Anelliptic approximation for P-wave phase velocity in orthorhombic media

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Outline

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- Anelliptic approximation
- Acoustic approximation
- Simplest approximation
- An application for pseudo-wave modeling
- Summary

- Orthorhombic media

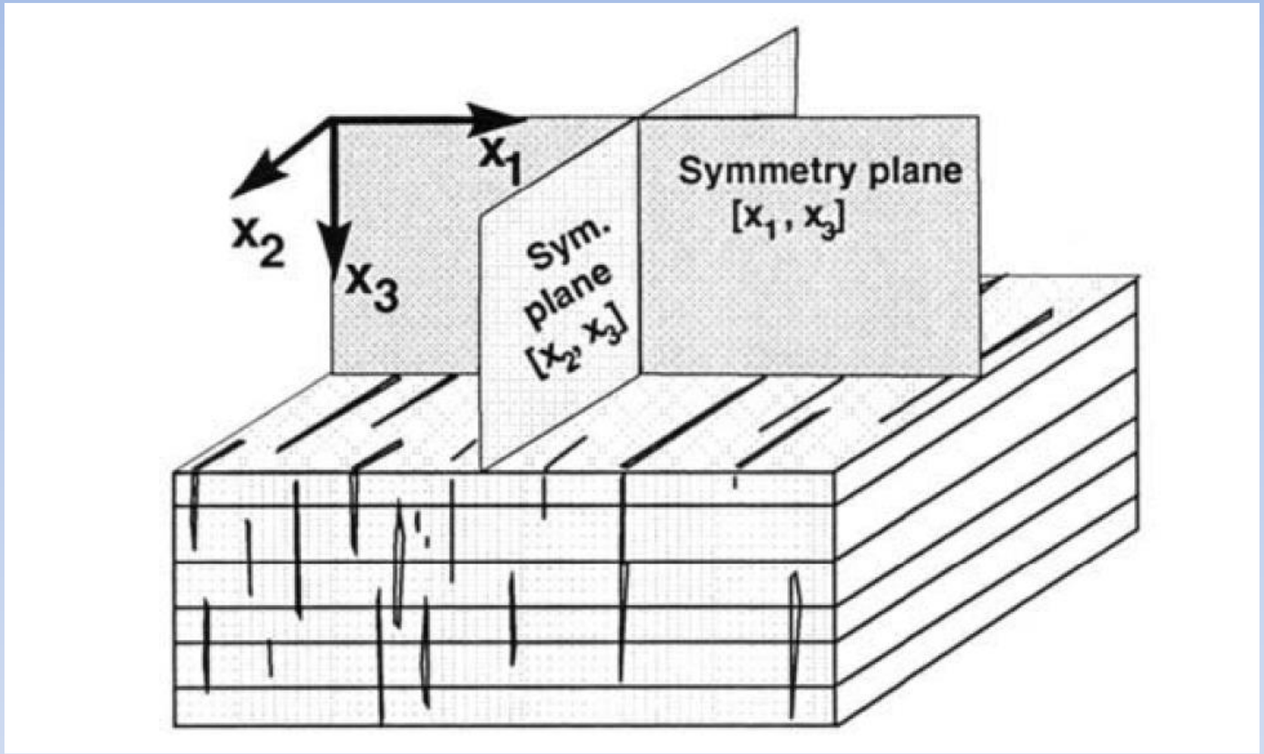


Figure 1: Orthorhombic model caused by parallel vertical fractures embedded in a finely layered medium.(Tsvankin, 2001). In this presentation, x_1-, x_2- and x_3-axes correspond to x-, y-, z-axes, respectively.

Exact expression for P-wave phase velocity

Stiffness matrix for orthorhombic medium

$$C^{ORT} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{55} & \\ & & & & & c_{66} \end{pmatrix}$$

P-wave phase velocity can be calculated from

$$\begin{vmatrix} G_{11} - \rho\upsilon^2 & G_{12} & G_{13} \\ G_{12} & G_{22} - \rho\upsilon^2 & G_{23} \\ G_{13} & G_{23} & G_{33} - \rho\upsilon^2 \end{vmatrix} = 0$$

where

$$G_{11} = c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2 \qquad G_{12} = (c_{12} + c_{66})n_1n_2$$

$$G_{22} = c_{66}n_1^2 + c_{22}n_2^2 + n_{44}n_3^2 \qquad G_{13} = (c_{13} + c_{55})n_1n_3$$

$$G_{33} = c_{55}n_1^2 + c_{44}n_2^2 + n_{33}n_3^2 \qquad G_{23} = (c_{23} + c_{44})n_2n_3$$

Here, (n_1, n_2, n_3) denotes the unit vector in P-wave propagating direction.

- Tsvankin (1997)'s notation for orthorhombic media

$$\upsilon_{p0} = \sqrt{c_{33}}$$

$$\upsilon_{s0} = \sqrt{c_{66}}$$

on z-axis

$$\varepsilon_1 = \frac{c_{22} - c_{33}}{2c_{33}}$$

$$\delta_1 = \frac{(c_{23} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}$$

$$\gamma_1 = \frac{c_{66} - c_{55}}{2c_{55}}$$

in [y, z] plane

$$\varepsilon_2 = \frac{c_{11} - c_{33}}{2c_{33}}$$

$$\delta_2 = \frac{(c_{13} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}(c_{33} - c_{55})}$$

$$\gamma_2 = \frac{c_{66} - c_{44}}{2c_{44}}$$

in [x, z] plane

$$\delta_3 = \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})}$$

in [x, y] plane

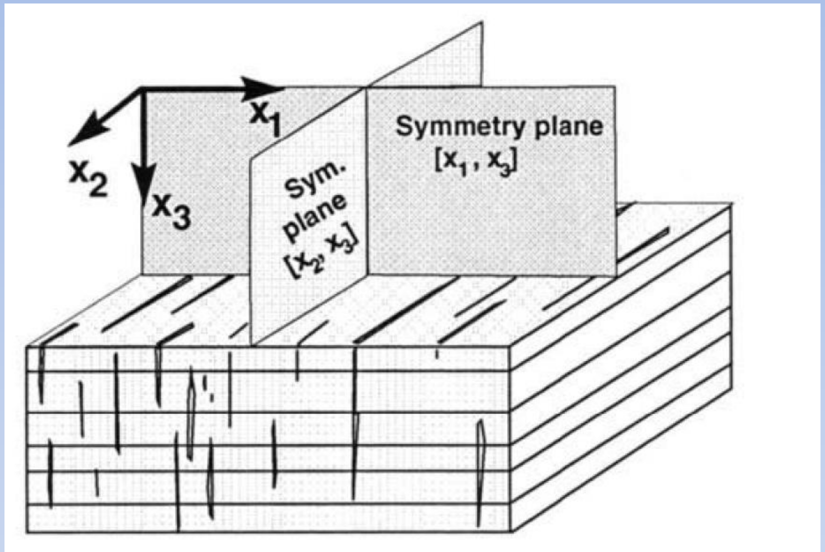


Figure 1: Orthorhombic model caused by parallel vertical fractures embedded in a finely layer medium.(Tsvankin, 2001). In this presentation, x1-, x2- and x3-axes corresponds to x-, y-, z-axex, respectively.

Anelliptic approximation for P-wave phase velocity

Fomel (2004) proposed the anelliptic approximation for P-wave velocity in VTI media.

We develop the anelliptic approximation for P-wave phase velocity in 3D orthorhombic media.

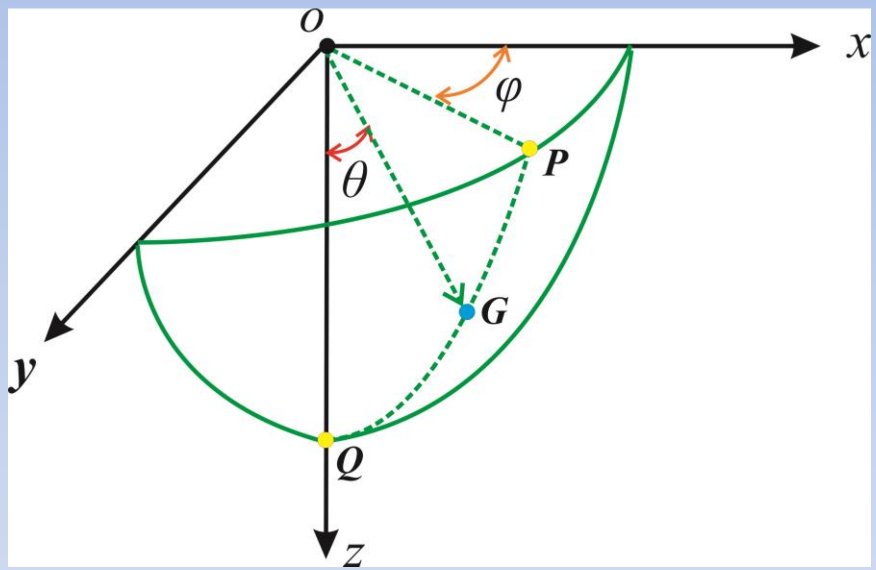


Figure 2: Schematic plot for the anelliptic approximation of P-wave phase velocity. G is a point on the P-wave phase velocity surface (formed by green solid lines).

For the phase-velocity approximation at a specified slice OPQ, we employ an anelliptic function of tilt θ. In this function, we use velocities at point P and Q and the second- and fourth-order derivatives with respect to θ at point Q.

The anelliptic approximation:

$$\upsilon_p^2(\varphi,\theta) = e(\varphi,\theta)\big(1 - s(\varphi)\big) + s(\varphi)\sqrt{e^2(\varphi,\theta) + \frac{2\alpha(\varphi)\upsilon_{p0}^2\upsilon_h^2(\varphi)}{s(\varphi)}\sin^2\theta\cos^2\theta}$$

where

$e(\varphi,\theta)$ is an elliptic function with respect to tilt ;

$s(\varphi)$ is an azimuth-dependent weight;

$\upsilon_h(\varphi)$ is the phase velocity in [x, y] plane;

$\alpha(\varphi)$ affects the approximation accuracy off z-axis.

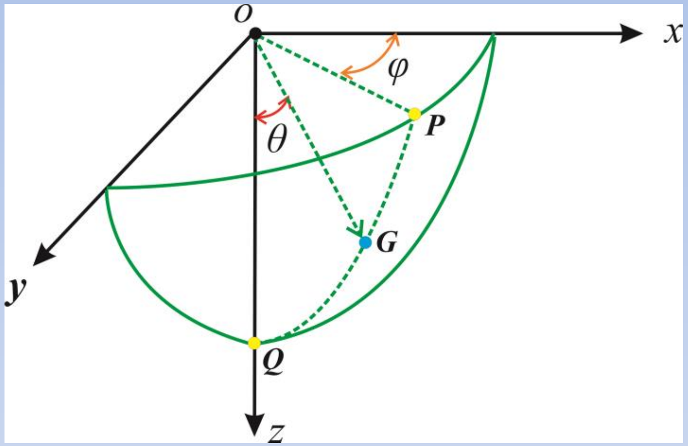


Figure 2: Schematic plot for the anelliptic approximation of P-wave phase velocity.

The elliptic function:

$$e(\varphi,\theta) = \upsilon_{p0}^2\cos^2\theta + \upsilon_h^2(\varphi)\sin^2\theta$$

All medium parameters are used in this approximation.

- Functions $S(\varphi)$ and $\alpha(\varphi)$

Series expansion for exact phase velocity:

$$\upsilon_p^2(\varphi,\theta) = \upsilon_{p0}^2 + \left.\frac{d(\upsilon_p^2)}{d(\sin^2\theta)}\right|_{\theta=0} \sin^2\theta + \frac{1}{2}\left.\frac{d^2(\upsilon_p^2)}{d(\sin^2\theta)^2}\right|_{\theta=0} \sin^4\theta + O(\sin^6\theta)$$

Series expansion for anelliptic approximation:

$$\upsilon_p^2(\varphi,\theta) = \upsilon_{p0}^2 + \left[-\upsilon_{p0}^2 + \left(1+\alpha(\varphi)\right)\upsilon_h^2(\varphi)\right]\sin^2\theta - \frac{1}{2}\frac{\alpha(\varphi)}{s(\varphi)}\frac{\upsilon_h^4(\varphi)}{\upsilon_{p0}^2}\left(2s(\varphi)+\alpha(\varphi)\right)\sin^4\theta + O(\sin^6\theta)$$

By fitting the series coefficients, we derive

$$\alpha(\varphi) = \frac{\upsilon_{p0}^2}{\upsilon_h^2(\varphi)}(1+\lambda(\varphi)) - 1 \qquad s(\varphi) = -\frac{\alpha^2(\varphi)\upsilon_h^4(\varphi)}{\mu(\varphi)\upsilon_{p0}^4 + 2\alpha(\varphi)\upsilon_h^4(\varphi)}$$

with

$$\lambda(\varphi) = \frac{1}{\upsilon_{p0}^2}\left.\frac{d\upsilon_p^2}{d(\sin^2\theta)}\right|_{\theta=0} \qquad \mu(\varphi) = \frac{1}{\upsilon_{p0}^2}\left.\frac{d^2(\upsilon_p^2)}{d(\sin^2\theta)^2}\right|_{\theta=0}$$

- Azimuth-dependent anellipticity

We extend the relation between parameter α and anellipticity η for 2D VTI case (Fomel, 2004) to 3D orthorhombic case,

$$\alpha(\varphi) = \frac{1}{1+2\eta(\varphi)} - 1$$

where parameter $\eta(\varphi)$ denotes the anellipticity in the vertical slice plane with azimuth φ .

Consequently, we derive

$$\eta(\varphi) = \frac{\upsilon_h^2(\varphi)}{2\upsilon_{p0}^2(1+\lambda(\varphi))} - \frac{1}{2}$$

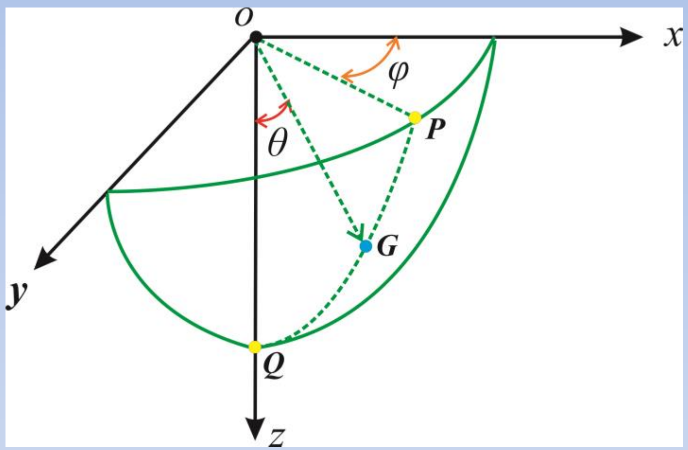


Figure 2: Schematic plot for the anelliptic approximation of P-wave phase velocity.

For $\varphi = 0$, $\eta(\varphi)$ becomes $\eta_2 = \dfrac{\varepsilon_2 - \delta_2}{1+2\delta_2}$ in [x, z] plane;

for $\varphi = \pi/2$, $\eta(\varphi)$ becomes $\eta_1 = \dfrac{\varepsilon_1 - \delta_1}{1+2\delta_1}$ in [y, z] plane.

Notation for acoustic apprxomiation

For the acoustic approximation, S-wave vertically velocity υ_{s0} is taken as 0 (Aklhalifah, 2003).

In this case, we have six indepdent medium parameters given by

$$\upsilon_{p0} = \sqrt{c_{33}}$$

on z-axis

$$r_1 = \frac{\upsilon_{nmo1}^2}{\upsilon_{p0}^2} = 1 + 2\delta_1$$

$$\eta_1 = \frac{\varepsilon_1 - \delta_1}{1 + 2\delta_1}$$

in [y, z] plane

$$r_2 = \frac{\upsilon_{nmo2}^2}{\upsilon_{p0}^2} = 1 + 2\delta_2$$

$$\eta_2 = \frac{\varepsilon_2 - \delta_2}{1 + 2\delta_2}$$

in [x, z] plane

$$r_3 = 1 + 2\delta_3$$

in [x, y] plane

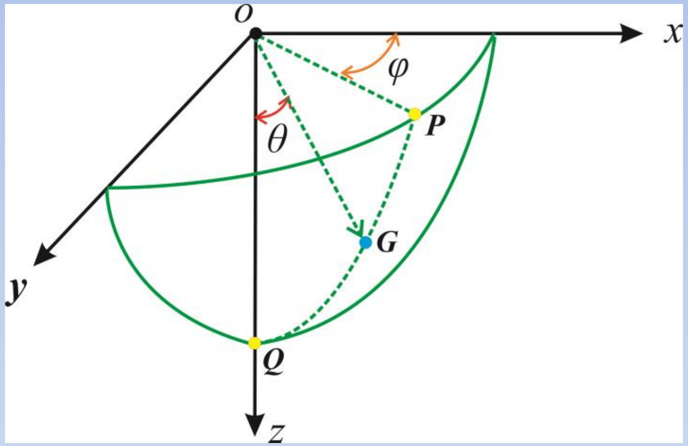


Figure 2: Schematic plot for the anelliptic approximation of P-wave phase velocity.

In this case, we derive

$$\upsilon_p^2(\varphi,\theta) = e(\varphi,\theta)\big(1-s(\varphi)\big) + s(\varphi)\sqrt{e^2(\varphi,\theta) + \frac{2\alpha(\varphi)\upsilon_{p0}^2\upsilon_h^2(\varphi)}{s(\varphi)}\sin^2\theta\cos^2\theta}$$

where

$$e(\varphi,\theta) = \upsilon_{p0}^2\cos^2\theta + \upsilon_h^2(\varphi)\sin^2\theta$$

$$\alpha(\varphi) = \frac{\upsilon_{p0}^2}{\upsilon_h^2(\varphi)}(r_1\sin^2\varphi + r_2 cos^2\varphi) - 1 \quad \Rightarrow \quad \eta(\varphi) = \frac{\upsilon_h^2(\varphi)}{2\upsilon_{p0}^2(r_1\sin^2\varphi + r_2 cos^2\varphi)} - \frac{1}{2}$$

$$s(\varphi) = -\frac{\alpha^2(\varphi)\upsilon_h^4(\varphi)}{\tilde{\mu}(\varphi)\upsilon_{p0}^4 + 2\alpha(\varphi)\tilde{\upsilon}_h^4(\varphi)} \quad \Rightarrow \quad s(\varphi) \approx 1/2$$

$$\upsilon_h^2(\varphi) = \frac{1}{2}\upsilon_{p0}^2\left(\tilde{v}_1\sin^2\varphi + \tilde{v}_2\cos^2\varphi + \sqrt{(\tilde{v}_1\sin^2\varphi + \tilde{v}_2\cos^2\varphi)^2 + \tilde{v}_3\sin^2 2\varphi}\right)$$

with

$$\tilde{\mu} = 4(r_1^2\eta_1 sin^4\varphi + r_2(-r_1 + \sqrt{r_1 r_2 r_3}(1+2\eta_2))sin^2\varphi cos^2\varphi + r_2^2\eta_2 cos^4\varphi)$$

$$\tilde{v}_1 = r_1(1+2\eta_1)$$

$$\tilde{v}_2 = r_2(1+2\eta_2)$$

$$\tilde{v}_3 = -r_2(1+2\eta_2)(r_1(1+2\eta_1) - r_2 r_3(1+2\eta_2))$$

The simplest approximation

We define a new Thomsen parameter in [x, y] plane,

$$\varepsilon_3 = \frac{c_{22} - c_{11}}{2c_{11}}$$

Anellipticity in [x, y] plane

$$\eta_3 = \frac{\varepsilon_3 - \delta_3}{1 + 2\delta_3}$$

We derive anellipticity in [x,y] plane for orthorhombic media (Vasconcelos and Tsvankin, 2006)

$$\eta_3 = \frac{\varepsilon_1 - \varepsilon_2 - \delta_3(1 + 2\varepsilon_2)}{(1 + 2\delta_3)(1 + 2\varepsilon_2)}$$

$$\Downarrow$$

$$\eta_3 = \frac{r_1(1 + 2\eta_1)}{2r_2 r_3(1 + 2\eta_2)} - \frac{1}{2}$$

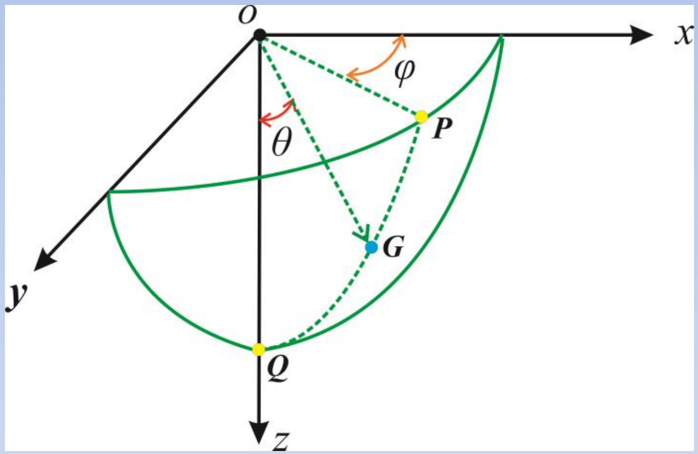


Figure 2: Schematic plot for the anelliptic approximation of P-wave phase velocity.

For the simplest approximation, anellipticity $\eta_3 = 0$ and parameter $s(\varphi) \approx 1/2$.

In this case, we derive

$$\upsilon_p^2(\varphi,\theta) = \frac{e(\varphi,\theta) + \sqrt{e^2(\varphi,\theta) - 8\upsilon_{p0}^4 \beta(\varphi)\sin^2\theta\cos^2\theta}}{2}$$

where

$$e(\varphi,\theta) = \upsilon_{p0}^2(\cos^2\theta + (r_1(1+2\eta_1)\sin^2\varphi + r_2(1+2\eta_2)\cos^2\varphi)\sin^2\theta)$$

$$\beta(\varphi) = r_1\eta_1\sin^2\varphi + r_2\eta_2\cos^2\varphi$$

The parameter $\eta(\varphi)$ describes the strength of anellipticity for the vertical plane with azimuth of φ,

$$\eta(\varphi) = \frac{r_1\eta_1\sin^2\varphi + r_2\eta_2\cos^2\varphi}{r_1\sin^2\varphi + r_2\cos^2\varphi}$$

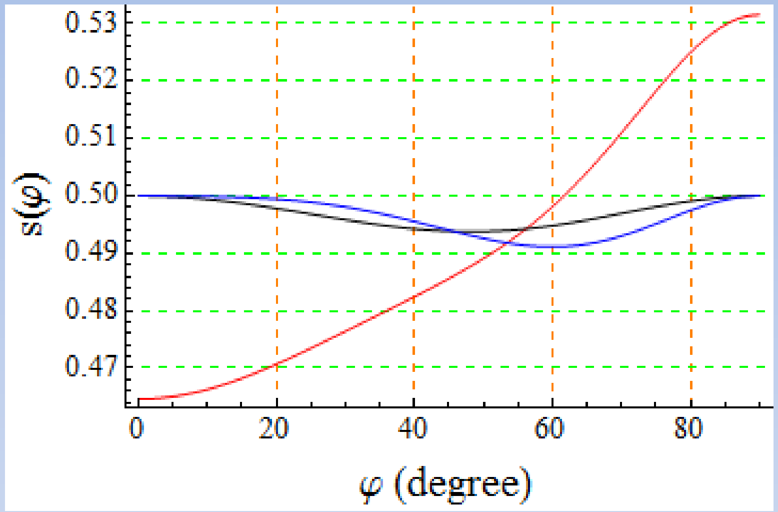
Numerical examples

- The variation of functions $s(\varphi)$ and $\eta(\varphi)$ versus azimuth.
- Accuracy test of our approximations.
- The influence of anellipticity parameter η_3 on the accuracy of acoustic and the simplest approximations.

Table 1. Parameters for an orthorhombic medium

υ_{p0}	υ_{s0}	ε_1	δ_1	γ_1	ε_2	δ_2	γ_2	δ_3
3km/s	1.2km/s	0.25	0.05	0.28	0.15	-0.1	0.15	0.15

- The variation of functions $s(\varphi)$ and $\eta(\varphi)$ versus azimuth.



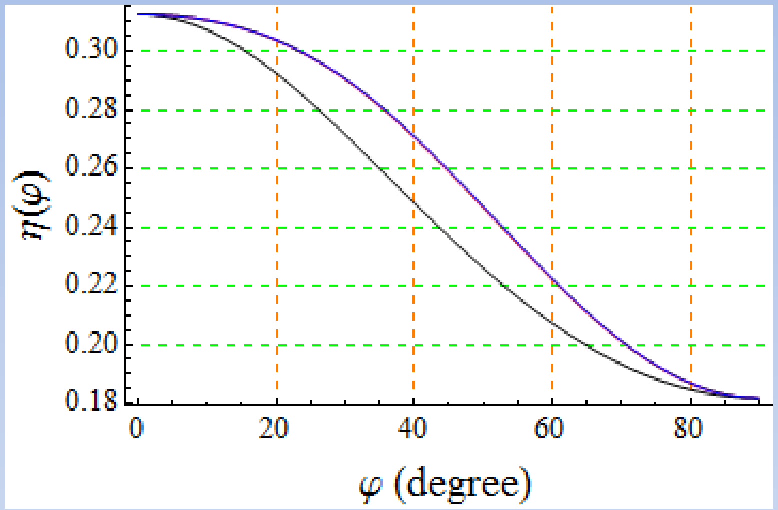


Figure 3: $s(\varphi)$ (left) and $\eta(\varphi)$ (right) calculated from anelliptic (red line), acoustic (blue line) and the simplest (black line) approximations.

- Accuracy test of our approximations.

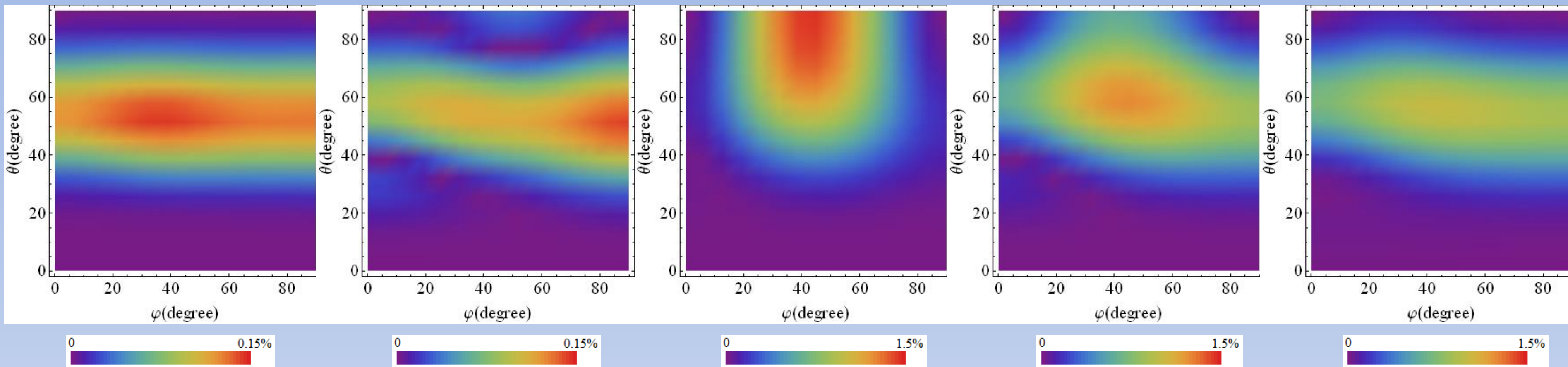


Figure 4: Relative absolute error of several approximations in an orthorhombic medium. From left to right, plots correspond to our anelliptic, acoustic and simplest approximations as well as Tsvankin (1997) and Daley (2004) approximations.

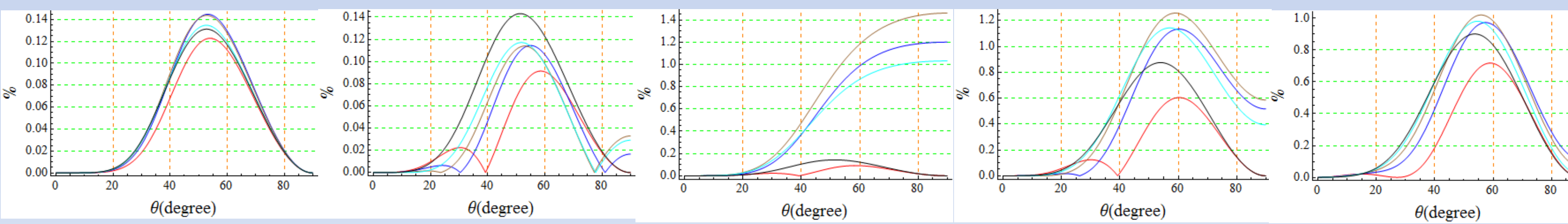


Figure 5: Error slices extracted from Figure 4 along azimuths of 0 (red line), $\pi/6$ (brown line), $\pi/3$ (cyan line) and $\pi/2$ (black line), respectively. The order of plots is the same as those in Figure 4.

Note that $s(\varphi) \approx 1/2$ is taken for our acoustic approximation.

- The influence of anellipticity parameter η_3 on the accuracy of acoustic and the simplest approximations

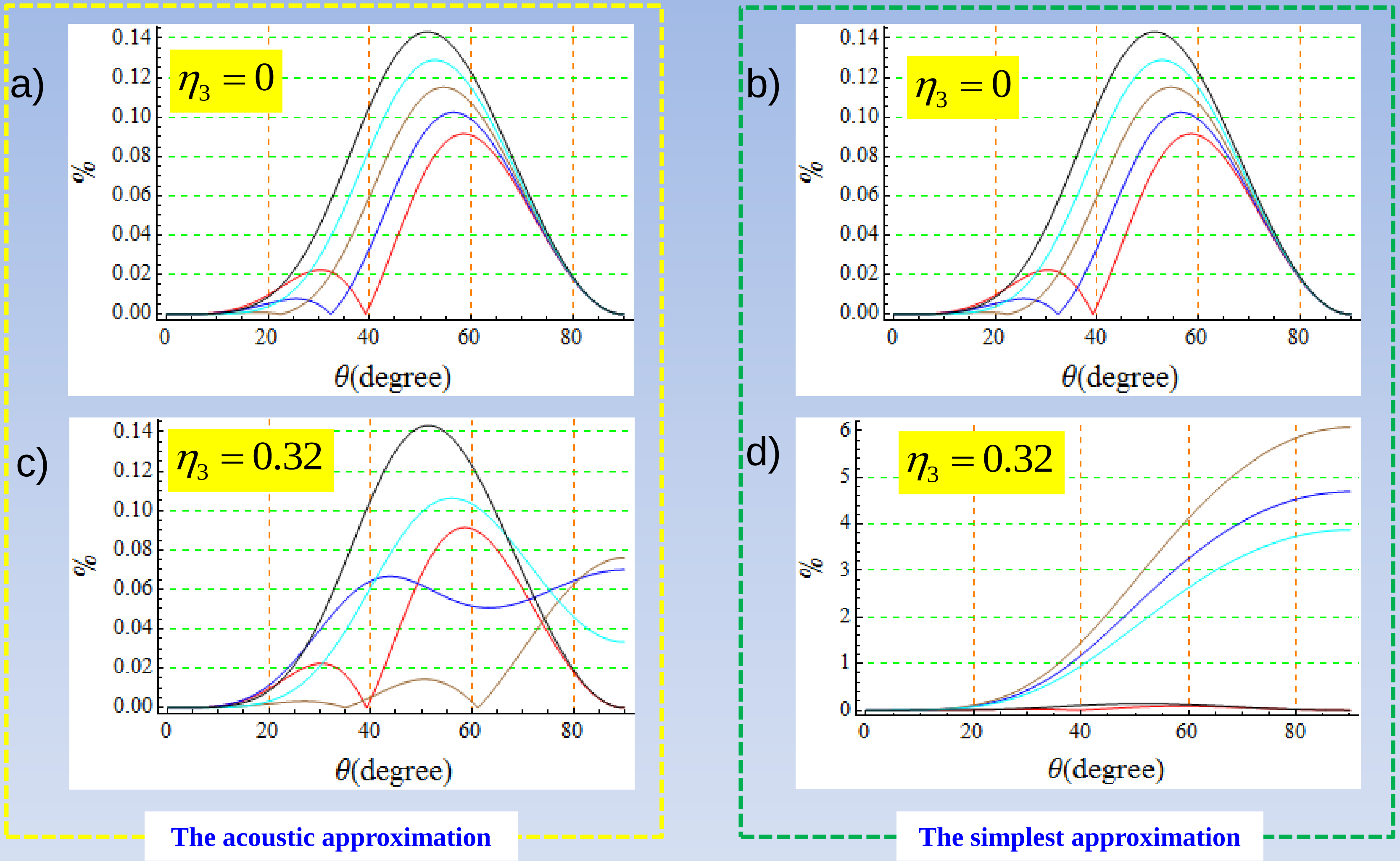


Figure 6: Relative absolute error of acoustic (plots a and c) and elliptic approximations (plots b and d) for two orthorhombic media with elliptical (plots a and b) and anelliptical (plot c and d) properties in [x, y] plane, respectively. The notation for colored lines is the same as Figure 5. Only medium parameter δ_3 given in “numerical examples” is adjusted to be 0.077 and -0.15 for orthorhombic medium with elliptical and strong anelliptical properties, respectively.

Application for pseudo-wave modeling

The pseudo-wave propagation operator (Song and Alkhalifah, 2013):

$$P(\boldsymbol{x},t+\Delta t) = -P(\boldsymbol{x},t-\Delta t) = 2\int_{-\infty}^{+\infty} P(\boldsymbol{k},t)\cos[\Phi(\boldsymbol{k},\boldsymbol{x})\Delta t]e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k}$$

with

$$\Phi(\boldsymbol{k},\boldsymbol{x}) = \boldsymbol{k}\upsilon(\boldsymbol{k},\boldsymbol{x})$$

Here,

$P(\boldsymbol{x},t)$ is wavefield;

$\boldsymbol{k},\ \boldsymbol{x}$ are wavenumber vector and spatial coordinate vector;

$\upsilon(\boldsymbol{k},\boldsymbol{x})$ is phase velocity.

From our acoustic approximation, we derive

$$\Phi^2(\boldsymbol{k},\boldsymbol{x}) = \frac{1}{2}E(\boldsymbol{k},\boldsymbol{x}) + \frac{1}{2}\sqrt{E^2(\boldsymbol{k},\boldsymbol{x}) + F(\boldsymbol{k},\boldsymbol{x})}$$

where

$$E(\boldsymbol{k},\boldsymbol{x}) = \upsilon_{p0}^2(\boldsymbol{x})k_z^2 + G(k_x,k_y,\boldsymbol{x})$$

$$F(\boldsymbol{k}) = 4\upsilon_{p0}^2(\boldsymbol{x})\left[\upsilon_{p0}^2(\boldsymbol{x})(r_1(\boldsymbol{x})k_y^2 + r_2(\boldsymbol{x})k_x^2) - G(k_x,k_y,\boldsymbol{x})\right]k_z^2$$

with

$$G(k_x,k_y,\boldsymbol{x}) = \frac{1}{2}\upsilon_{p0}^2(\boldsymbol{x})\left[\tilde{v}_1(\boldsymbol{x})k_y^2 + \tilde{v}_2(\boldsymbol{x})k_x^2 + \sqrt{[\tilde{v}_1k_y^2(\boldsymbol{x}) + \tilde{v}_2(\boldsymbol{x})k_x^2]^2 + 4\tilde{v}_3k_x^2k_y^2}\right]$$

For isotropic media, the phase function is reduced to $\Phi^2(\boldsymbol{k},\boldsymbol{x}) = \upsilon_{p0}^2(\boldsymbol{x})(k_x^2 + k_y^2 + k_z^2)$

- Two orthrohombic models

A homogeneous orthorhombic medium:

Table 2. Parameters for an homogenous orthorhombic medium

υ_{p0}	ε_1	δ_1	ε_2	δ_2	δ_3
3km/s	0.25	0.05	0.15	-0.1	0.15

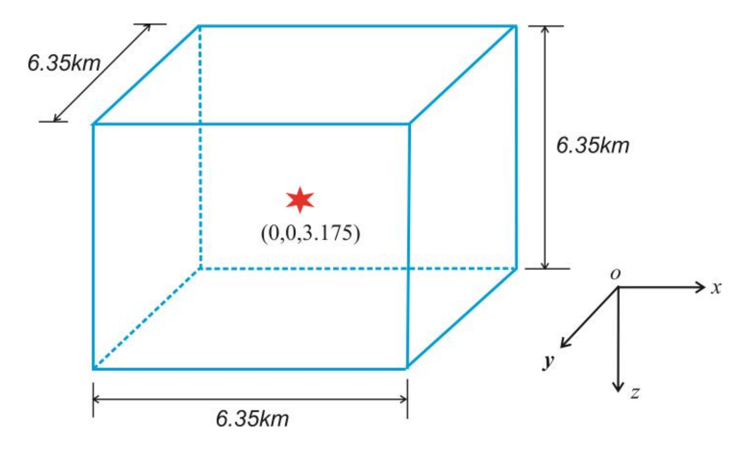


Figure 7: Setup of both models

A vertically factorized orthorhombic medium:

$$\upsilon_{p0}(z) = 1.5 + 0.472441z$$

where the units for υ_{p0} and z are "*km/s*" and "*km*".

For $z = 0km$ (top), $\upsilon_{p0} = 1.5km/s$

For $z = 6.35km$ (bottom), $\upsilon_{p0} = 4.5km/s$

Other parameters are taken from Tabel 2.

Both models have the same value of $\upsilon_{p0} = 3\mathrm{km/s}$ at the depth of source.

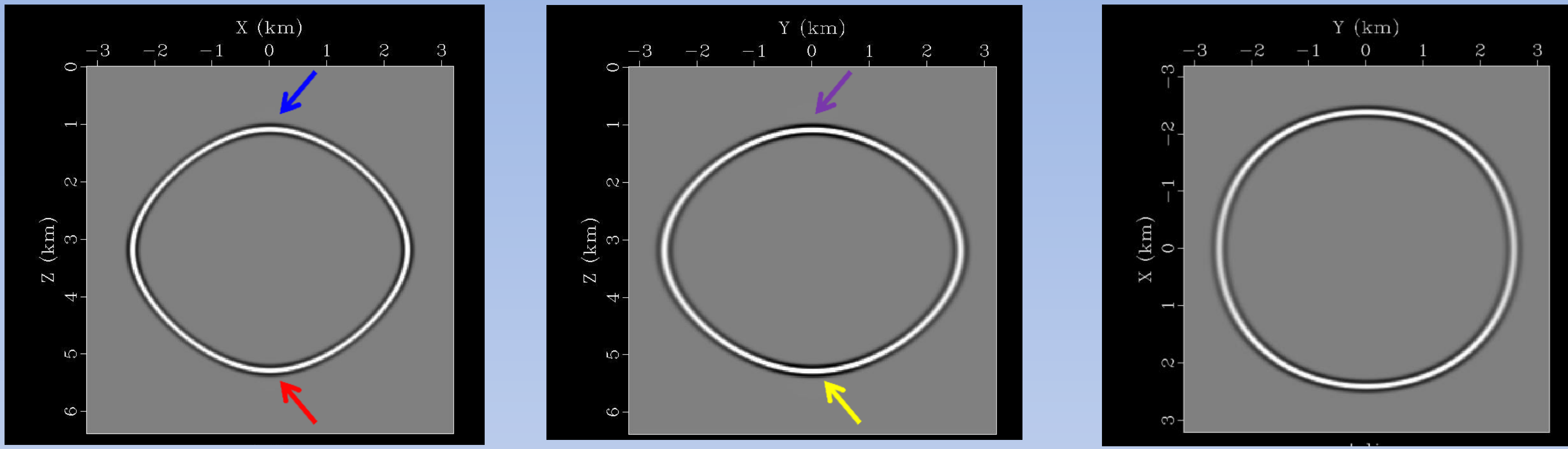


Figure 8: Three slices for the wavefield snapshot at t=0.8s in a 3D homogneous orthorhombic medium. All slices cross the position of source.

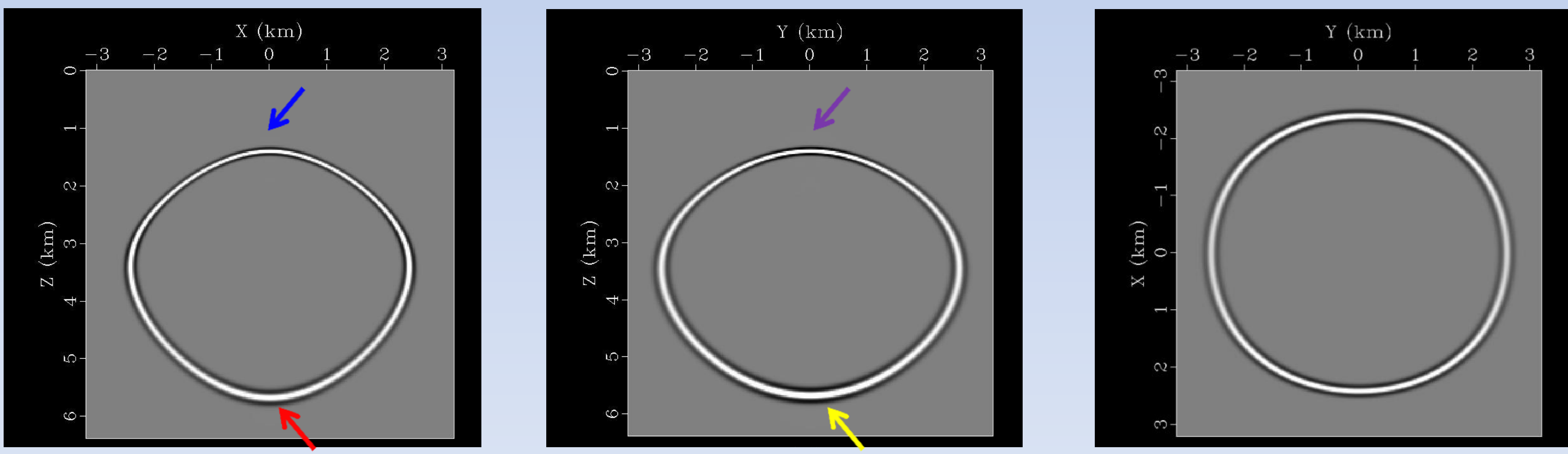


Figure 9: Three slices for the wavefield snapshot at t=0.8s in a 3D vertically factorized orthorhombic medium. All slices cross the position of source.

Summary

- We develop an anelliptic approximation (with nine independent medium parameters) for P-wave phase velocity in orthorhombic media. The simplified formulas are obtained for acoustic and simplest approximations, respectively.
- The acoustic approximation (with six independent medium parameters) has good accuracy even for the orthorhombic media with a strongly anelliptical property in horizontal plane.
- The simplest approximation (with five independent medium parameters) is valid under assumption of weak anellipticity in the horizontal plane.
- Our phase velocity approximations can be applied to the problem of pseudo-wave modeling and reverse time migration in orthorhombic media.

Acknowledgements

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