# First-order ray tracing for P and S waves in inhomogeneous weakly anisotropic media

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## Outline:

- Introduction
- Basic formulae
- FORT and FODRT for P waves
- FORT and FODRT for S waves
- Conclusions
- Future plans

# Introduction

- anisotropy often weak
- standard "anisotropic" ray tracers
  - too complicated
  - often collapse during S-wave treatment
  - do not take into account S-wave coupling

Solution: consider weak anisotropy
as a perturbation of isotropy

# Basic formulae

## Perturbation of an isotropic reference medium

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl} \ , \qquad a^0_{ijkl} = (\alpha^2 - 2\beta^2)\delta_{ij}\delta_{kl} + \beta^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$a_{ijkl}$ - density-normalized elastic moduli in a weakly anisotropic medium

$a^0_{ijkl}$ - density-normalized elastic moduli in a reference isotropic medium ($\alpha$, $\beta$ - P, S reference velocities)

$\Delta a_{ijkl}$ - perturbation of density-normalized elastic moduli $a^0_{ijkl}$

$|\Delta a_{ijkl}|/|a_{ijkl}|$ - small parameter

# Basic formulae

## Christoffel matrix, its eigenvalues, eigenvectors

$\Gamma_{ik}(\mathbf{x},\mathbf{p}) = a_{ijkl}p_j p_l$ - generalized Christoffel matrix

$G(\mathbf{x},\mathbf{p}) = \Gamma_{ik}(\mathbf{x},\mathbf{p})g_i g_k$ - eigenvalue of $\mathbf{\Gamma}(\mathbf{x},\mathbf{p})$

$\mathbf{g}$ - eigenvector of $\mathbf{\Gamma}(\mathbf{x},\mathbf{p})$

$\mathbf{p}$ - slowness vector

# Basic formulae

**Basic idea of FORT and FODRT:**

**replace exact eigenvalue of $\boldsymbol{\Gamma}(\mathbf{x},\mathbf{p})$ by its first-order approximation!**

## Perturbation of an eigenvalue

$$G(\mathbf{x},\mathbf{p}) \sim G^{(0)}(\mathbf{x},\mathbf{p}) + \Delta G(\mathbf{x},\mathbf{p})$$

$G(\mathbf{x},\mathbf{p})$ - first-order approximation of exact eigenvalue

$G^{(0)}(\mathbf{x},\mathbf{p})$ - eigenvalue of $\boldsymbol{\Gamma}$ in reference isotropic medium

$\Delta G(\mathbf{x},\mathbf{p})$ - first-order perturbation of $G^{(0)}(\mathbf{x},\mathbf{p})$

# Basic formulae

## FORT for P waves

$G^{[3]}(\mathbf{x}, \mathbf{p})$ - first-order approximation of the largest eigenvalue

## FORT for S waves

$$G^{[\mathcal{M}]}(\mathbf{x}, \mathbf{p}) = \frac{1}{2}[G^{[1]}(\mathbf{x}, \mathbf{p}) + G^{[2]}(\mathbf{x}, \mathbf{p})]$$

$G^{[\mathcal{M}]}(\mathbf{x}, \mathbf{p})$ - first-order approximation of an exact mean eigenvalue

$G^{[I]}(\mathbf{x}, \mathbf{p})$ - S-wave first-order eigenvalues of $\mathbf{\Gamma}$, $\quad I = 1, 2$

# Basic formulae

$G(\mathbf{x}, \mathbf{p})$ - first-order approximation of an exact eigenvalue

Eikonal equation: $G(\mathbf{x}, \mathbf{p}) = 1$

Ray-tracing equations (FORT):

$$dx_i/d\tau = \tfrac{1}{2}\partial G/\partial p_i \ , \qquad dp_i/d\tau = -\tfrac{1}{2}\partial G/\partial x_i$$

$x_i$ - ray coordinates of the first-order ray

$p_i$ - components of the first-order slowness vector $\mathbf{p}$

$\tau$ - first-order traveltime

# Basic formulae

## Dynamic ray-tracing equations (FODRT):

$$dX_i^{(I)}/d\tau = \frac{1}{2}\left(\frac{\partial^2 G(\mathbf{x},\mathbf{p})}{\partial p_i \partial x_j} X_j^{(I)} + \frac{\partial^2 G(\mathbf{x},\mathbf{p})}{\partial p_i \partial p_j} Y_j^{(I)}\right)$$

$$dY_i^{(I)}/d\tau = -\frac{1}{2}\left(\frac{\partial^2 G(\mathbf{x},\mathbf{p})}{\partial x_i \partial x_j} X_j^{(I)} + \frac{\partial^2 G(\mathbf{x},\mathbf{p})}{\partial x_i \partial p_j} Y_j^{(I)}\right)$$

$$X_i^{(I)} = [\partial x_i / \partial \gamma^{(I)}]_{\tau=const} \ , \qquad Y_i^{(I)} = [\partial p_i / \partial \gamma^{(I)}]_{\tau=const}$$

$\gamma^{(I)}$ - ray parameters (e.g., take-off angles)

$\mathcal{L} = |\mathbf{X}^{(1)} \times \mathbf{X}^{(2)}|^{1/2}$ - first-order geometrical spreading

# Basic formulae

## Displacement vector of P wave

$$\mathbf{u}(\tau,\omega) = \mathcal{C}(\tau)\mathbf{f}^{[3]}(\tau)\,\exp(\mathrm{i}\omega\tau)$$

$\mathbf{u}(\tau,\omega)$ - displacement vector of P wave

$\tau$ - second-order traveltime along P-wave ray

$\mathcal{C}(\tau)$ - amplitude term, proportional to $\mathcal{L}^{-1}$

$\mathbf{f}^{[3]}(\tau)$ - second-order P-wave polarization vector

# Basic formulae

## Displacement vector of coupled S waves

$$\mathbf{u}(\tau,\omega) = [\mathcal{A}(\tau)\mathbf{f}^{[1]}(\tau) + \mathcal{B}(\tau)\mathbf{f}^{[2]}(\tau)]\ D^{[\mathcal{M}]}(\tau)\ \exp(\mathrm{i}\omega\tau)$$

$\mathbf{u}(\tau,\omega)$ - displacement vector of coupled S waves

$\tau$ - second-order traveltime along common S-wave ray

$\mathcal{A}(\tau)$, $\mathcal{B}(\tau)$ - amplitude terms

$\mathbf{f}^{[K]}(\tau)$ - define the second-order S-wave polarization plane

$D^{[\mathcal{M}]}(\tau)$ - common S-wave ray amplitude, proportional to $\mathcal{L}^{-1}$

# Basic formulae

## Second-order coupling equations

$$\begin{pmatrix} d\mathcal{A}/d\tau \\ d\mathcal{B}/d\tau \end{pmatrix} = -\mathrm{i}\omega/2 \begin{pmatrix} M_{11}(\mathbf{x},\mathbf{p}^{[\mathcal{M}]})-1 & M_{12}(\mathbf{x},\mathbf{p}^{[\mathcal{M}]}) \\ M_{12}(\mathbf{x},\mathbf{p}^{[\mathcal{M}]}) & M_{22}(\mathbf{x},\mathbf{p}^{[\mathcal{M}]})-1 \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}$$

$$M_{KL}(\mathbf{x},\mathbf{p}^{[\mathcal{M}]}) = B_{KL} - B_{K3}B_{L3}/(B_{33}-1)$$

$B_{kl} = B_{kl}(\mathbf{x},\mathbf{p}^{[\mathcal{M}]})$ - weak anisotropy matrix

# Basic formulae

$$B_{mn} = B_{mn}(\mathbf{x}, \mathbf{p}) = \Gamma_{ik}(\mathbf{x}, \mathbf{p}) e_i^{[m]}(\mathbf{x}) e_k^{[n]}(\mathbf{x})$$ - weak anisotropy matrix

$\Gamma_{ik}$ - elements of Christoffel matrix $\mathbf{\Gamma}$

$\mathbf{e}^{[m]}$ - triplet of orthonormal vectors

$\mathbf{e}^{[3]} = c\mathbf{p}$, $\mathbf{e}^{[1]}, \mathbf{e}^{[2]}$ arbitrarily in the plane $\perp \mathbf{e}^{[3]}$

$\mathbf{p}$ - first-order P- or common S-wave slowness vector

$c$ - first-order P- or common S-wave phase velocity

# Basic formulae

## S-wave polarization plane

$$\mathbf{f}^{[K]}(\mathbf{p}^{[\mathcal{M}]}) = \mathbf{e}^{[K]}(\mathbf{p}^{[\mathcal{M}]}) + \mathbf{e}^{[3]}(\mathbf{p}^{[\mathcal{M}]}) B_{K3}(\mathbf{p}^{[\mathcal{M}]})/(1 - B_{33}(\mathbf{p}^{[\mathcal{M}]}))$$

$\mathbf{f}^{[K]}(\mathbf{p}^{[\mathcal{M}]})$ define S-wave polarization plane perpendicular to

$$\mathbf{f}^{[3]}(\mathbf{p}^{[\mathcal{M}]}) = \mathbf{e}^{[3]}(\mathbf{p}^{[\mathcal{M}]}) + (B_{13}(\mathbf{p}^{[\mathcal{M}]})\mathbf{e}^{[1]}(\mathbf{p}^{[\mathcal{M}]}) + B_{23}(\mathbf{p}^{[\mathcal{M}]})\mathbf{e}^{[2]}(\mathbf{p}^{[\mathcal{M}]}))/ \\ [1 - \tfrac{1}{2}(B_{11}(\mathbf{p}^{[\mathcal{M}]}) + B_{22}(\mathbf{p}^{[\mathcal{M}]}))]$$

$\mathbf{f}^{[i]}(\mathbf{p}^{[\mathcal{M}]})$ - non-perpendicular, non-unit vectors

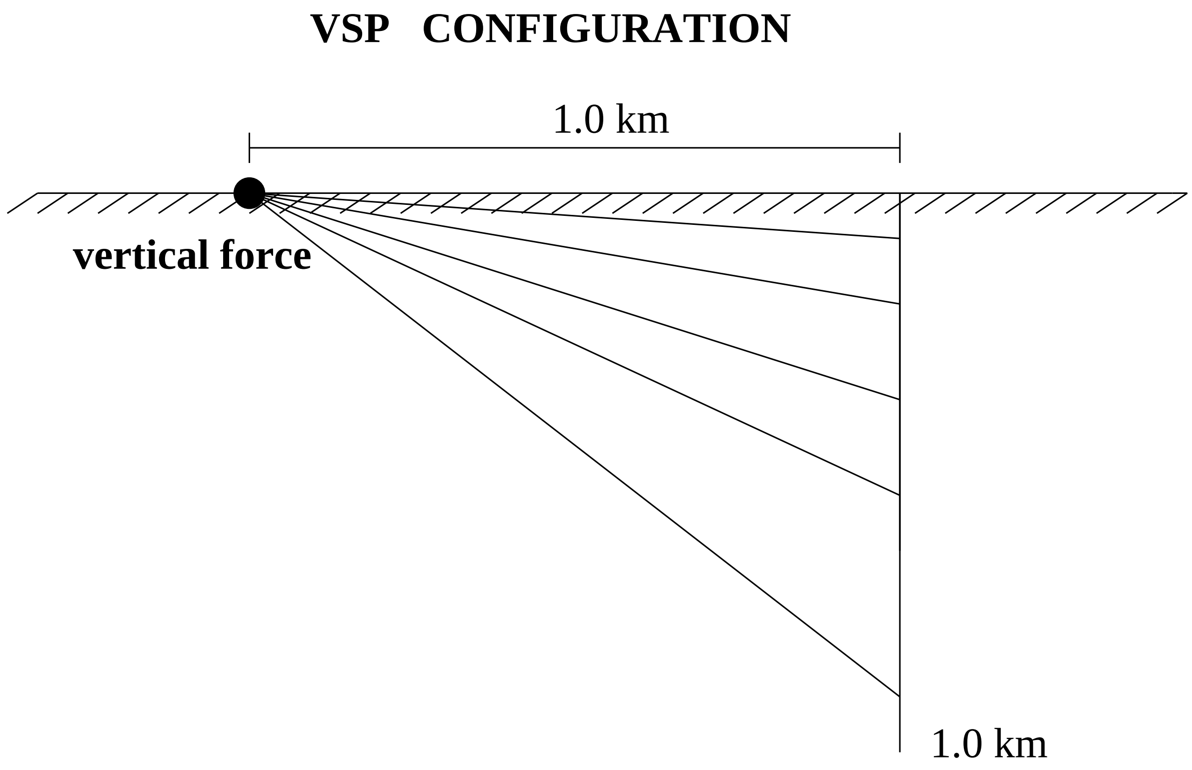
# Basic formulae

## P-wave polarization vector

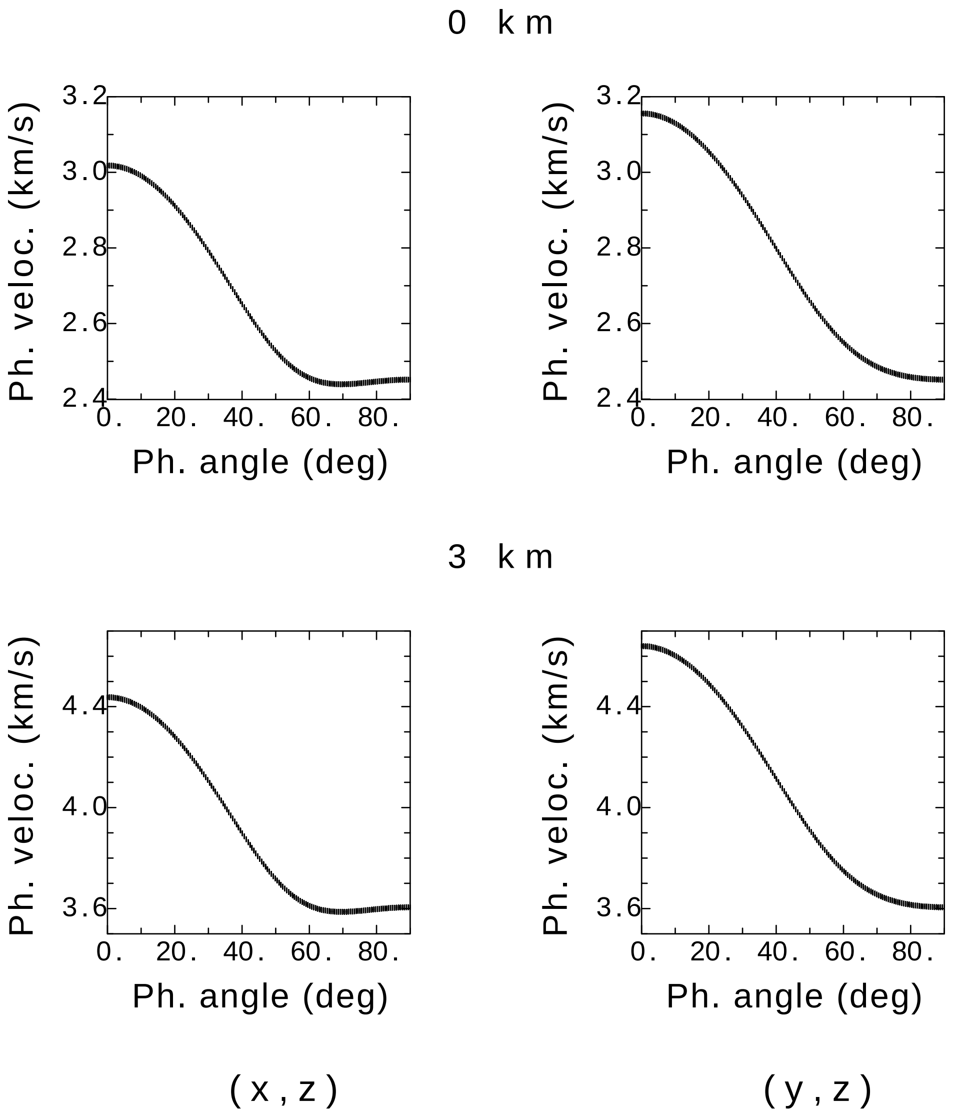
$$\mathbf{f}^{[3]}(\mathbf{p}^{[3]}) = \mathbf{e}^{[3]}(\mathbf{p}^{[3]}) + (B_{13}(\mathbf{p}^{[3]})\mathbf{e}^{[1]}(\mathbf{p}^{[3]}) + B_{23}(\mathbf{p}^{[3]})\mathbf{e}^{[2]}(\mathbf{p}^{[3]}))/$$

$$[1 - \tfrac{1}{2}(B_{11}(\mathbf{p}^{[3]}) + B_{22}(\mathbf{p}^{[3]}))]$$

# Configuration

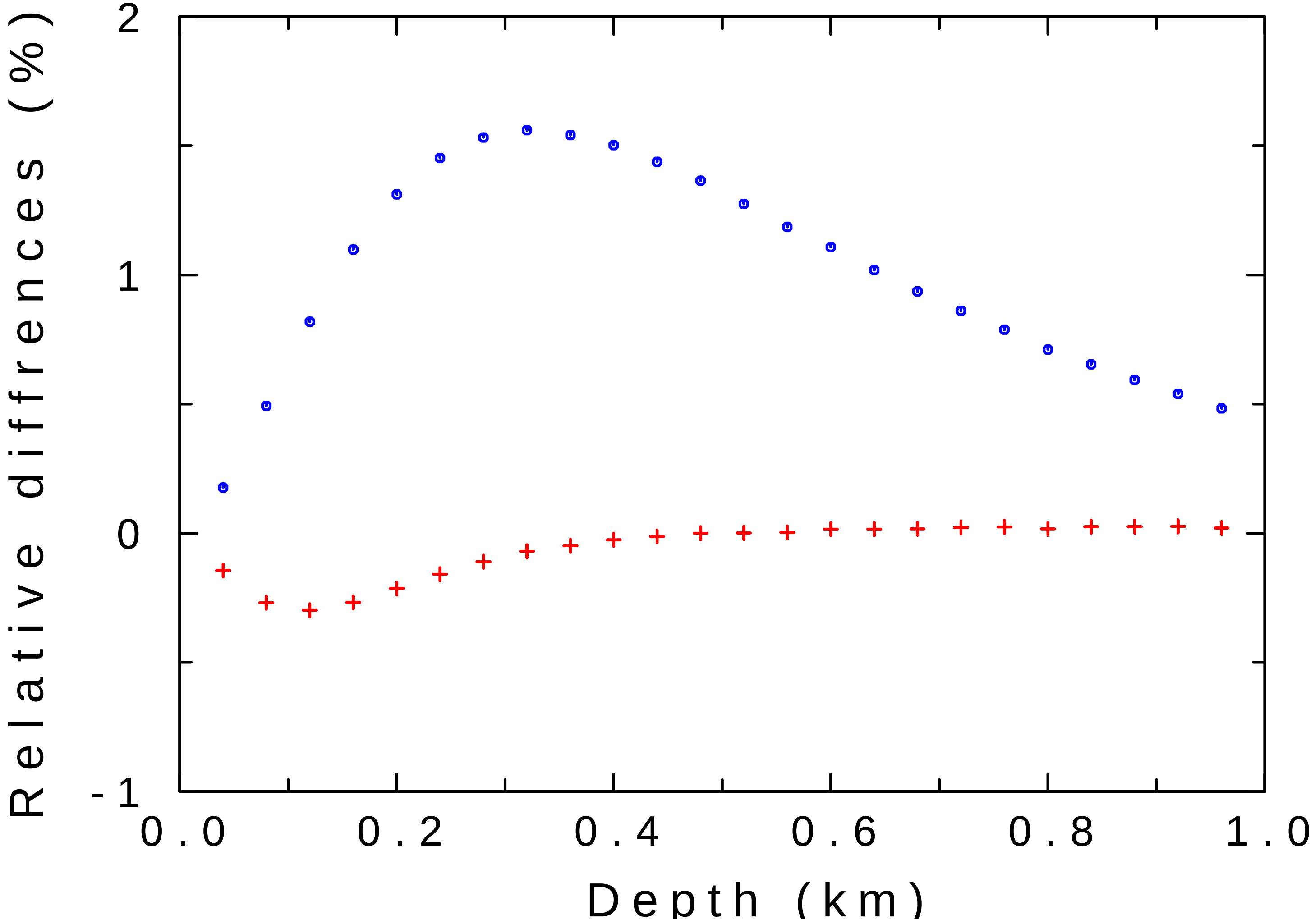


# P waves - ORT model (∼ 20% anisotropy)

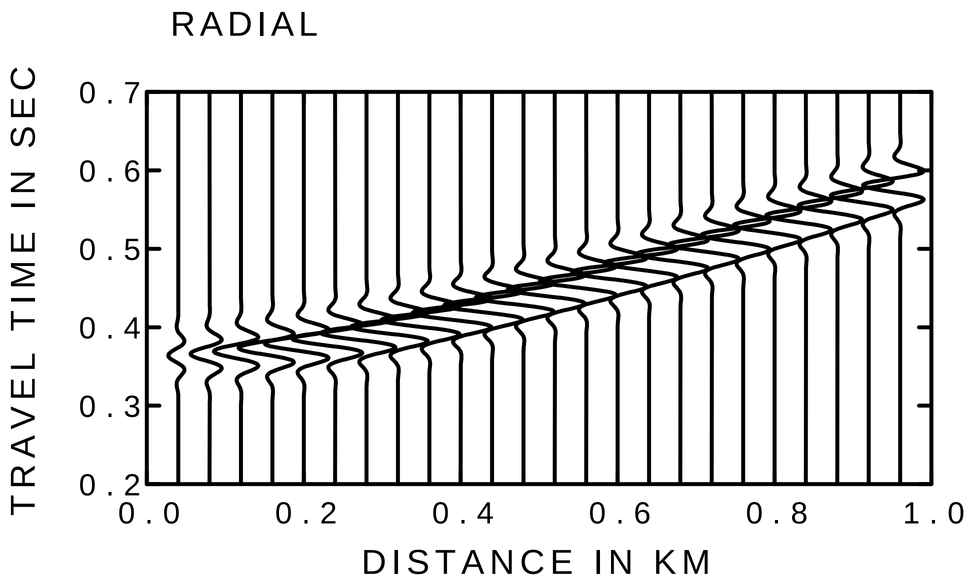


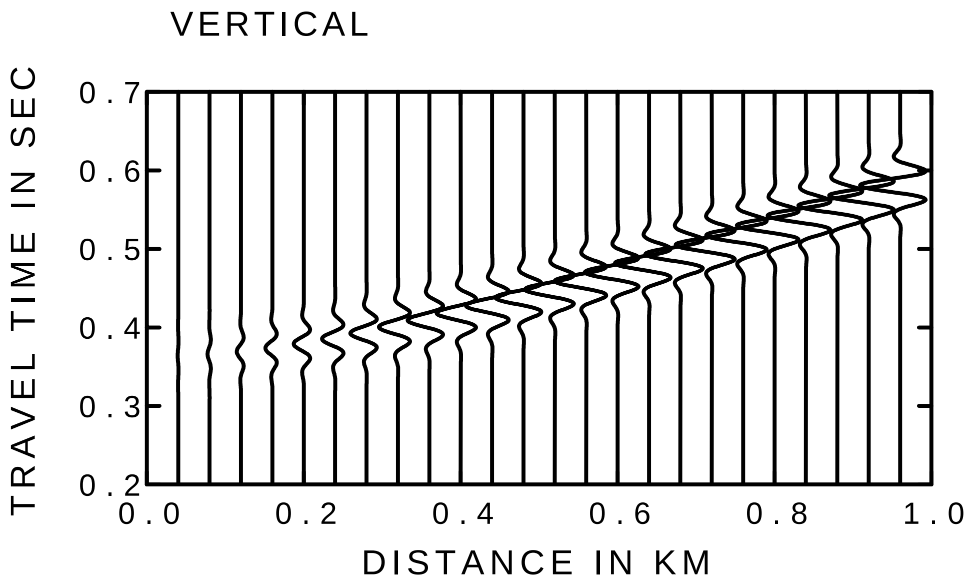
# P waves - ORT model

first- and second-order traveltime (comparison with ANRAY)

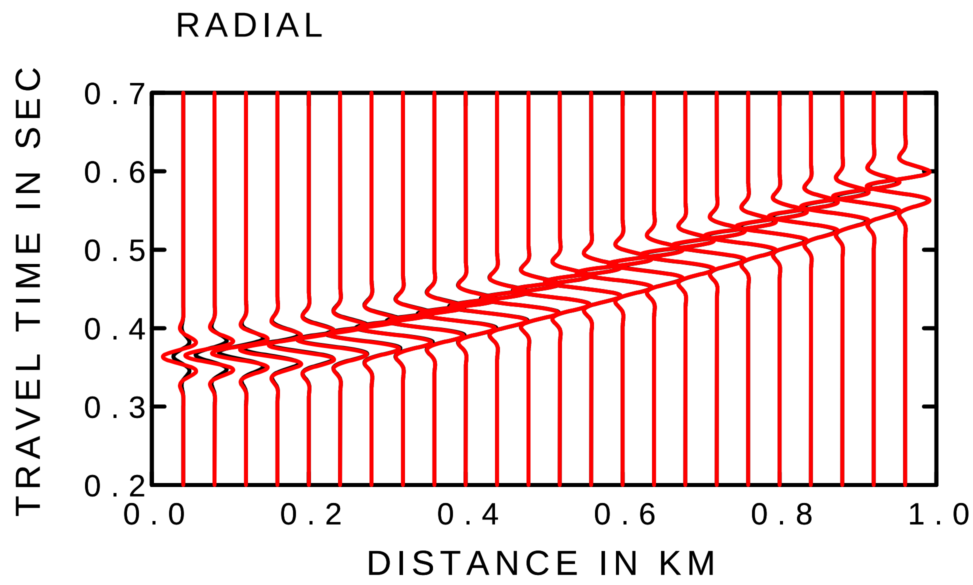


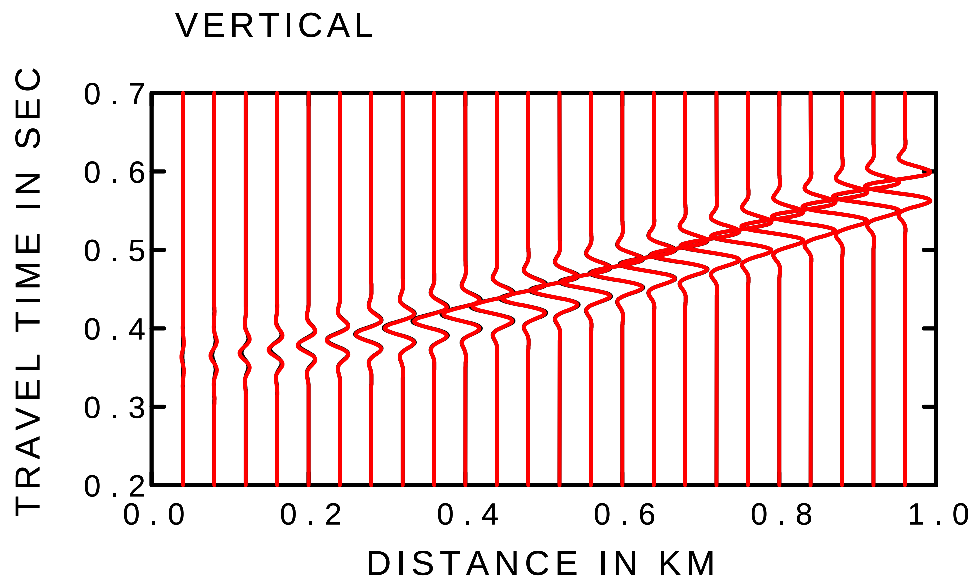
# P waves - ORT model- ANRAY



# P waves - ORT model- ANRAY, FORT



# S waves - comparisons

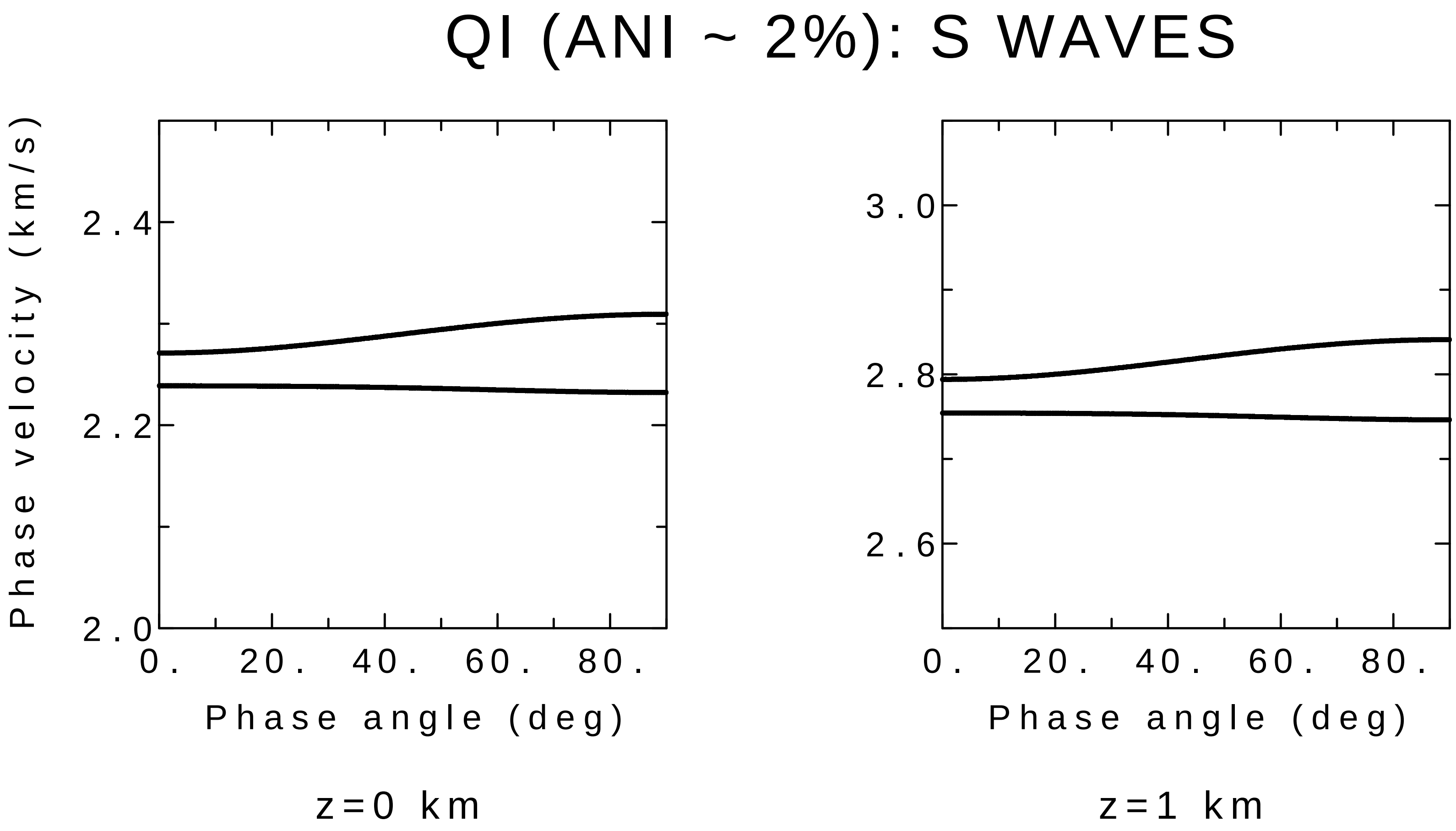
Comparison of

- the Fourier pseudospectral method (FM)
- the ray theory (RT)
- the FORT coupling ray theory (CRT)

for the VSP experiment

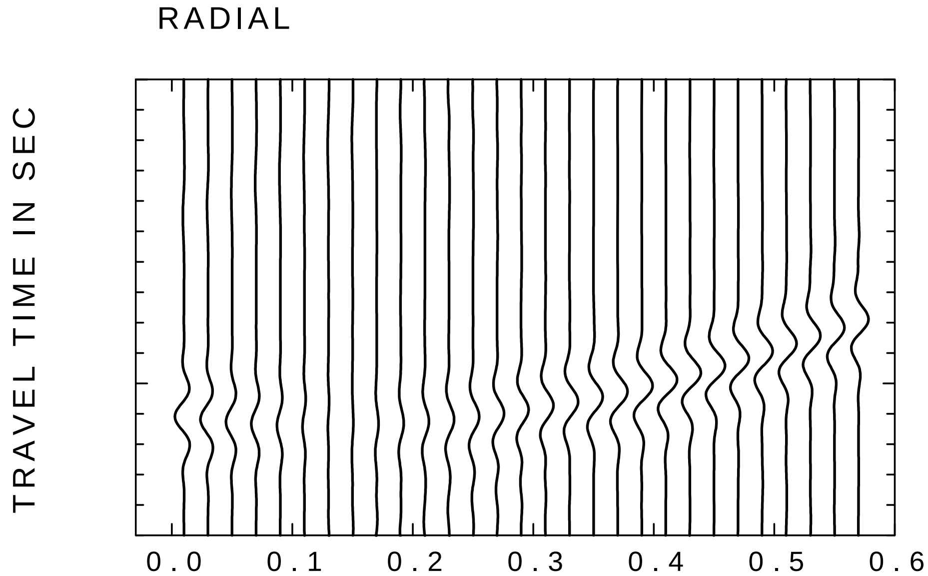
# Comparisons - weak anisotropy

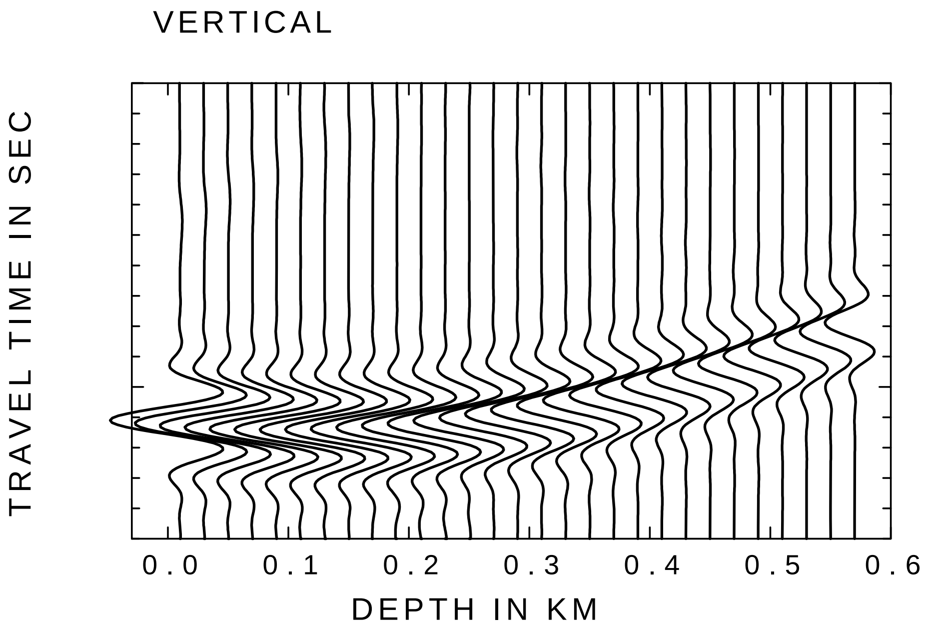
HTI model with axis of symmetry $45^0$ off profile (Klimeš & Bulant, 2004)

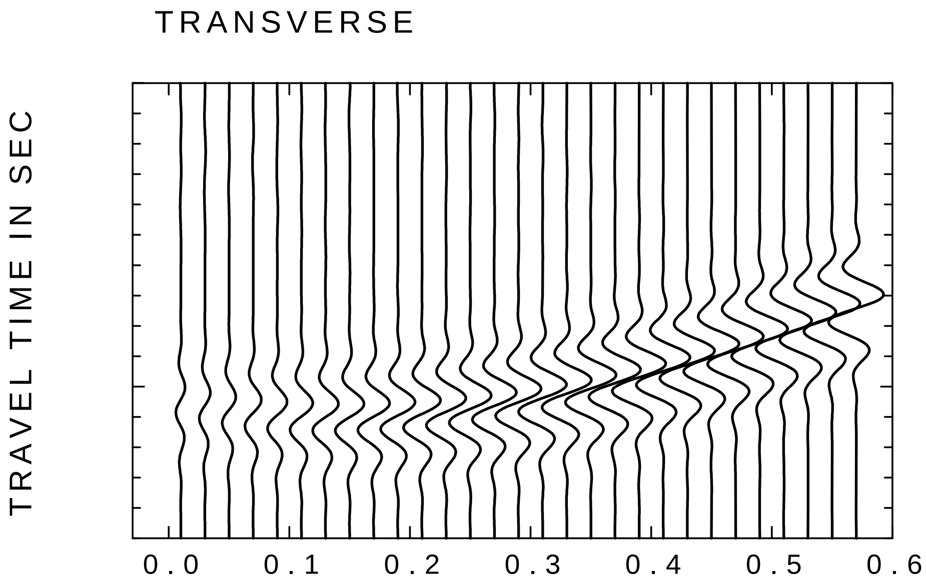


# Comparisons

QI: FM

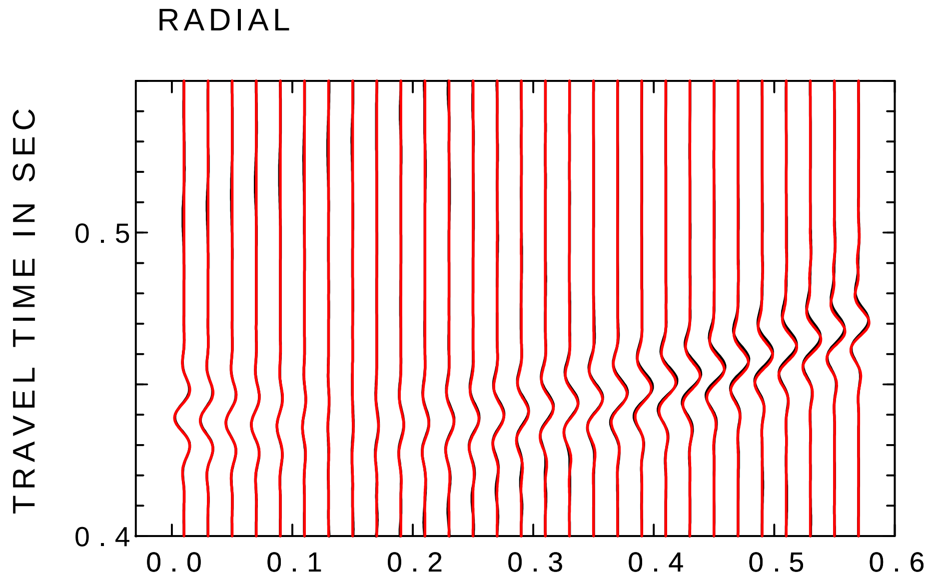


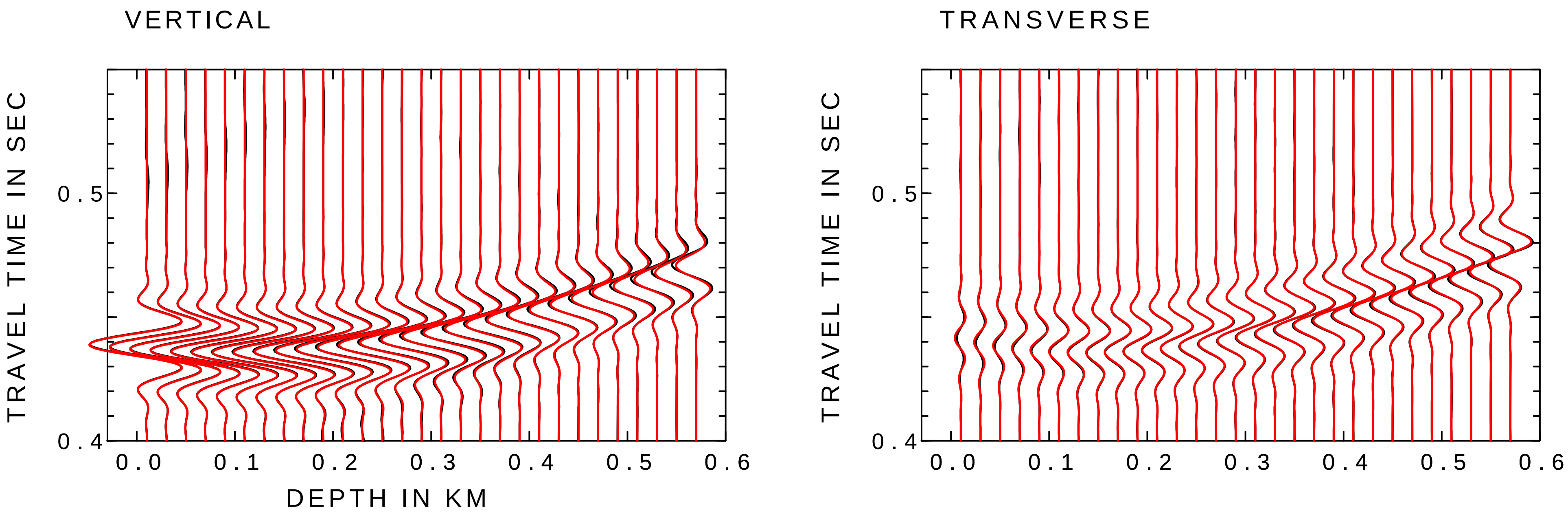



# Comparisons

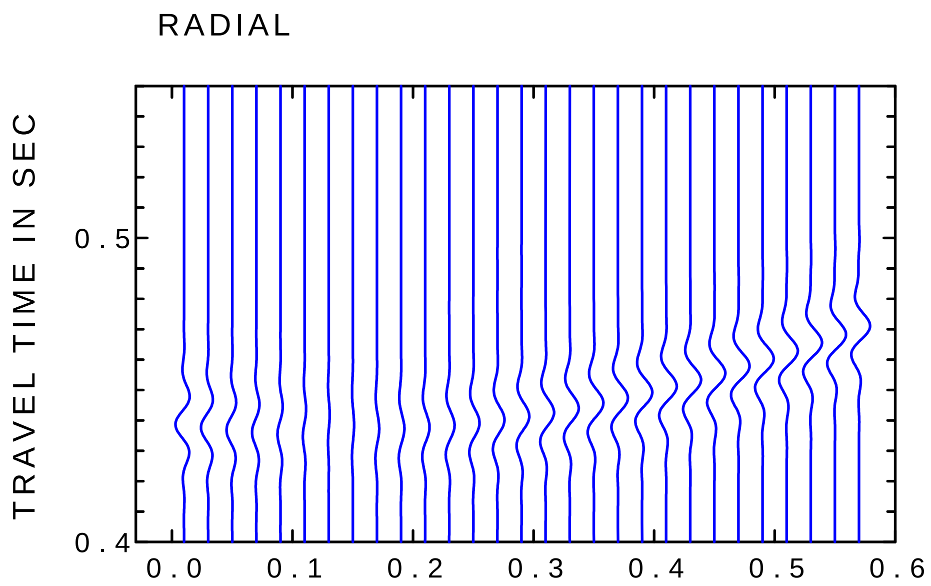
QI: FM CRT

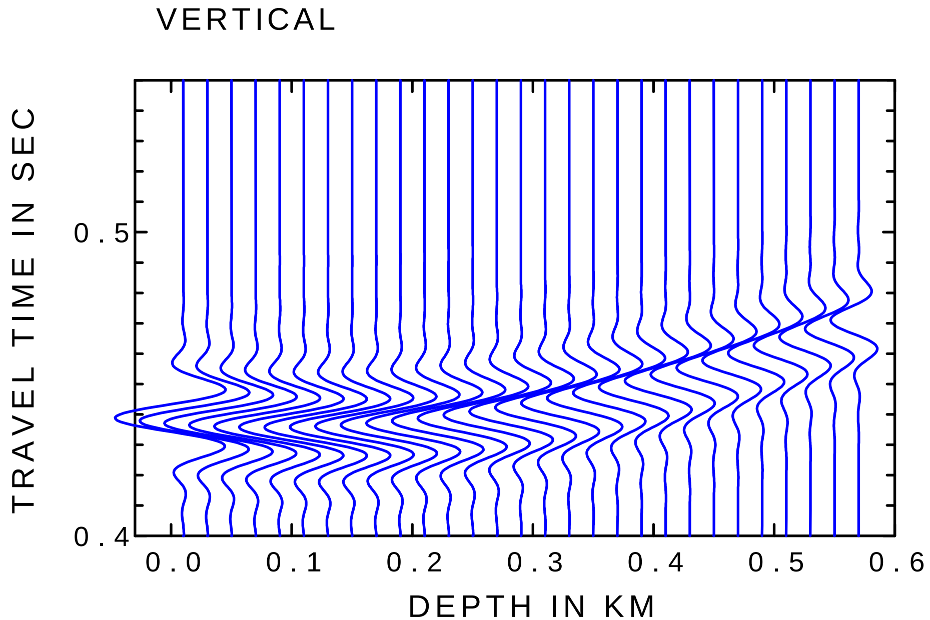


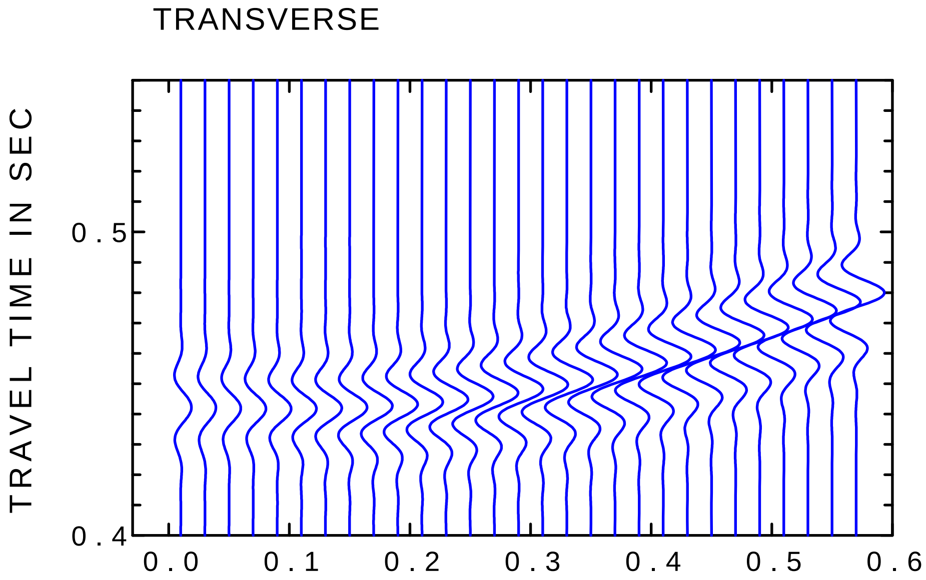



# Comparisons

QI: RT

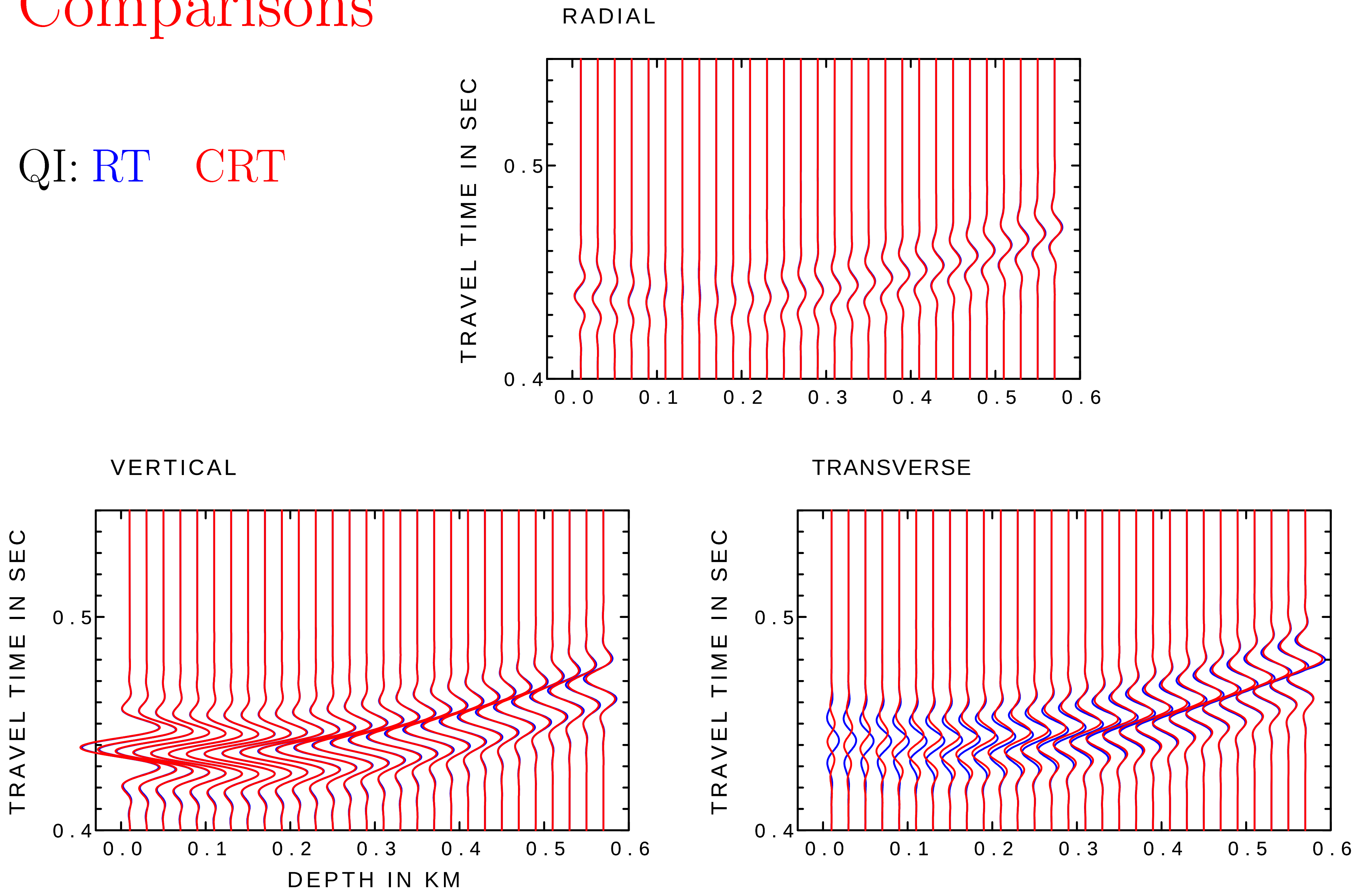


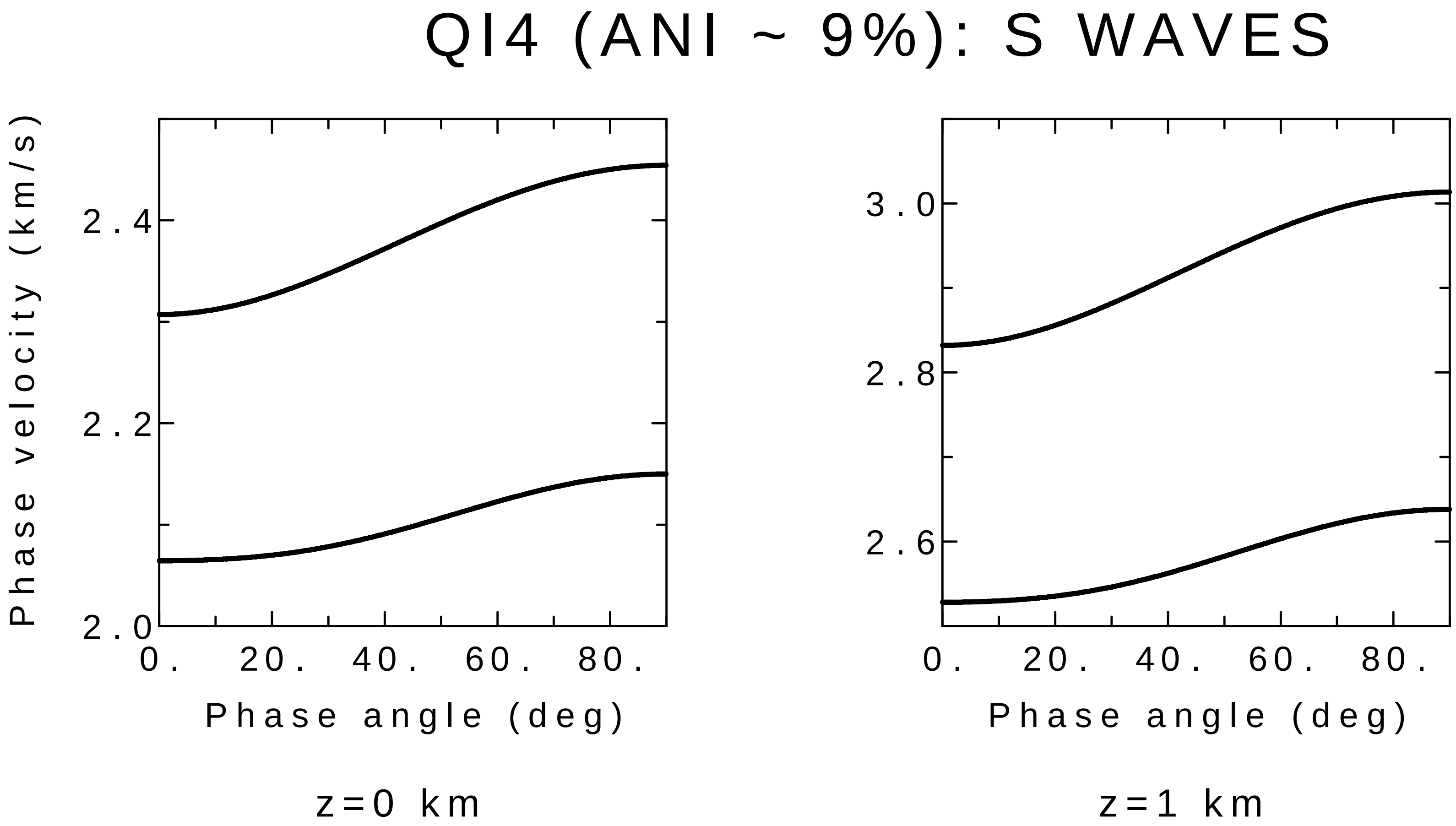
# Comparisons

QI: RT CRT



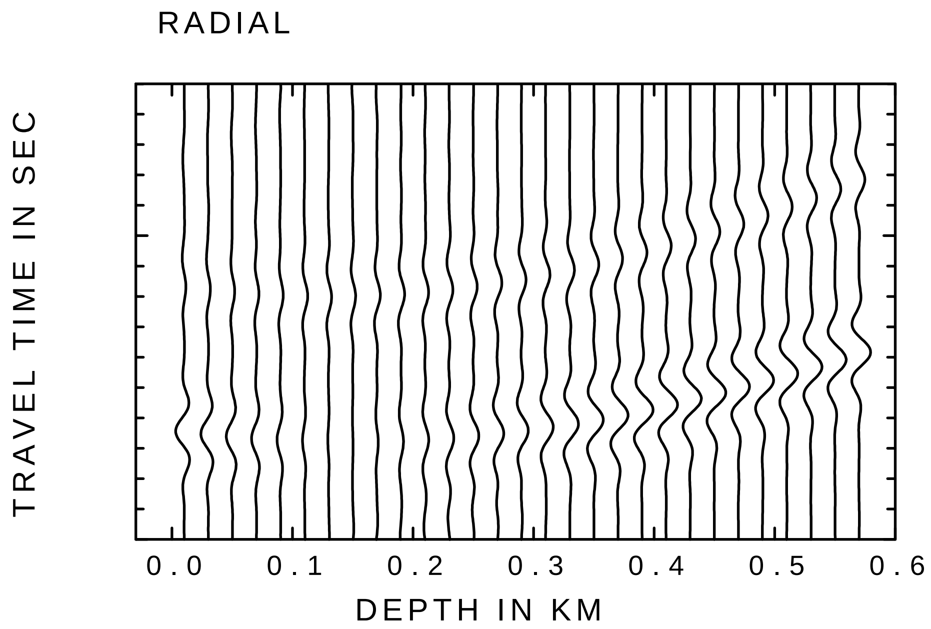
# Comparisons - stronger anisotropy

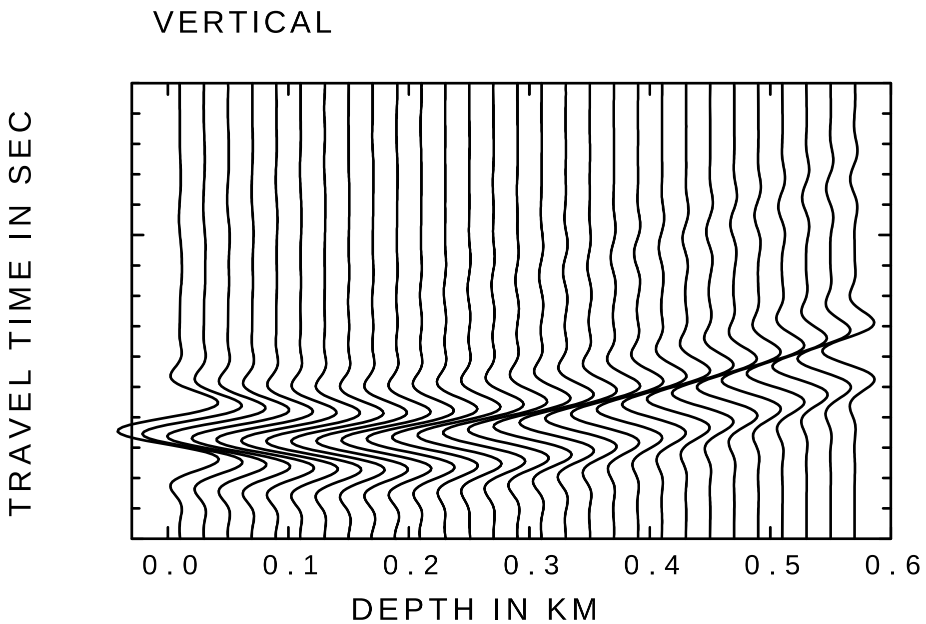
HTI model with axis of symmetry $45^0$ off profile (Klimeš & Bulant, 2004)

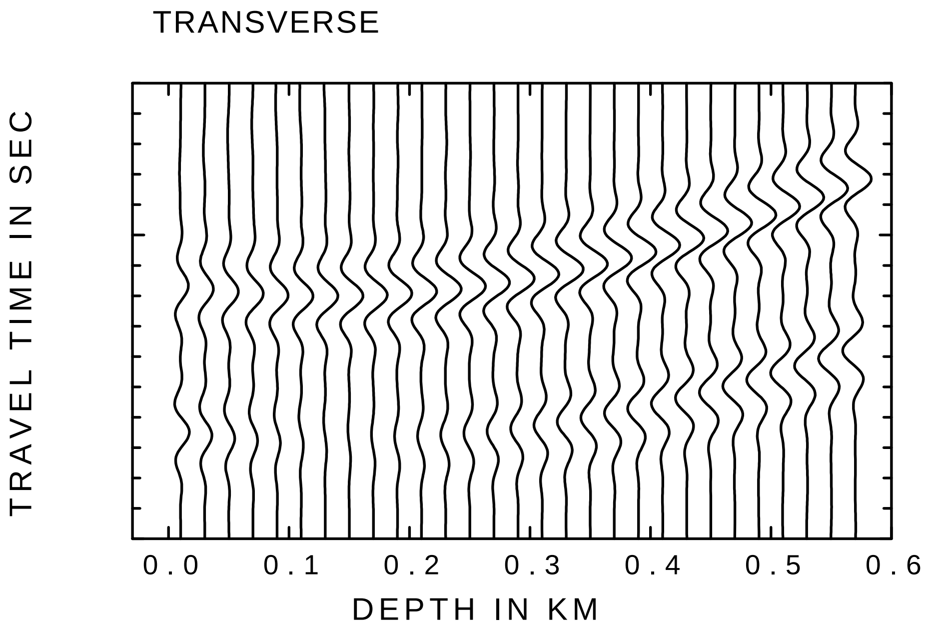


# Comparisons

QI4: FM

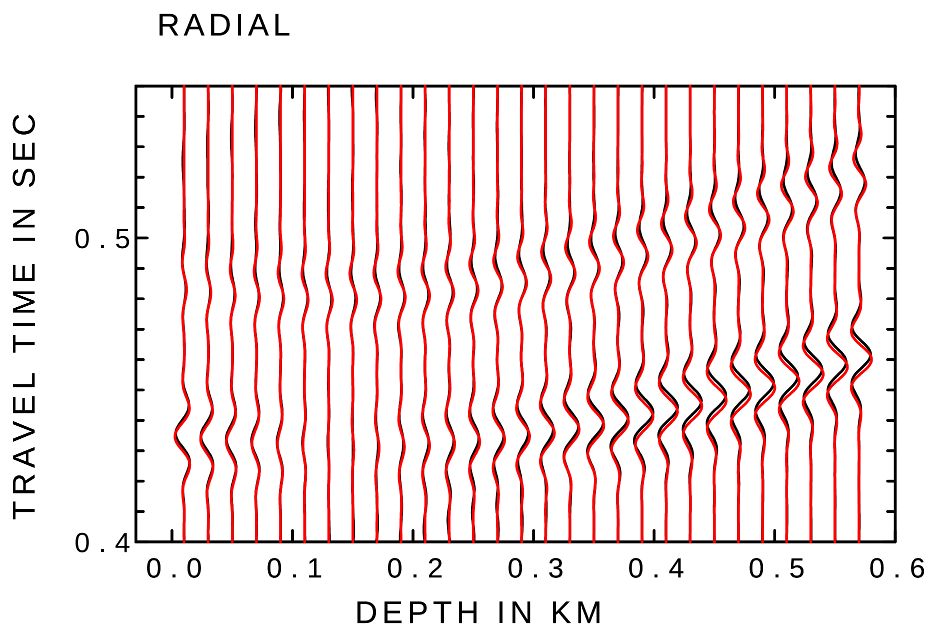


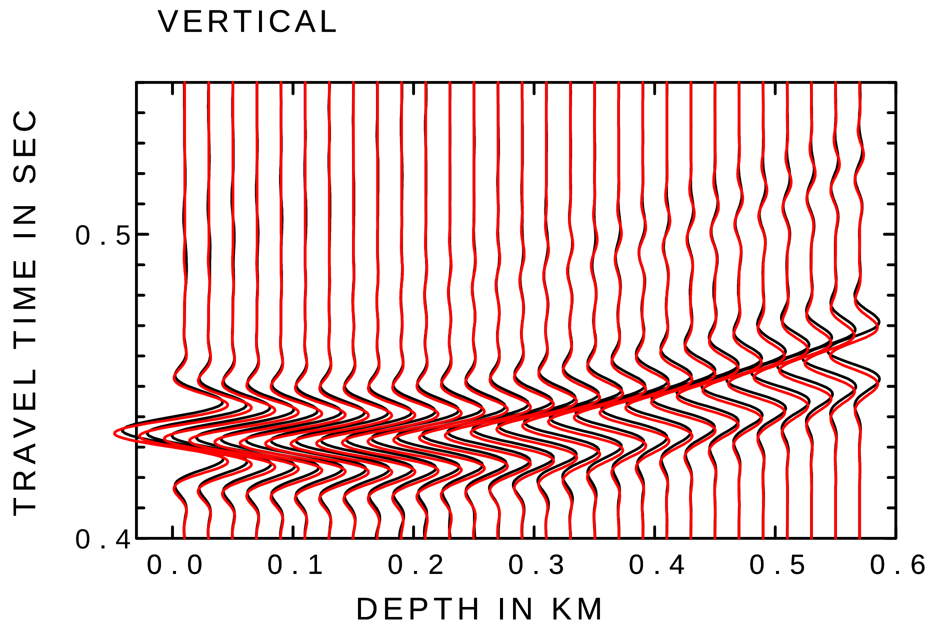



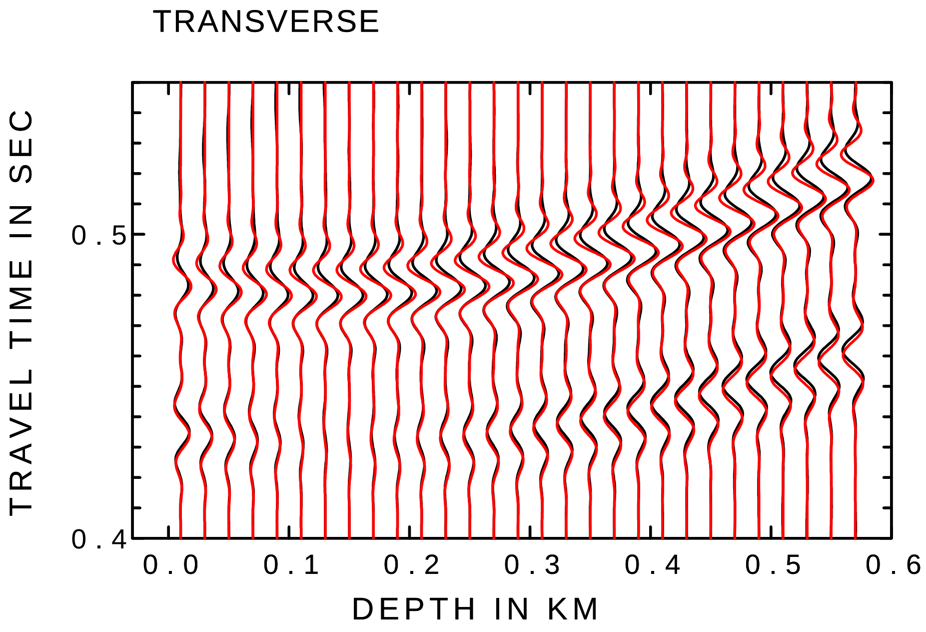



# Comparisons

QI4: FM CRT

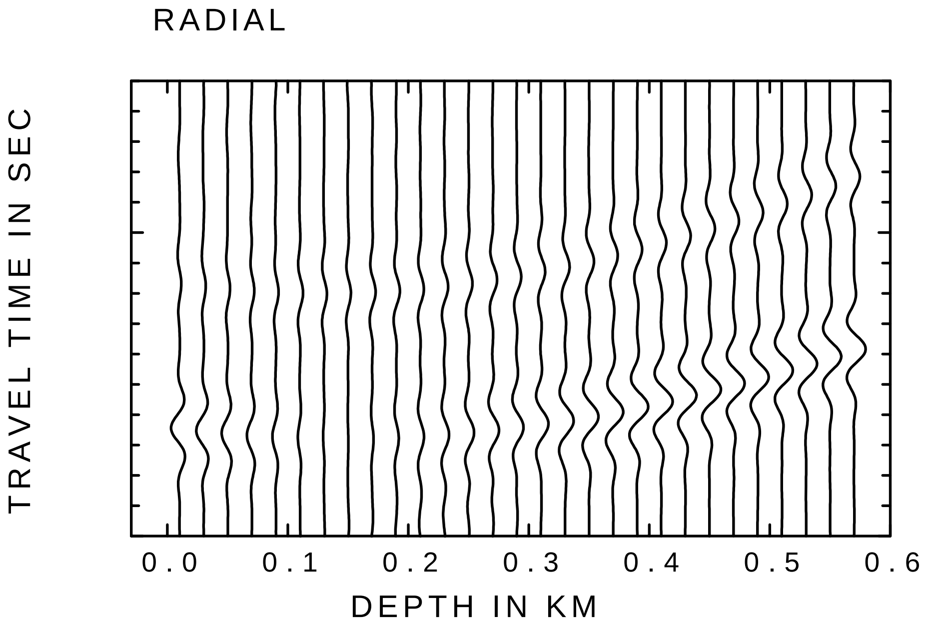


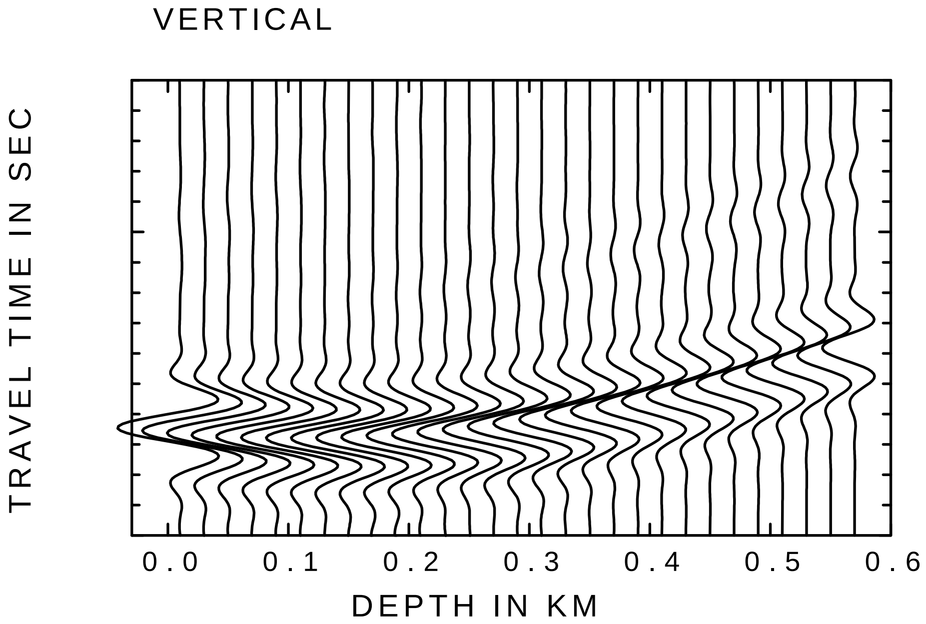



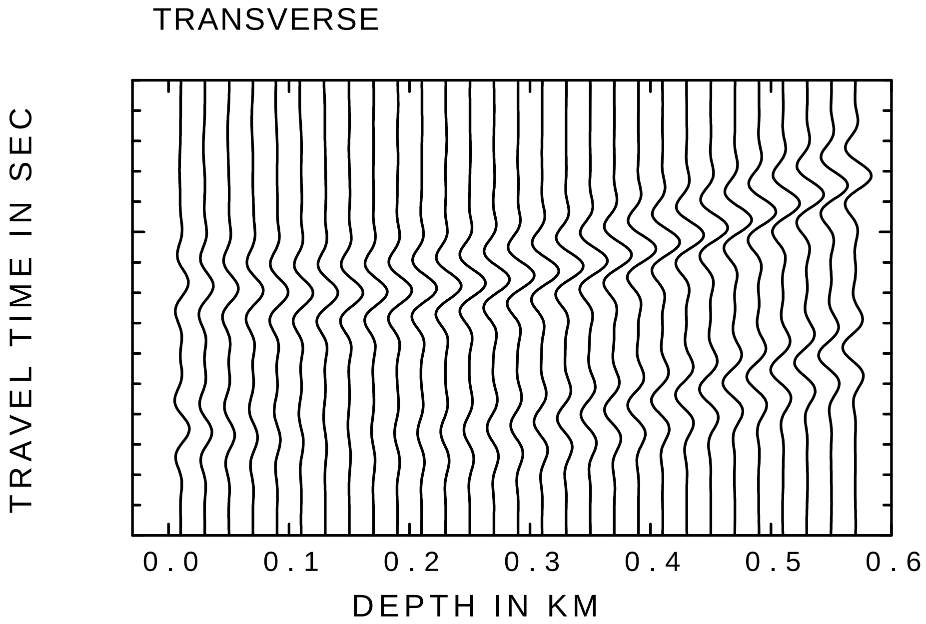



# Comparisons

QI4: FM

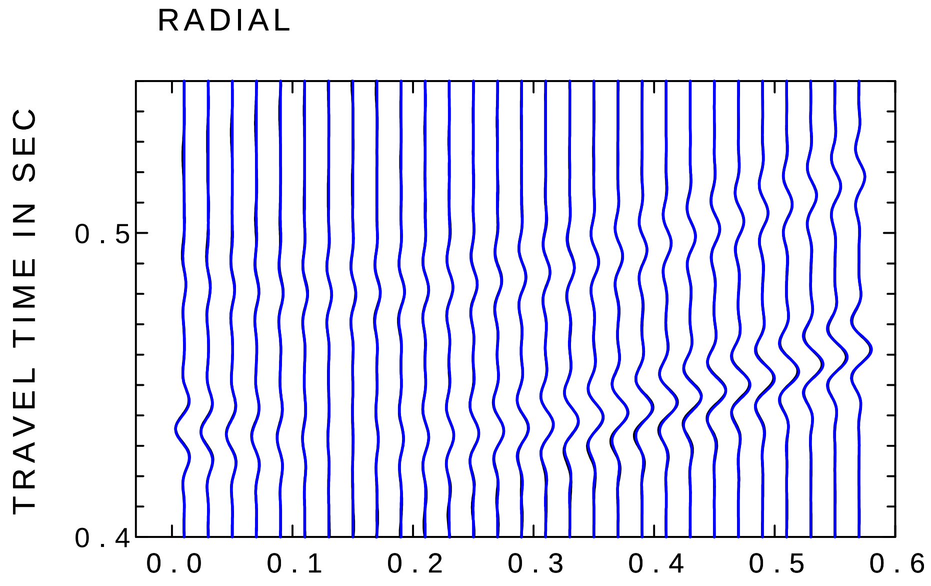


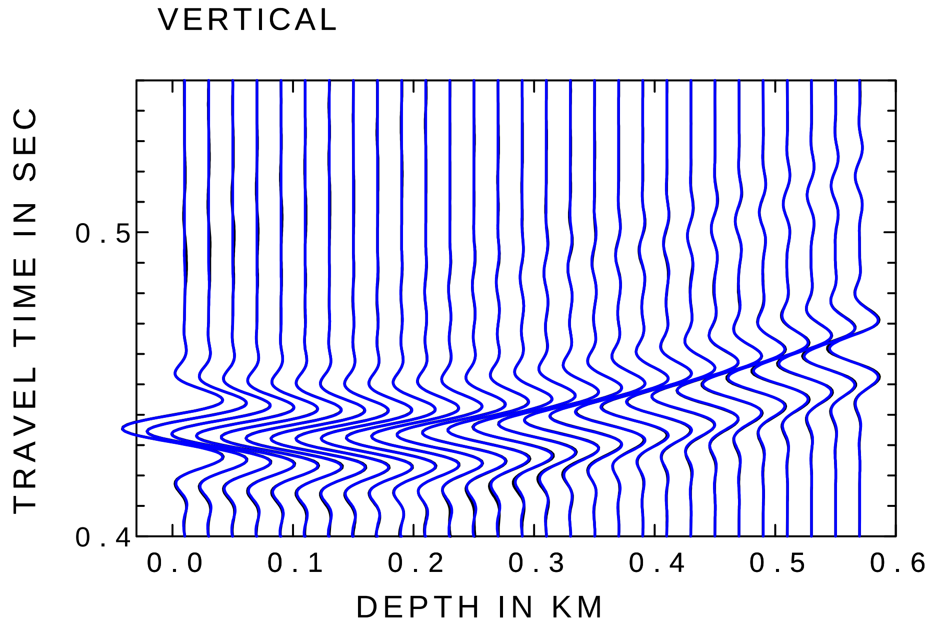



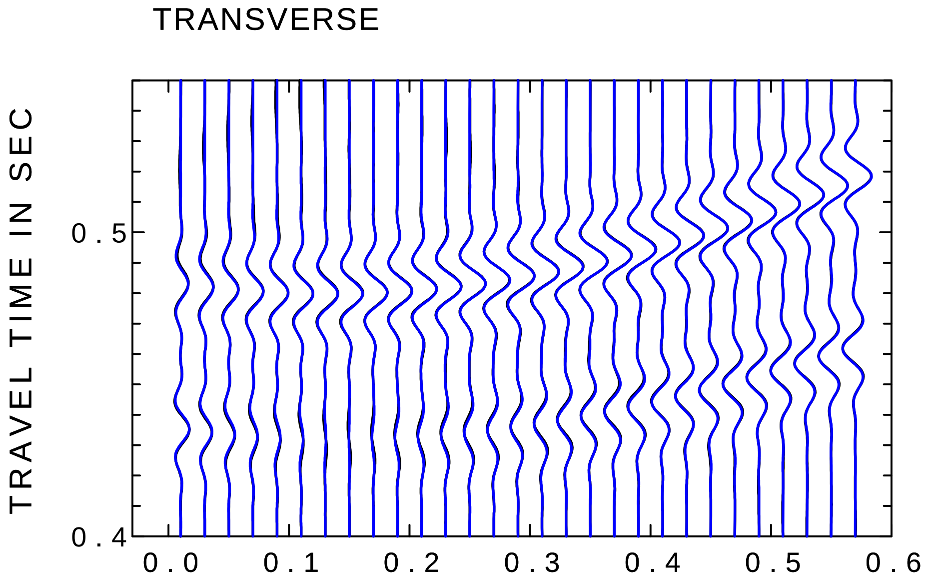



# Comparisons

QI4: FM RT



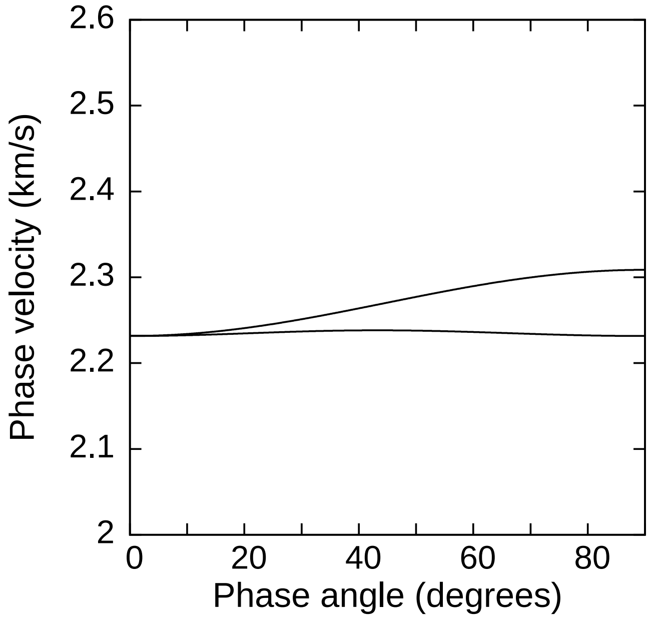



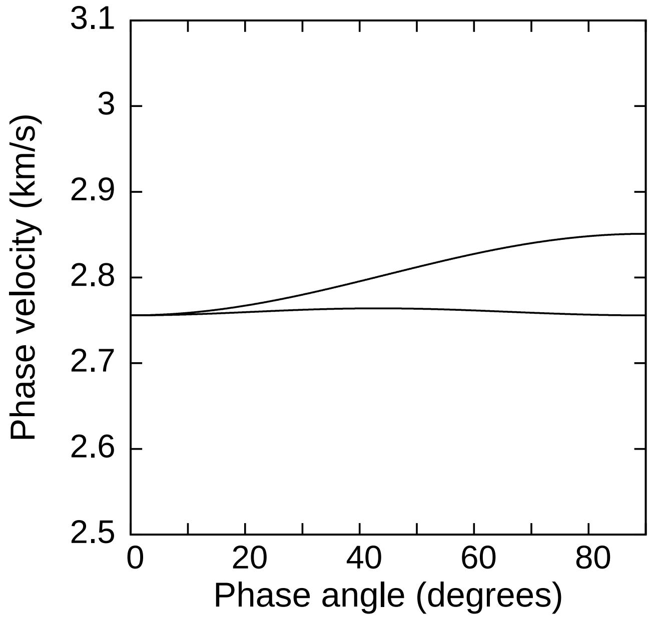
# Comparisons - kiss singularity

model QI rotated by $44^0$ in the horizontal plane $\Rightarrow$

$1^0$ between the profile and the axis of symmetry



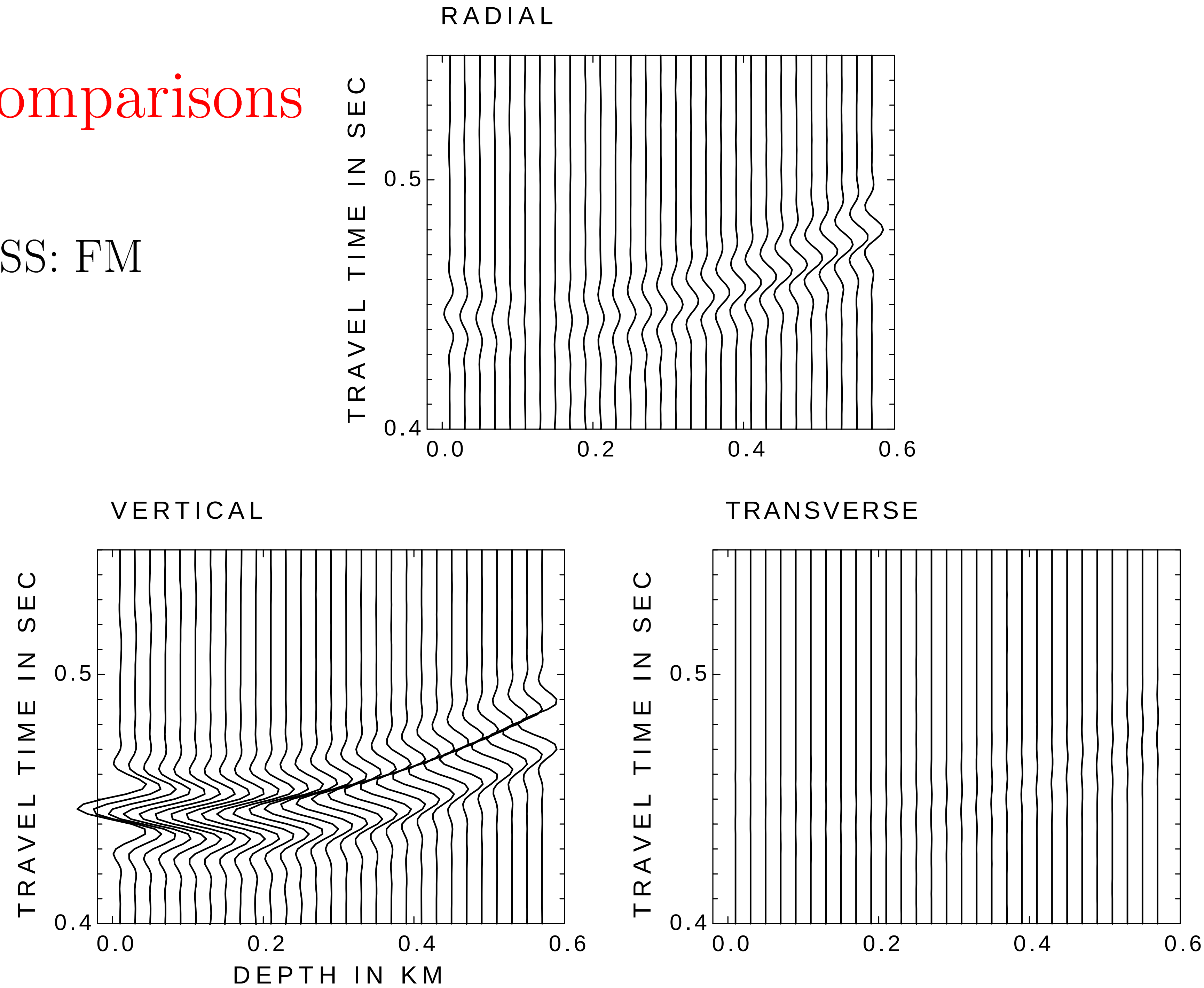

z=0 km




z=1 km

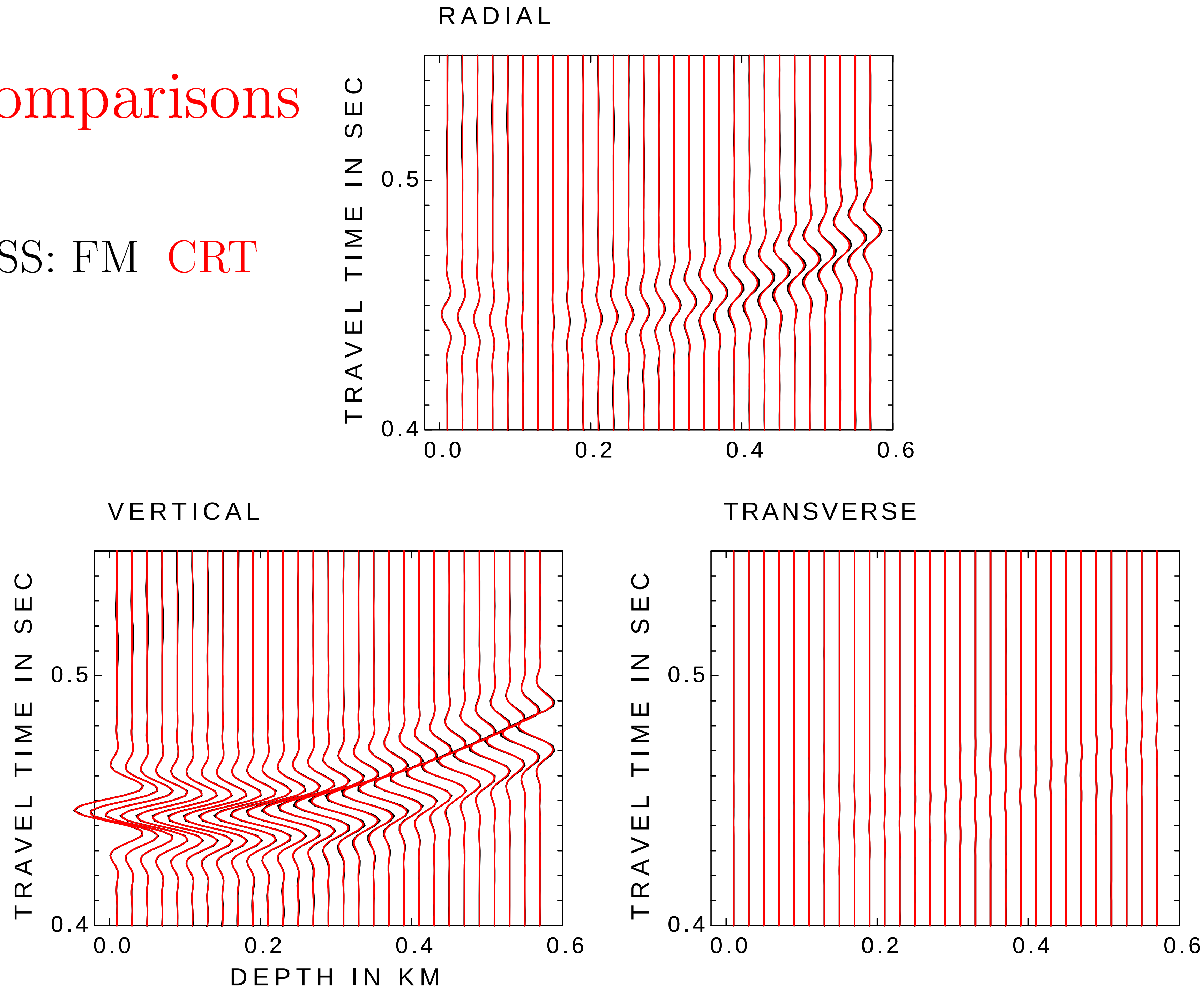
# Comparisons

KISS: FM



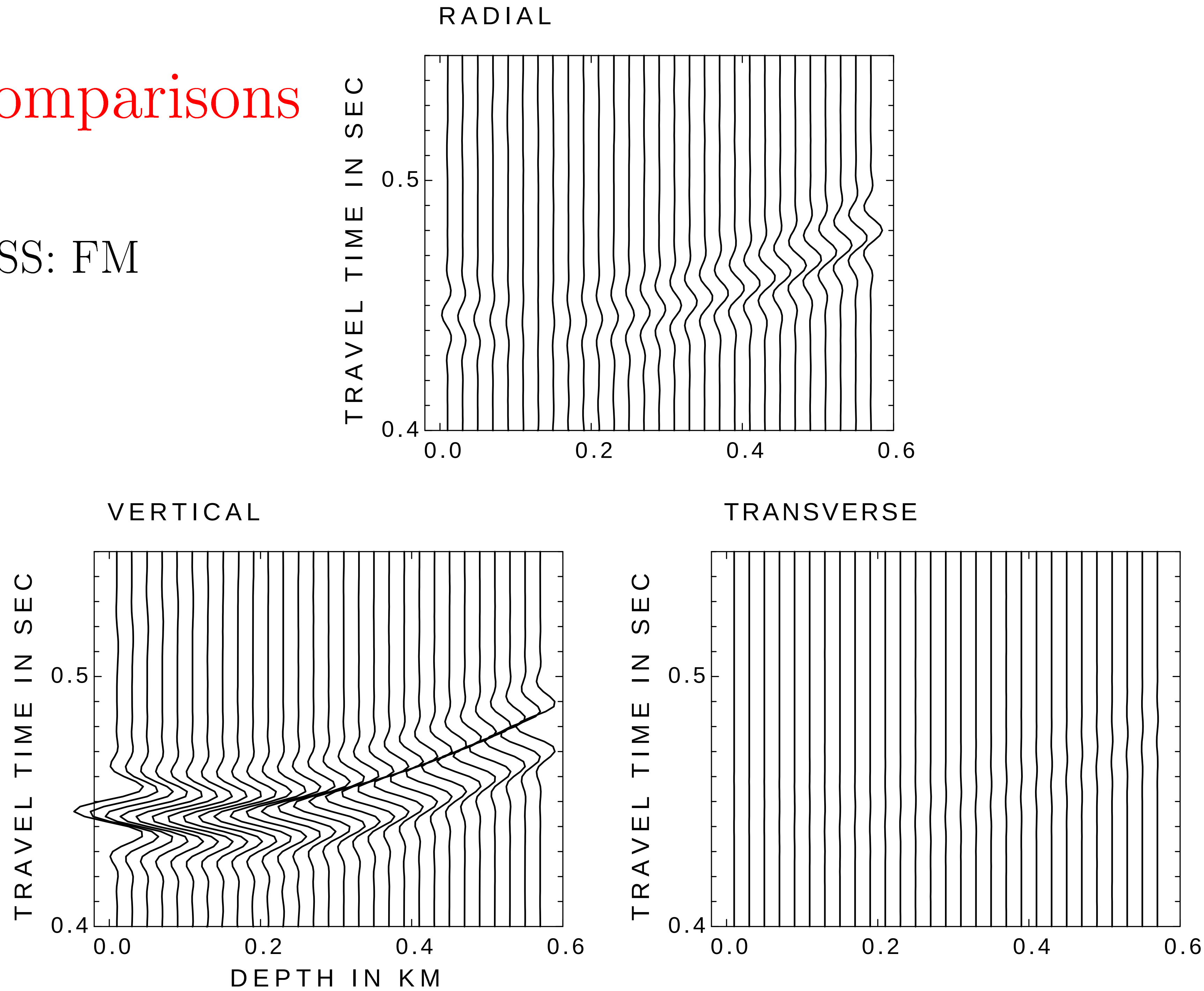
# Comparisons

KISS: FM CRT



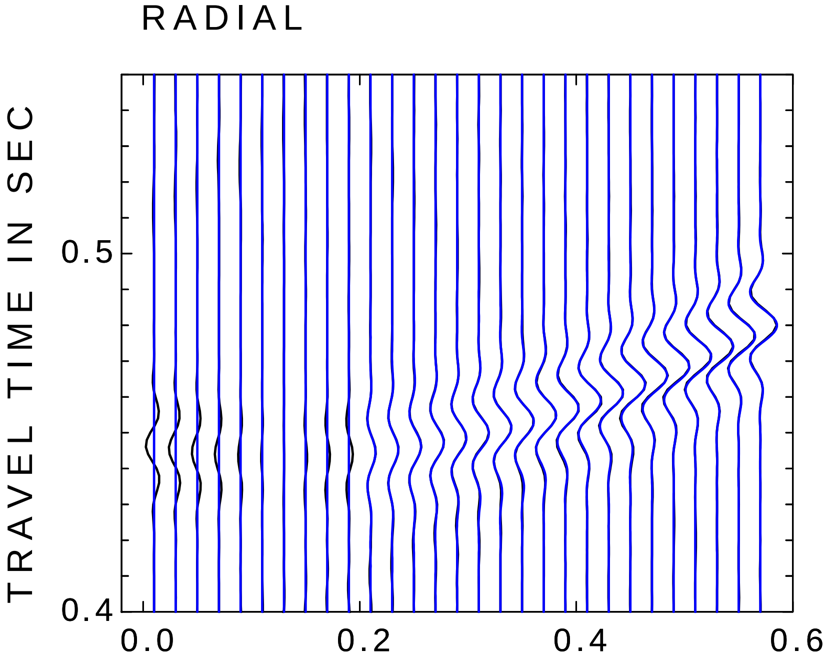
# Comparisons

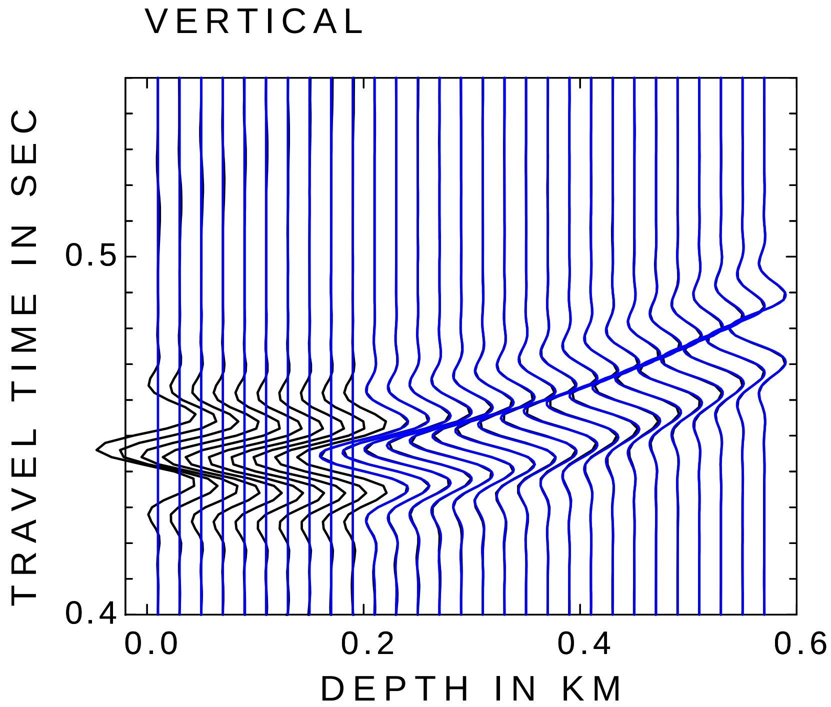
KISS: FM

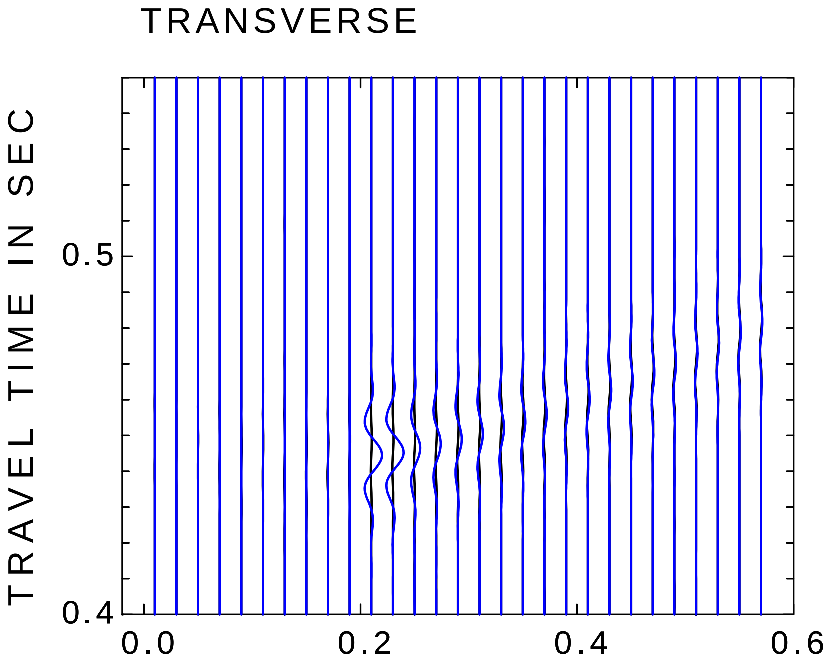


# Comparisons

KISS: FM RT

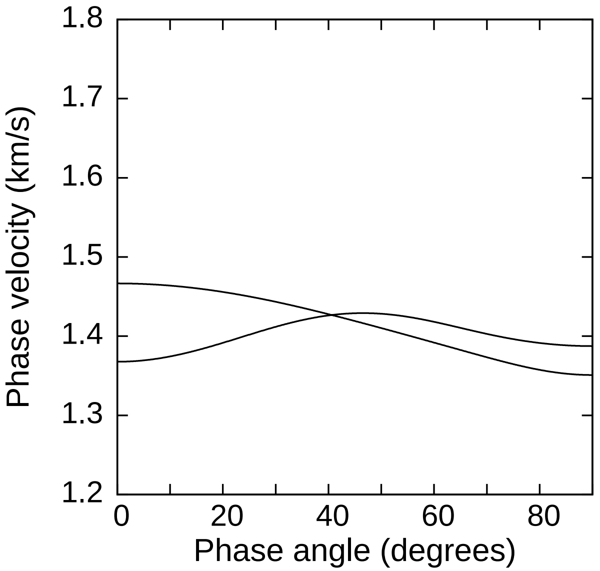


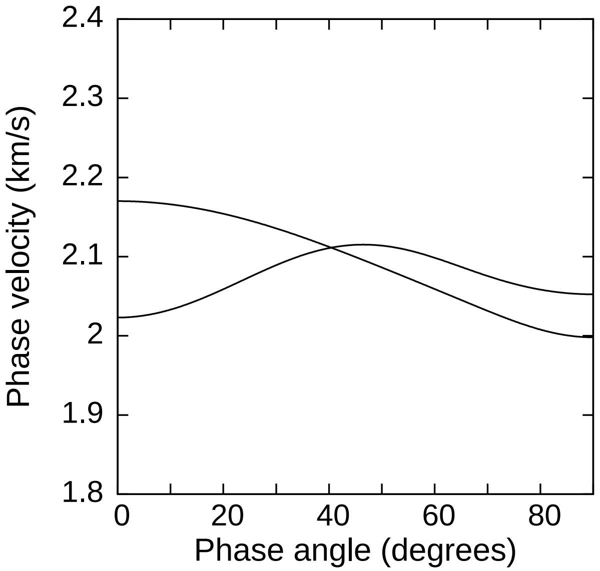



# Comparisons - conical singularity

model ORT: modified model of Schoenberg & Helbig (1997)



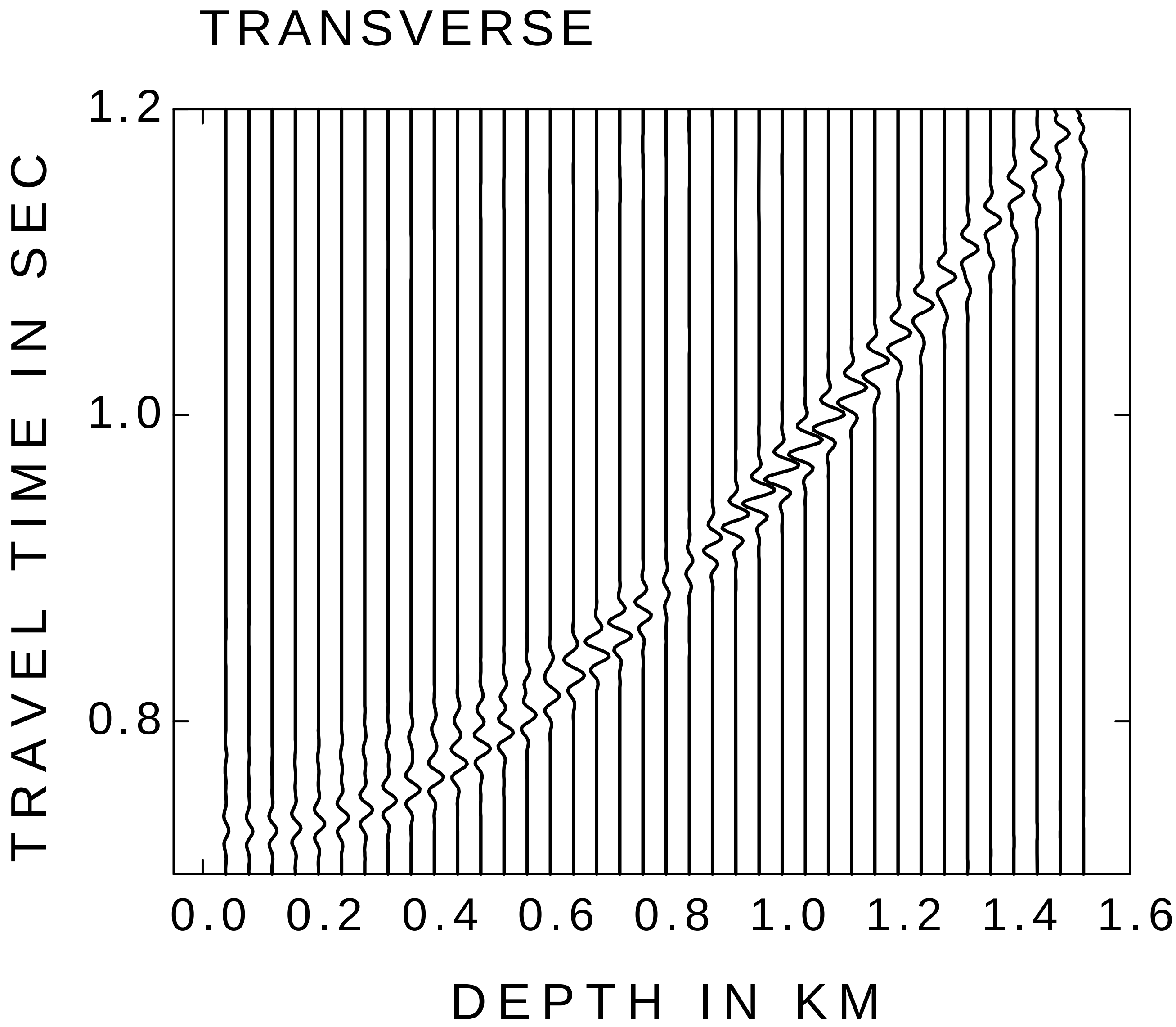

z=0 km




z=3 km

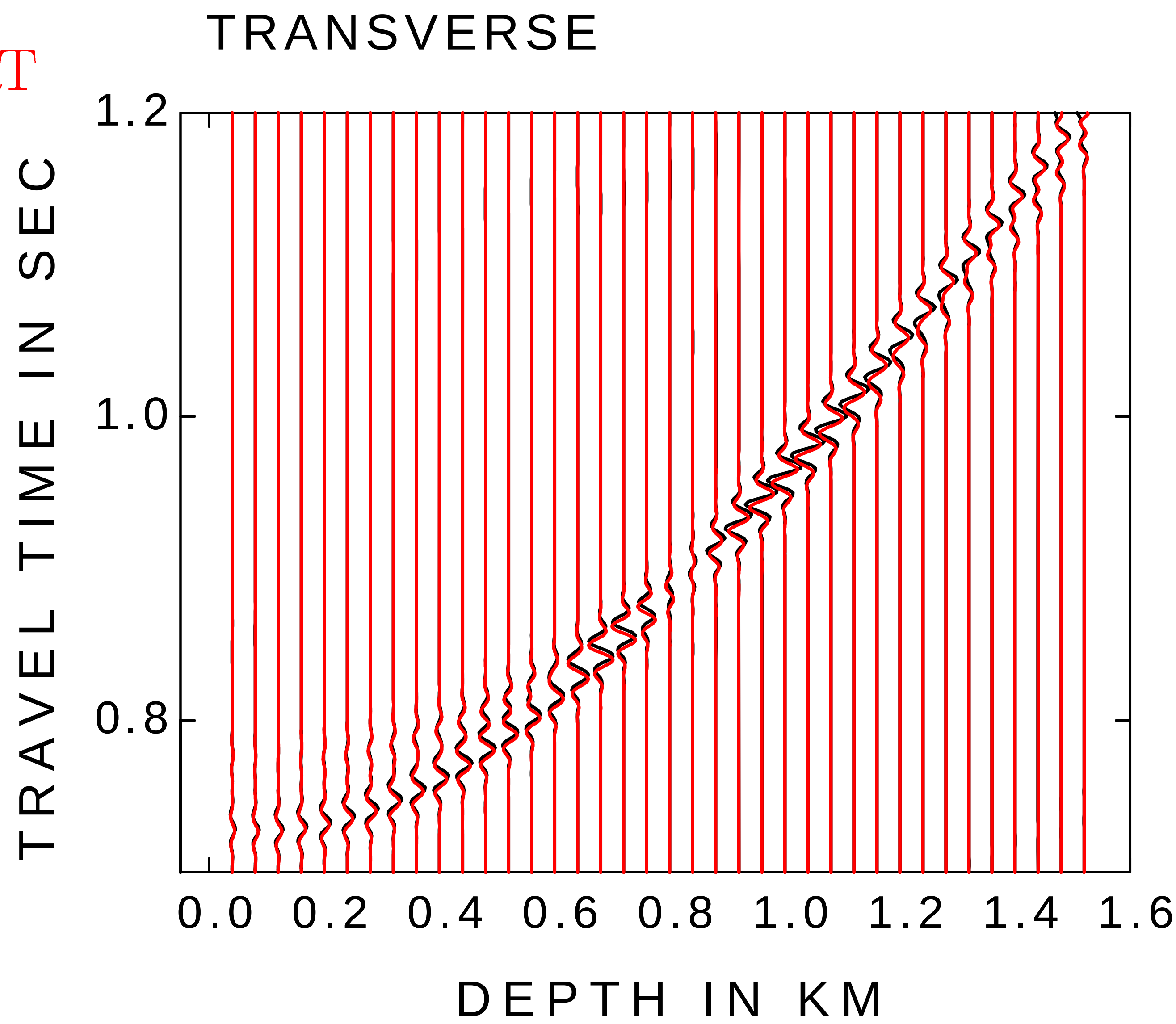
# Comparisons

ORT: FM



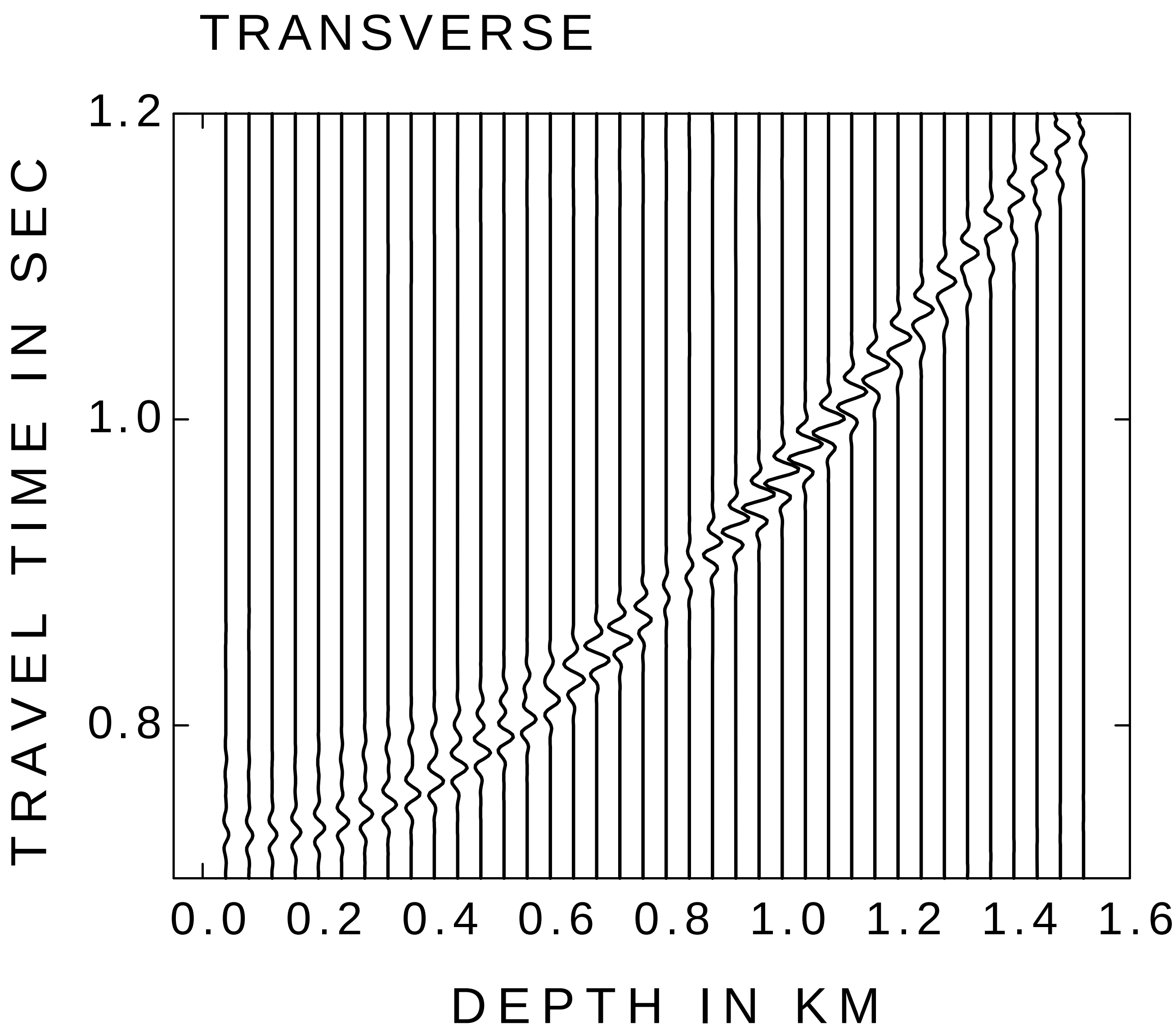
# Comparisons

ORT: FM CRT



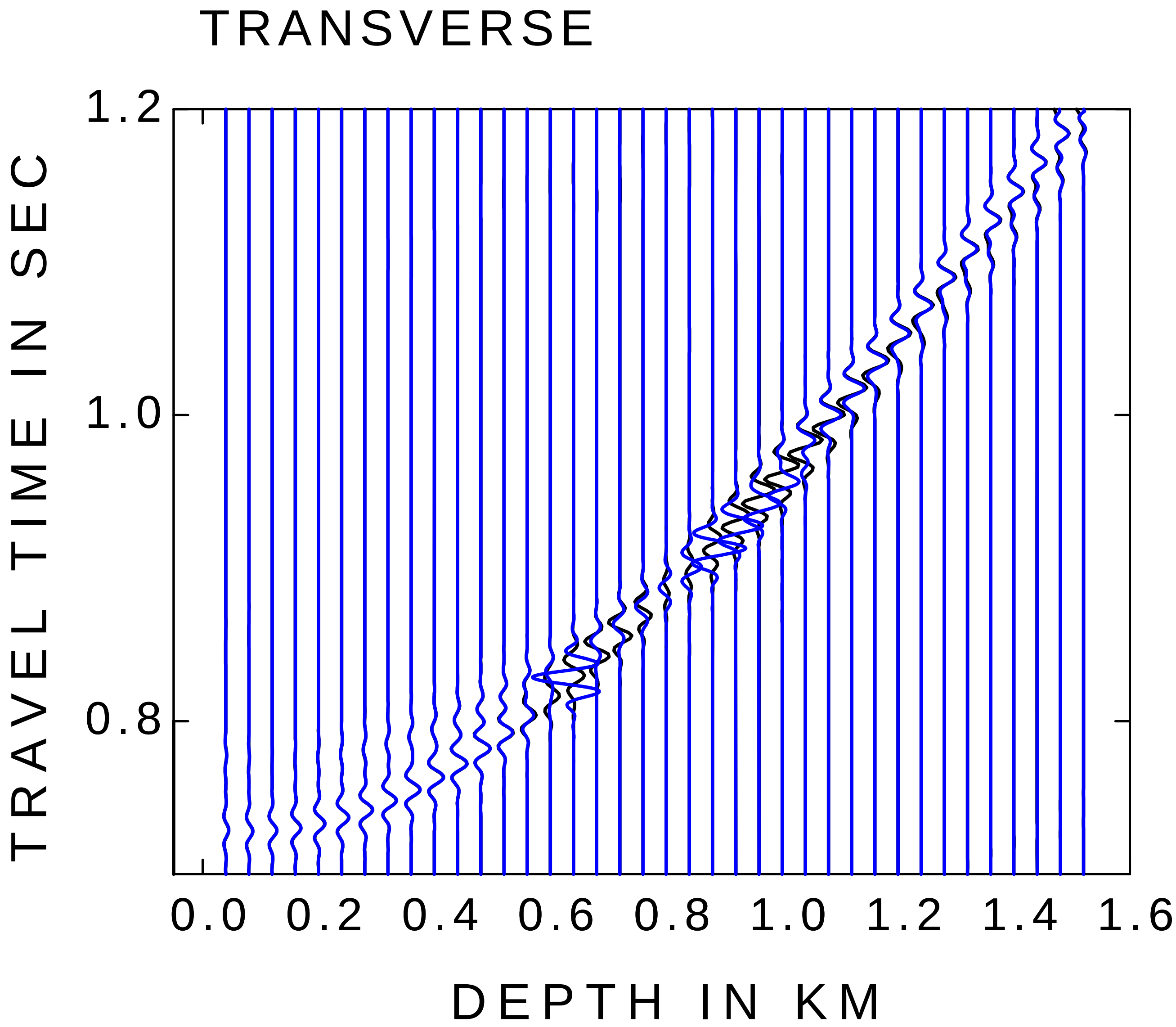
# Comparisons

ORT: FM



# Comparisons

ORT: FM RT



# Conclusions

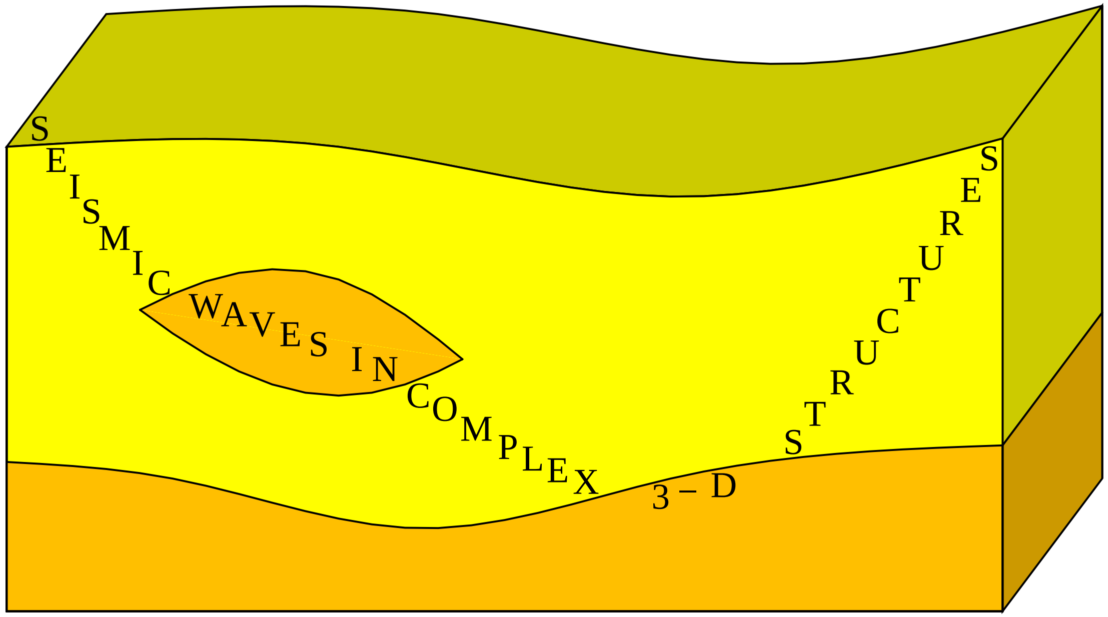
- FORT and FODRT approximate in weakly anisotropic media
- FORT and FODRT exact in isotropic media
- applicable to any anisotropic symmetry
- P and S waves treated separately
  - no need for pseudoacoustic or any other approximation
- S-wave coupling included
- no collapses in S-wave singular regions
- simple structure of FORT and FODRT equations

# Conclusions

- linear dependence of RT and DRT on WA parameters;
  no dependence on a reference medium
- 15 different independent coefficients of RT and DRT
  for P and S waves in general anisotropy
- easy generalization for layered media
- second-order traveltime corrections
  for P and common S-wave rays
- natural replacement in routine processing codes
- CPU for 1 section: FORT $\sim$ 30 sec; FM $\sim$ 4 hours

# Future plans

- generalization for layered media
- prevailing frequency coupled S-waves concept
- generalization for weakly attenuating media
- higher-symmetry anisotropy
  with varying symmetry elements


SEISMIC WAVES IN COMPLEX 3 – D STRUCTURES