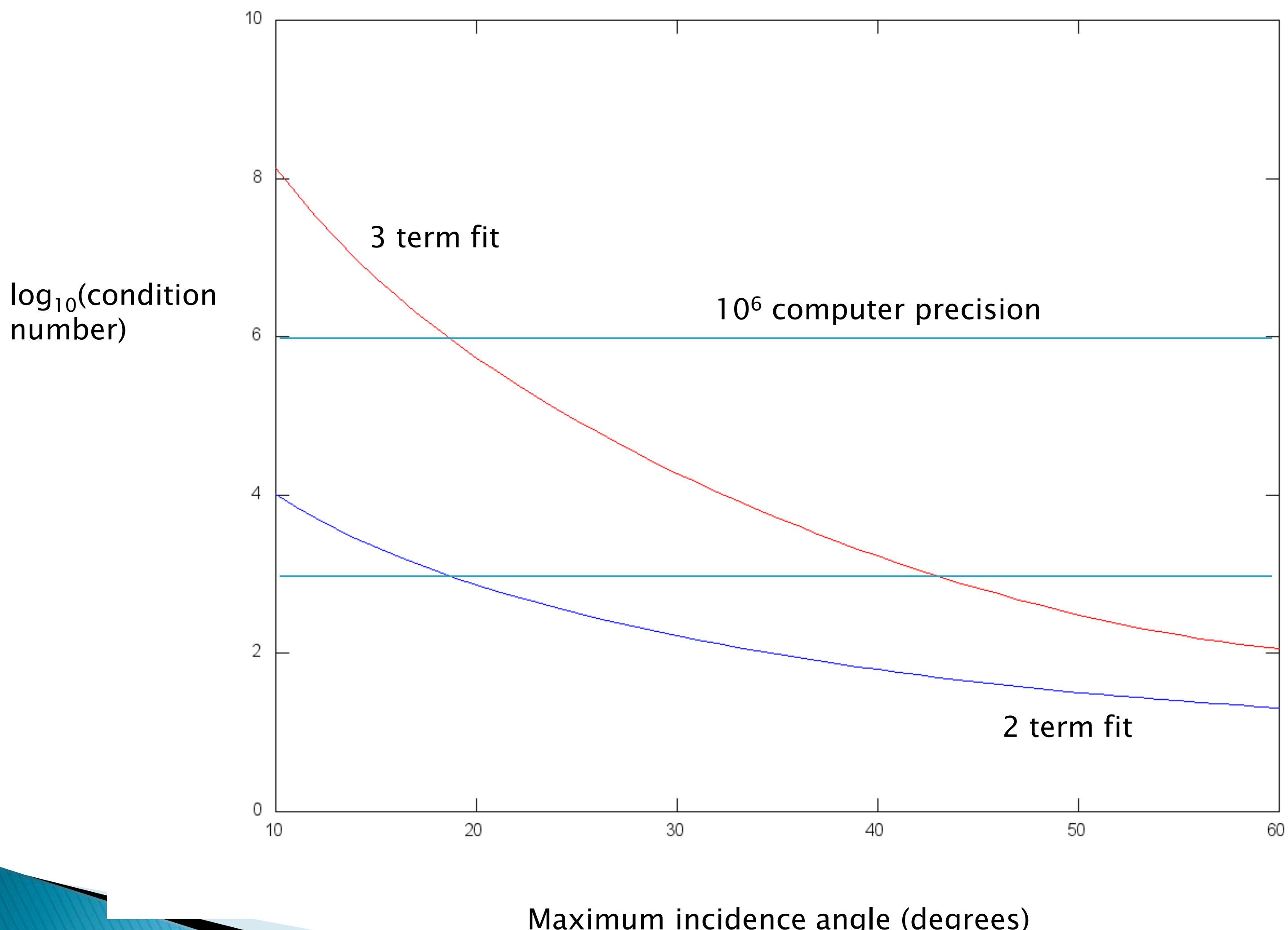
Maximum incidence angle (degrees)

Curve fitting: 2 term v 3 term

Similar analysis is intractable for the 3 term fit. However we still expect the maximum incidence angle to be the key controlling parameter.



Curve fitting: 2 term v 3 term condition numbers



Robust curve fitting

In least-squares curve fitting, outliers in the data exert excessive leverage on the solutions.
Robust methods diminish the influence of outliers.

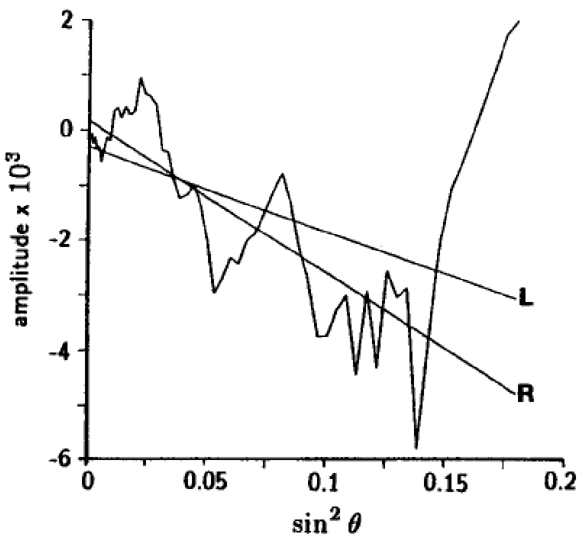


FIG. 5a.

FIG. 5. Comparison of AVO intercept and slopes estimated by least-squares and by the robust technique on seismic data set 1. The upper panel shows a section of the gather and the lower panel shows rescaled amplitudes at $t = 3.644$ s plotted against $\sin^2 \theta$; the marked lines are fitted by least-squares (L) and robustly (R). The CMPs are (a) 770 and (b) 772.

Walden, A.T., 1991, Making AVO sections more robust, Geophysical Prospecting, 39, 915-942

Robust curve fitting

It reduces correlations between parameters compared to least-squares.
But the condition number is worse.

Does it help with 3 term fit?
NO

Anisotropy

Two kinds of effect:

1) Kinematic. Overburden anisotropy changes the offset-angle mapping at the target. A smooth anisotropy model is needed from the surface down to the target.

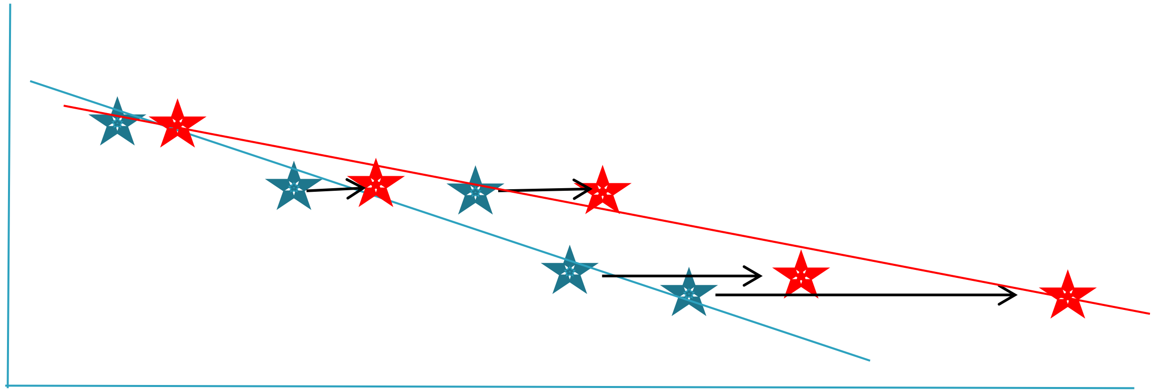
2) Dynamic. At the target, the AVA response depends on the anisotropy of upper and lower layers.

Anisotropy: Kinematic effect

Velocity is angle-dependent:

$$V_P(\vartheta) = V_{P0}\left[1 + \delta \sin^2\vartheta \cos^2\vartheta + \varepsilon \sin^4\vartheta\right]$$

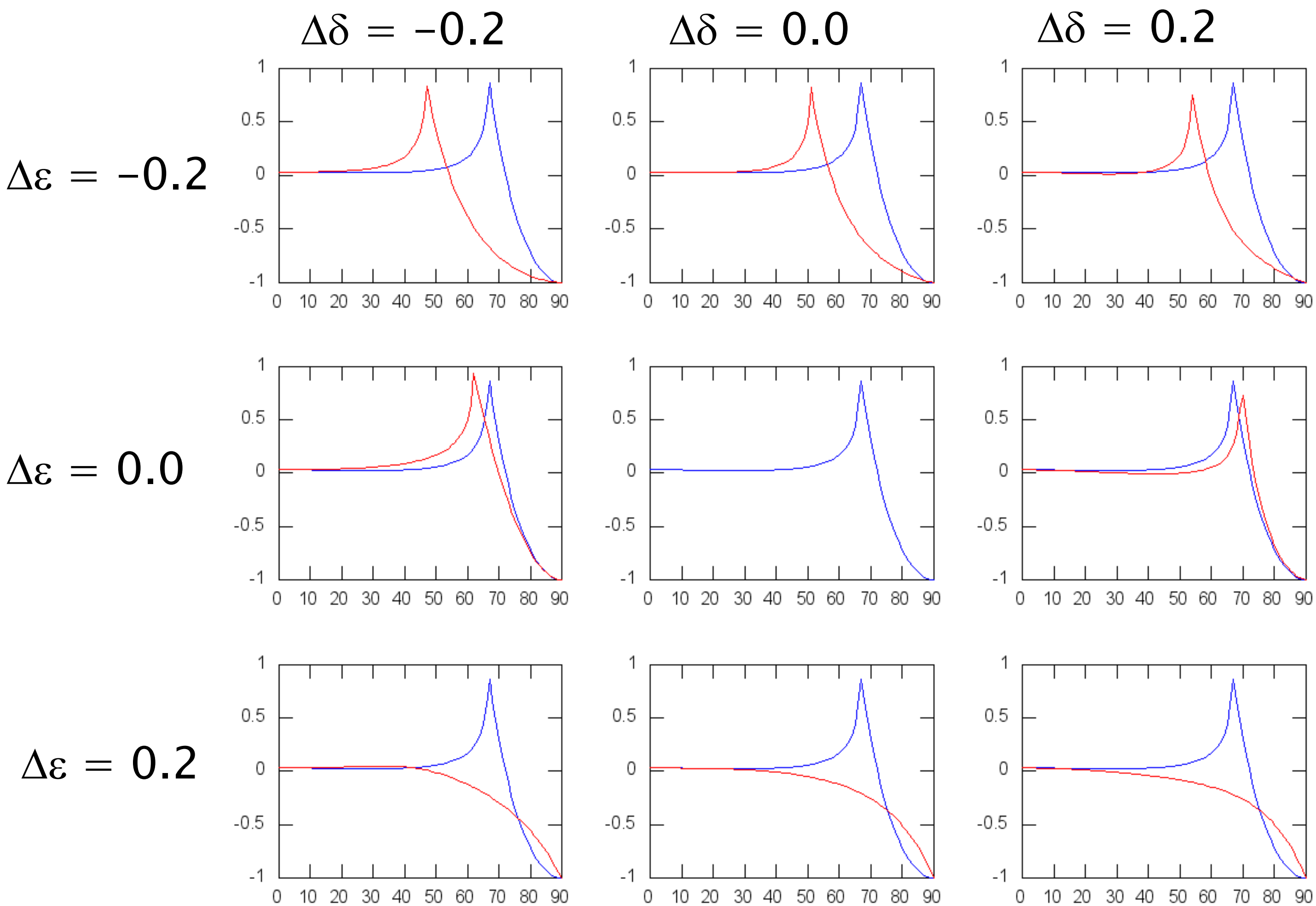
Overburden anisotropy changes the offset-angle relationship.



The angle axis is stretched depending on anisotropy
Wrong gradient (& curvature)

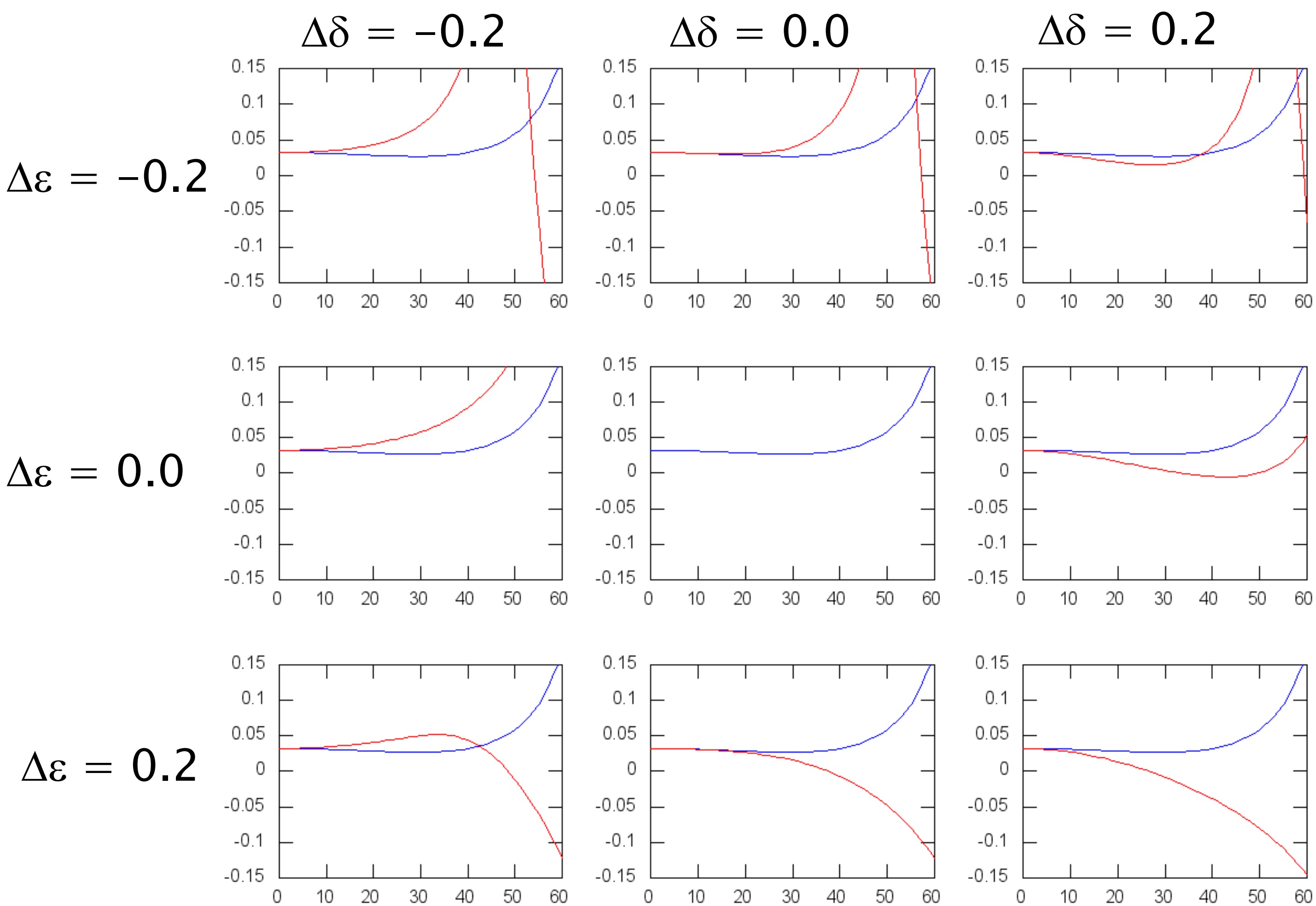
Anisotropy: Dynamic effect

Anisotropic PP reflection coefficients, Vp, Vs, ρ fixed in both layers



Anisotropy: Dynamic effect

Anisotropic PP reflection coefficients, Vp, Vs, ρ fixed in both layers



Anisotropy: Dynamic effect

Isotropic Shuey

$$r_{PP}(\vartheta) \approx \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right) + \left[\frac{1}{2}\frac{\Delta V_P}{V_P} - 2\frac{V_S^2}{V_P^2}\left(2\frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}\right)\right]\sin^2\vartheta + \frac{1}{2}\frac{\Delta V_P}{V_P}\left(\tan^2\vartheta - \sin^2\vartheta\right)$$

Anisotropic (Rüger)

$$r_{PP}(\vartheta) \approx \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right) + \left[\frac{1}{2}\frac{\Delta V_P}{V_P} - 2\frac{V_S^2}{V_P^2}\left(2\frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}\right)\right]\sin^2\vartheta + \frac{1}{2}\frac{\Delta V_P}{V_P}\left(\tan^2\vartheta - \sin^2\vartheta\right)$$
$$+ (\delta_2 - \delta_1)\sin^2\vartheta + (\varepsilon_2 - \varepsilon_1)\left(\tan^2\vartheta - \sin^2\vartheta\right)$$

δ controls the mid-angle reflectivity and modifies the AVA gradient
ε influences the higher angles and affects the curvature.

Assumptions

- Seismic data
  - No multiples or mode conversions
  - Relative amplitudes are correct
  - Stationary wavelet
  - Wavelet is known
  - No residual moveout
  - Data are correctly positioned (migrated)

- Geology
  - Horizontal, plane layers
  - Very slow lateral velocity variation
  - Thin-bed effects are not too significant

- Wave propagation
  - Plane wave reflection coefficients
  - (Small angle assumptions)
  - Approximations to Zoeppritz
  - No significant scattering

- Inversion
  - Noise is uncorrelated, iid
  - Linearised
  - Regularisation -> Solution is smooth

Elastic Impedance Inversion

$$r_{PP}(\vartheta) \approx C_1 \frac{\Delta R_1}{R_1} + C_2 \frac{\Delta R_2}{R_2} + C_3 \frac{\Delta R_3}{R_3}$$

Take a fixed angle and integrate over depth

$$\int_{z_0}^{z_1} r_{PP}(\vartheta; z)dz = \int_{z_0}^{z_1} \left[C_1 \frac{\Delta R_1}{R_1} + C_2 \frac{\Delta R_2}{R_2} + C_3 \frac{\Delta R_3}{R_3} \right] dz = C_1 \ln\left(\frac{R_1(z_1)}{R_1(z_0)} \right) + C_2 \ln\left(\frac{R_2(z_1)}{R_2(z_0)} \right) + C_3 \ln\left(\frac{R_3(z_1)}{R_3(z_0)} \right)$$

We have assumed that Vs/Vp is constant with depth!
Take exponential:

$$EI(\vartheta; z_1) = \left(\frac{R_1(z_1)}{R_1(z_0)} \right)^{C_1} \left(\frac{R_2(z_1)}{R_2(z_0)} \right)^{C_2} \left(\frac{R_3(z_1)}{R_3(z_0)} \right)^{C_3}$$

This can be done with the two term equation as well.

Elastic Impedance Inversion

$$EI(\vartheta; z_1) = \left(\frac{R_1(z_1)}{R_1(z_0)}\right)^{C_1} \left(\frac{R_2(z_1)}{R_2(z_0)}\right)^{C_2} \left(\frac{R_3(z_1)}{R_3(z_0)}\right)^{C_3}$$

This can be done with the two term equation as well.

$$EI(\vartheta) = \left(\frac{V_P}{V_{P0}}\right)^{(1+\tan^2\vartheta)} \left(\frac{\rho}{\rho_0}\right)^{(1-4\gamma^2\sin^2\vartheta)} \left(\frac{V_S}{V_{S0}}\right)^{-8\gamma^2\sin^2\vartheta}$$

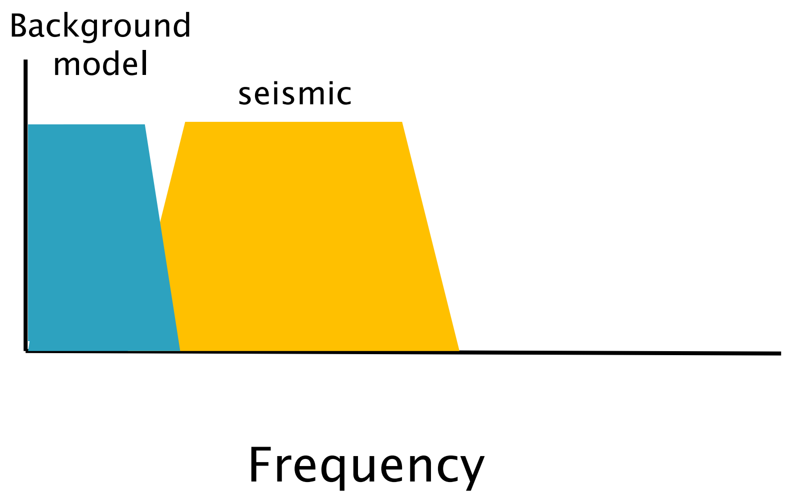
Extended elastic impedance

$$EI(\chi) = \left(\frac{V_P}{V_{P0}}\right)^{(\cos\chi+\sin\chi)} \left(\frac{\rho}{\rho_0}\right)^{(\cos\chi-4\gamma^2\sin\chi)} \left(\frac{V_S}{V_{S0}}\right)^{-8\gamma^2\sin\chi}$$

χ is a non-physical angle used to make combinations of the elastic properties. Particular choices of χ can be made to maximise sensitivity to fluids or to lithologies.

Prestack AVA Inversion

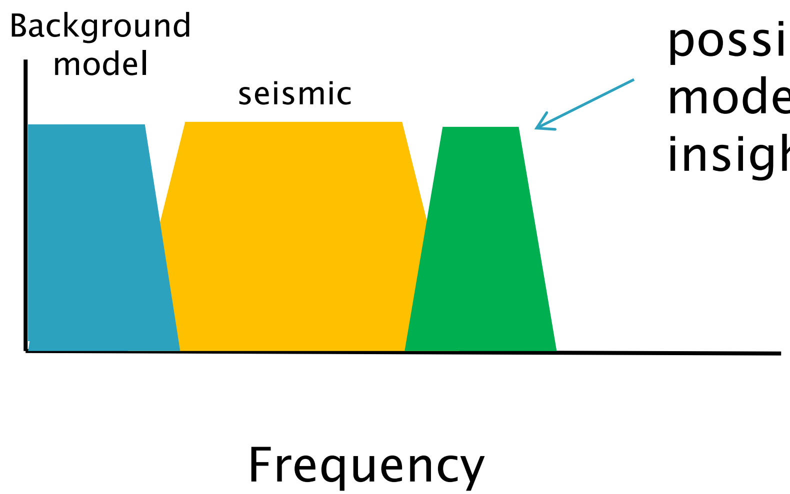
Minimise the misfit between data and synthetics by adjusting model.
Constraints may be used to control model values or relationships between them (for example, use Gardner as a soft constraint on density).
There is often an assumption of model smoothness, which further stabilises the inversion.
Some form of background model is needed to supply low frequency information.



Stochastic AVA Inversion

Geostatistical methods (eg Dubrule et al)
Bayesian (Buland et al)
Various sampling methods (MCMC, simulated annealing, etc)

Aim is to get better idea of uncertainty & ambiguity in the results.



Can explore the range of possible high frequency models. Needs geological insight.

Inversion for lithology & fluid properties

Relate elastic properties to lithology and fluid properties through rock physics models, empirical relationships, and probabilistic models. (Buland et al, Coleou et al, many others ...)

Typically porosity, saturation, Vshale are the properties of interest.
Results can include probabilistic facies classifications.

Since they include uncertainties they are useful for risk analysis.