GEOMECHANICS FOR GEOPHYSICISTS

Basics of Rock Mechanics & Geomechanics

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Basics of Rock Mechanics & Geomechanics

- Stresses, strains, elastic moduli, Hooke's law & wave equations
    - ✓ Isotropic & Anisotropic solids
- Poroelasticity, effective stress
- Rock failure

Linear Elasticity

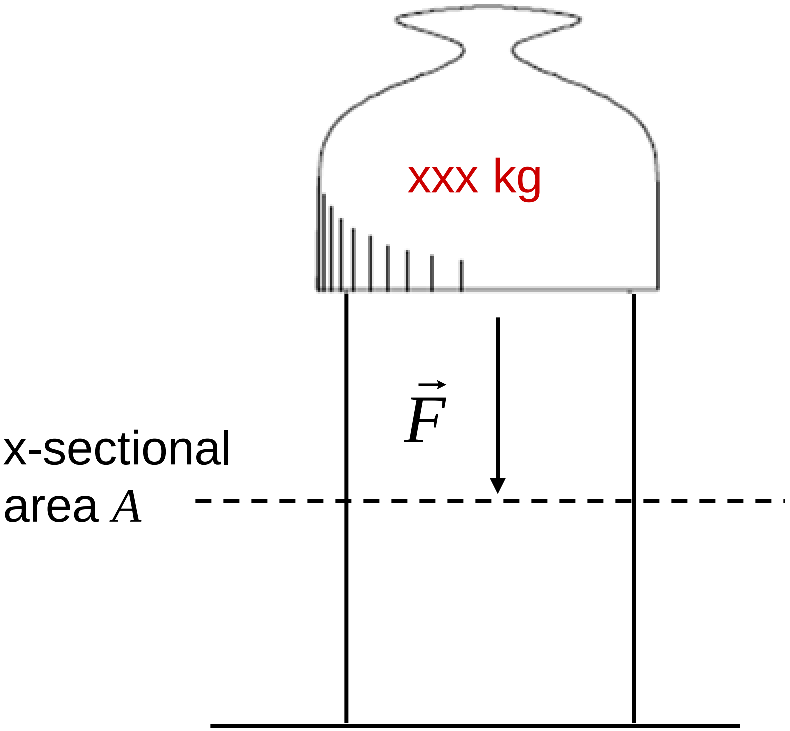
Stress

Stress: $$\sigma = \frac{F}{A}$$

Unit: Pa (Pascal) = N/m^2

or: psi (pounds per square inc)

1 kpsi = 6.895 MPa

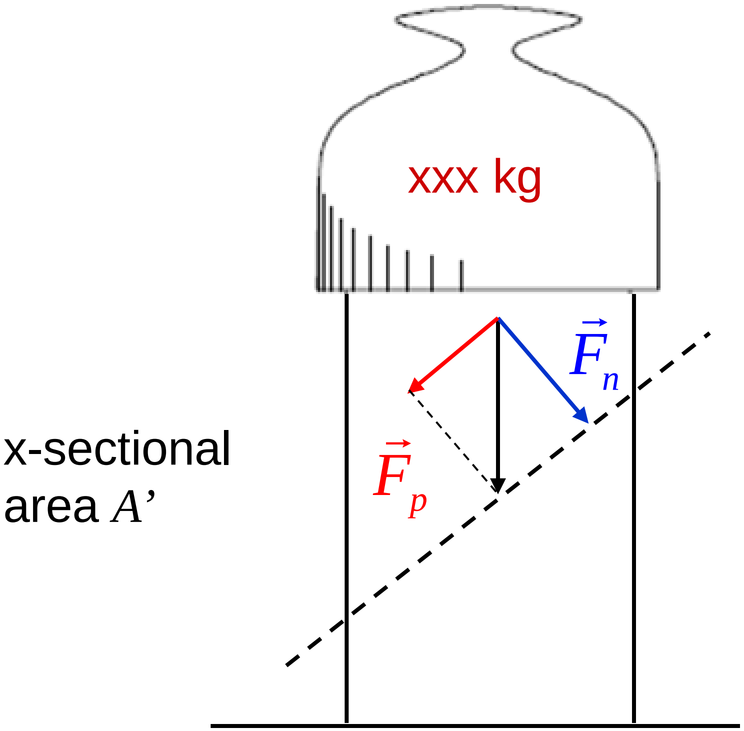


Rock Mechanics sign convention: Compressive stresses are positive

Stress

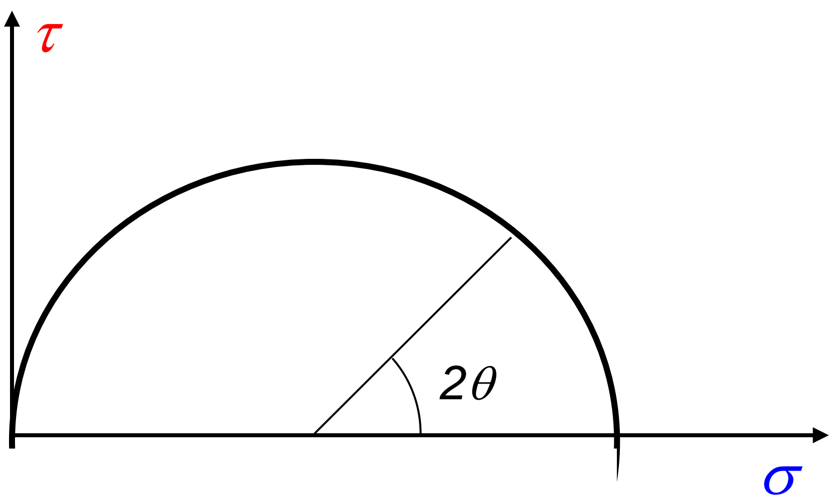
Normal stress $\sigma = \frac{F_n}{A'}$

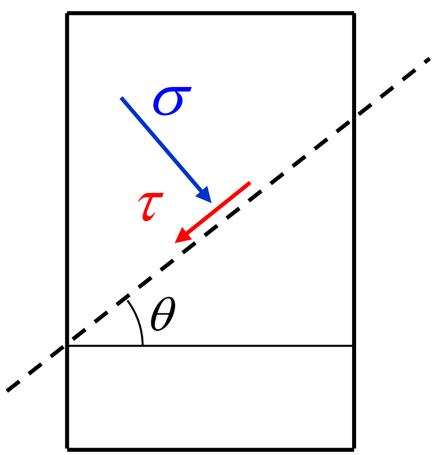
Shear stress $\tau = \frac{F_p}{A'}$



Stress

Plotting corresponding values of σ and τ



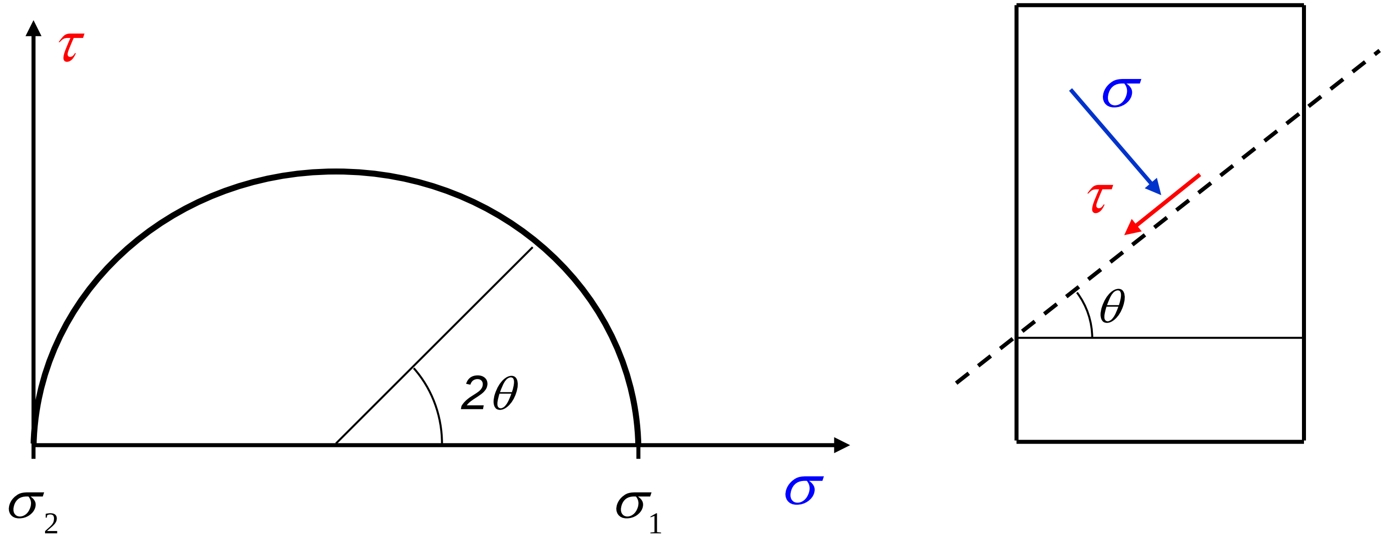


σ is *normal stress* and τ is *shear stress*

Stress

$\tau = 0$ for $\sigma = \sigma_1$ or σ_2

σ_1 and σ_2 are *principal stresses*



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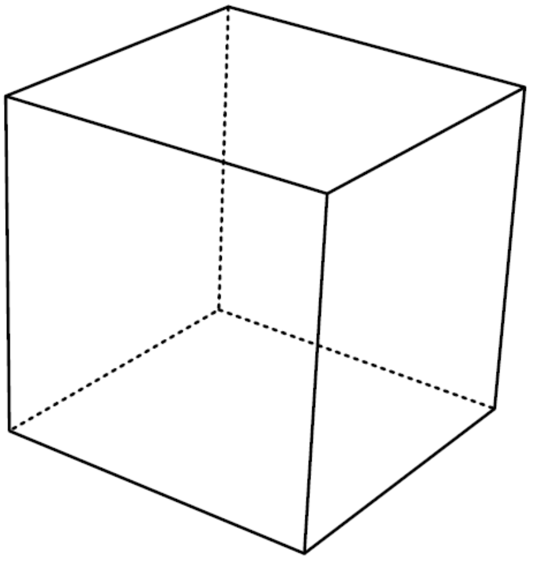
Stress



Complete description of the stress state: **the stress tensor**

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

When a body is at rest, not net translational or rotational forces can act on it

$$\Rightarrow \quad \tau_{yx} = \tau_{xy} \qquad \tau_{zx} = \tau_{xz} \qquad \tau_{zy} = \tau_{yz}$$

The stress tensor is symmetric

$$\sum_j \frac{\partial \sigma_{ji}}{\partial x_j} = 0$$

Equations of static equilibrium

Stress

The components of the stress tensor
depends on our choice of coordinate system

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

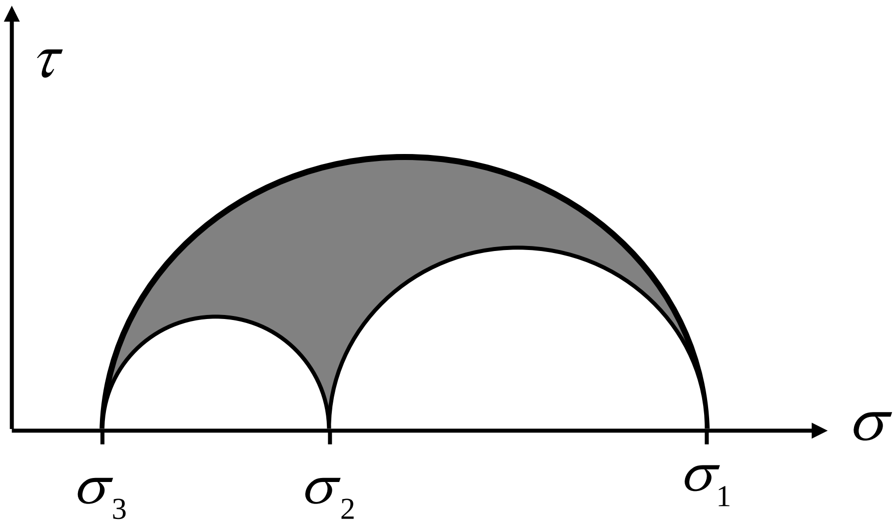
Some combinations of the components are invariant
to a rotation of the coordinate system, for instance:

The mean stress $\bar{\sigma} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) \qquad (= p)$

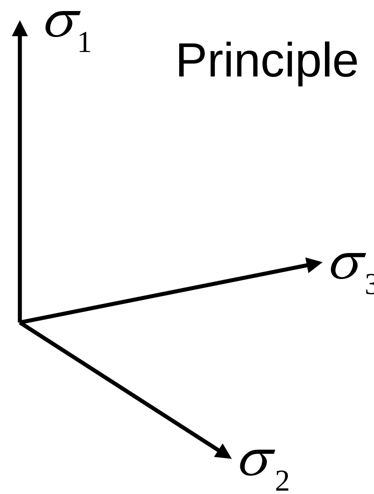
The generalized shear stress $q = \sqrt{\frac{3}{2}\left[\left(\sigma_1 - \bar{\sigma}\right)^2 + \left(\sigma_2 - \bar{\sigma}\right)^2 + \left(\sigma_3 - \bar{\sigma}\right)^2\right]}$

Stress

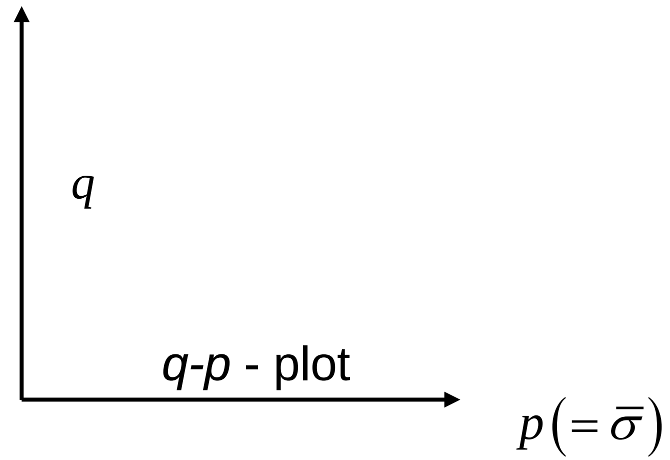
Graphical representations of a stress state



Mohr circle(s)



Principle stress plot



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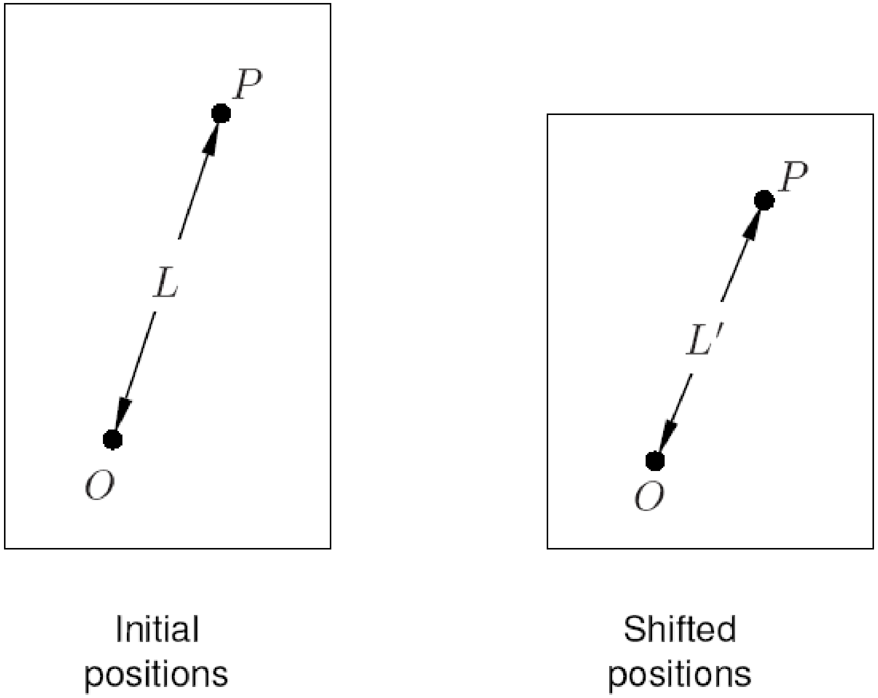
Strain

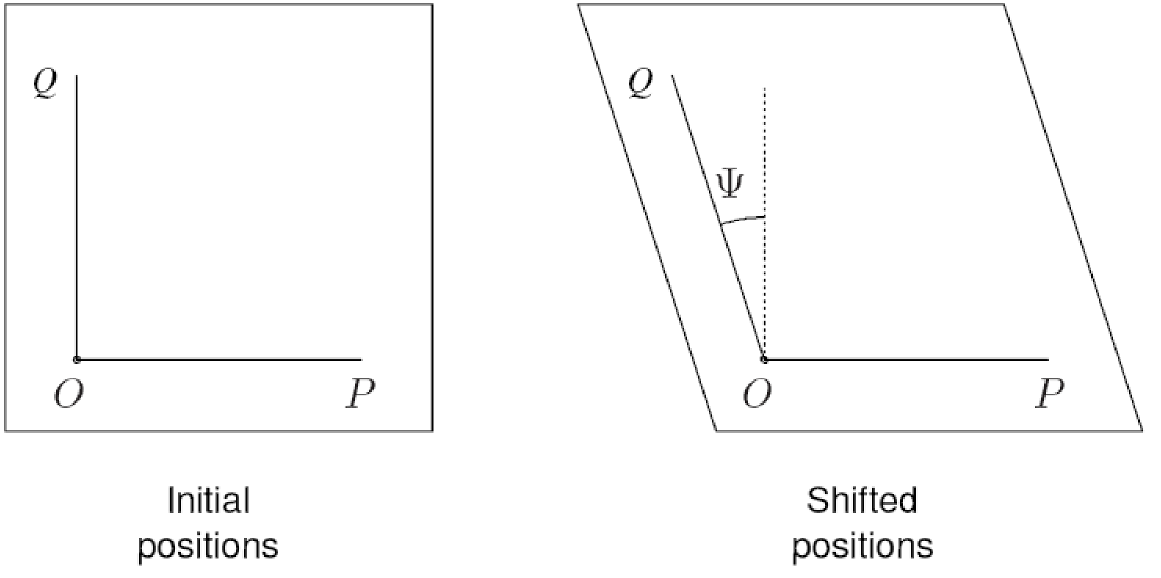


Elongation

$$\varepsilon = \frac{L - L'}{L} = -\frac{\Delta L}{L}$$

Rock mechanics sign convention: Compression ($L' < L$) is positive

Strain

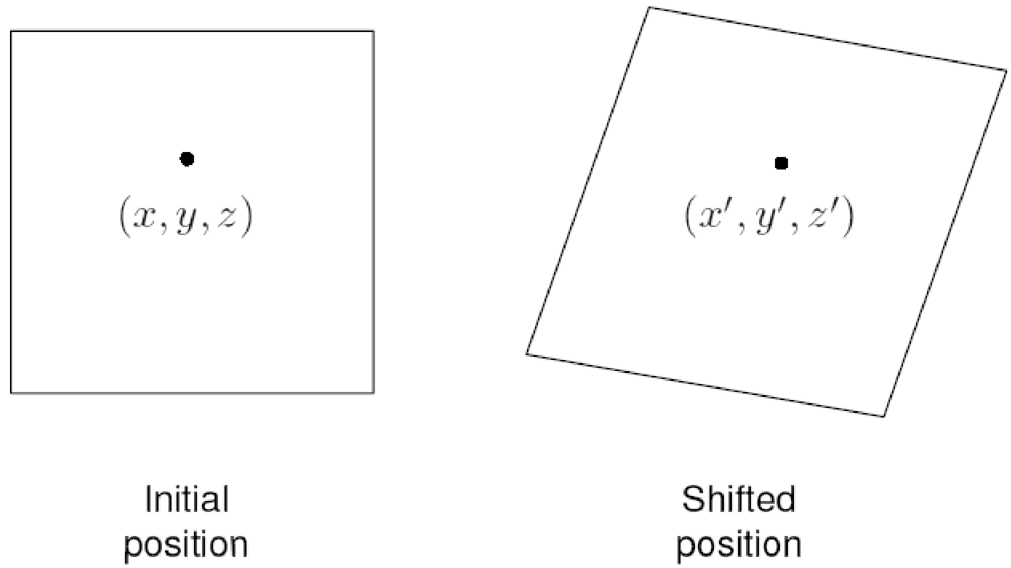


Shear strain

$$\Gamma = \frac{1}{2}\tan\Psi$$

Strain

$$u = x - x'$$
$$v = y - y'$$
$$w = z - z'$$



$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \Gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \text{etc.}$$

General form:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Strain

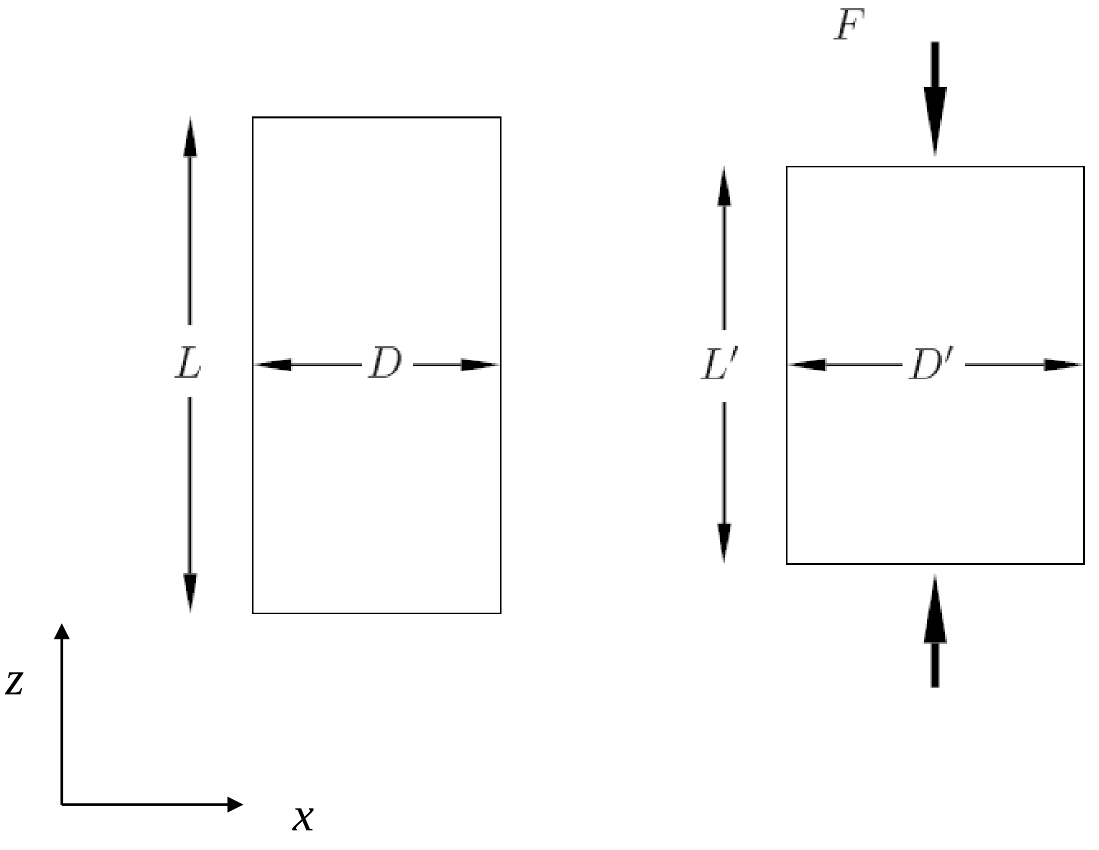
Strain tensor

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_x & \Gamma_{xy} & \Gamma_{xz} \\ \Gamma_{yx} & \varepsilon_y & \Gamma_{yz} \\ \Gamma_{zx} & \Gamma_{zy} & \varepsilon_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

Some combinations of the components are invariant to a rotation of the coordinate system, for instance:

Volumetric strain $\varepsilon_{vol} = \varepsilon_x + \varepsilon_y + \varepsilon_z = -\frac{\Delta V}{V}$

Elastic moduli



Hooke's law:

$$\varepsilon_z \propto \sigma_z$$

$$\varepsilon_z = \frac{\sigma_z}{E}$$

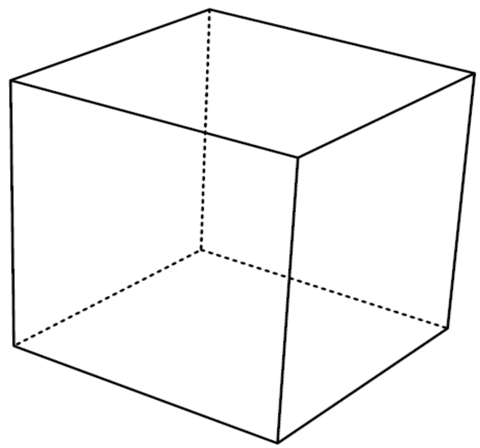
E = Young's modulus

$[E]$ = GPa

Poisson's ratio ν

$$\nu = -\frac{\varepsilon_x}{\varepsilon_z}$$

Elastic moduli



Bulk modulus:

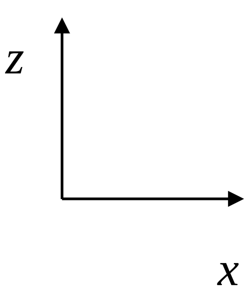
$$K = \frac{\overline{\sigma}}{\varepsilon_{vol}}$$

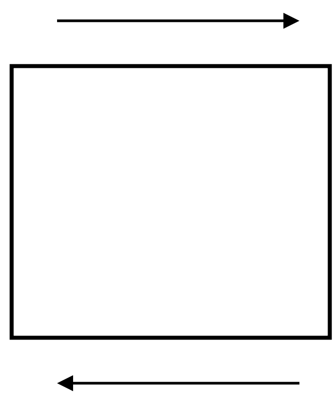
$[K]$ = GPa

If

$$\sigma_x = \sigma_y = \sigma_z = \sigma_p$$

$$K = \frac{\sigma_p}{\varepsilon_{vol}}$$





$$\tau_{xz} = 2G\Gamma_{xz}$$

Shear modulus G

$[G]$ = GPa

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Elastic moduli

Hooke's law –

general version:

$$\varepsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z$$

$$\varepsilon_y = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{\nu}{E}\sigma_z$$

$$\varepsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y + \frac{1}{E}\sigma_z$$

$$\Gamma_{yz} = \frac{1+\nu}{E}\tau_{yz} \equiv \frac{1}{2G}\tau_{yz}$$

$$\Gamma_{xz} = \frac{1+\nu}{E}\tau_{xz} \equiv \frac{1}{2G}\tau_{xz}$$

$$\Gamma_{xy} = \frac{1+\nu}{E}\tau_{xy} \equiv \frac{1}{2G}\tau_{xy}$$

Elastic moduli

Hooke's law –

Inverse general version:

$$\sigma_x = (\lambda + 2G)\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z$$

$$\sigma_y = \lambda\varepsilon_x + (\lambda + 2G)\varepsilon_y + \lambda\varepsilon_z$$

$$\sigma_z = \lambda\varepsilon_x + \lambda\varepsilon_y + (\lambda + 2G)\varepsilon_z$$

$$\tau_{yz} = 2G\Gamma_{yz}$$

$$\tau_{xz} = 2G\Gamma_{xz}$$

$$\tau_{xy} = 2G\Gamma_{xy}$$

Important footnote: When using Hooke's law in geomechanical applications, remember that strain is a relative quantity (change in length or angle) and is associated with a change in stress.

Relations between elastic moduli for isotropic solids

Modulus	$\lambda;G$	$H;G$	$K;G$	$E;\nu$
Plane wave modulus *H*	$\lambda+2G$	H	$K+\frac{4}{3}G$	$E\frac{(1-\nu)}{(1+\nu)(1-2\nu)}$
Shear modulus *G*	G	G	G	$E\frac{1}{2(1+\nu)}$
Bulk modulus *K*	$\lambda+\frac{2}{3}G$	$H-\frac{4}{3}G$	K	$E\frac{1}{3(1-2\nu)}$
Young's modulus *E*	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{G(3H-4G)}{H-G}$	$\frac{9KG}{3K+G}$	E
Lamè coefficient λ	λ	$H-2G$	$K-\frac{2}{3}G$	$E\frac{\nu}{(1+\nu)(1-2\nu)}$
Poisson's ratio ν	$\frac{\lambda}{2(\lambda+G)}$	$\frac{H-2G}{2(H-G)}$	$\frac{3K-2G}{2(3K+G)}$	ν

Fluids:
G = E = 0
K = λ
ν = ½

The Wave Equation

- Dynamic equilibrium: Newton's 2nd law combined with Hooke's law

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$$

- Solutions: Plane waves

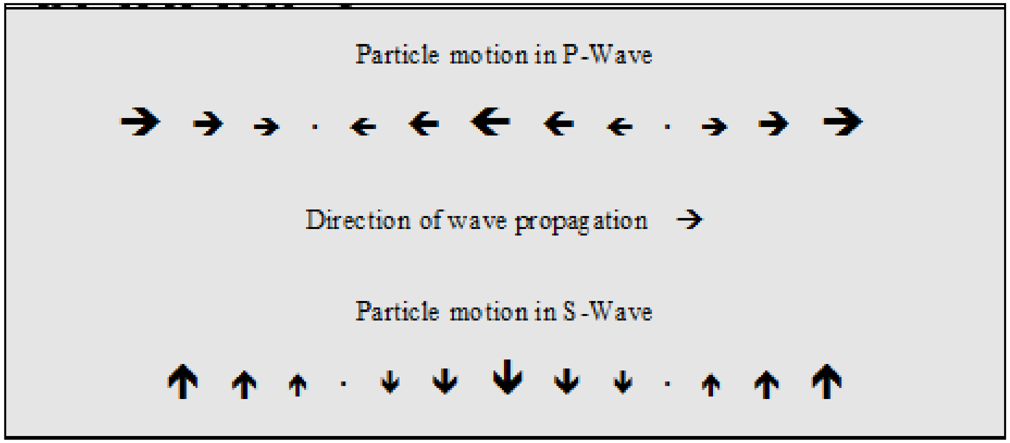
  e.g. for propagation || x: $u_x = u_x^0 e^{j(\omega t - q_P x)}$ (P-wave)

  $u_y = u_y^0 e^{j(\omega t - q_S x)}$ (S-wave)

  ω: Angular frequency $=2\pi f$

  q: Wavenumber $=2\pi/\lambda_w$; λ_w is the wavelength

P- & S-Wave Velocities



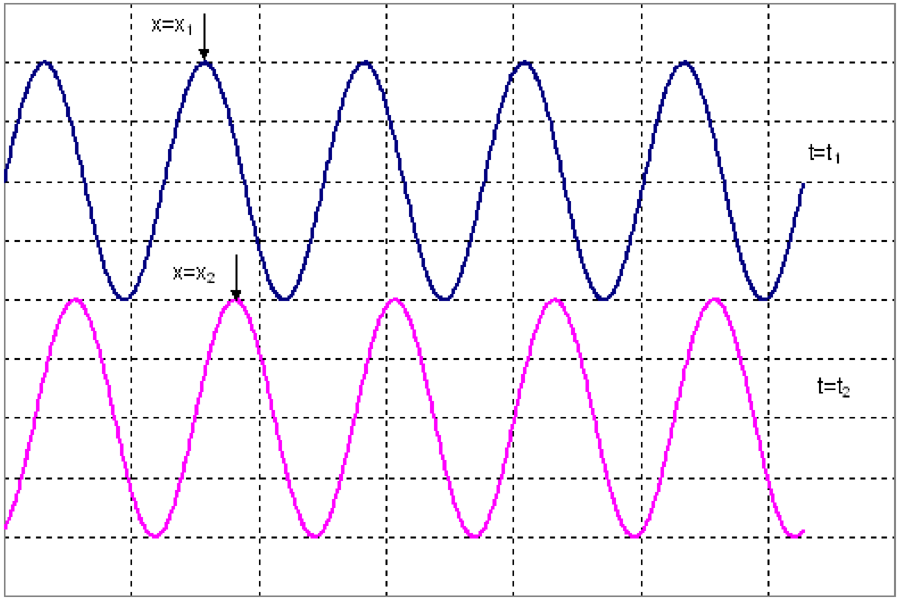
- In isotropic solids:

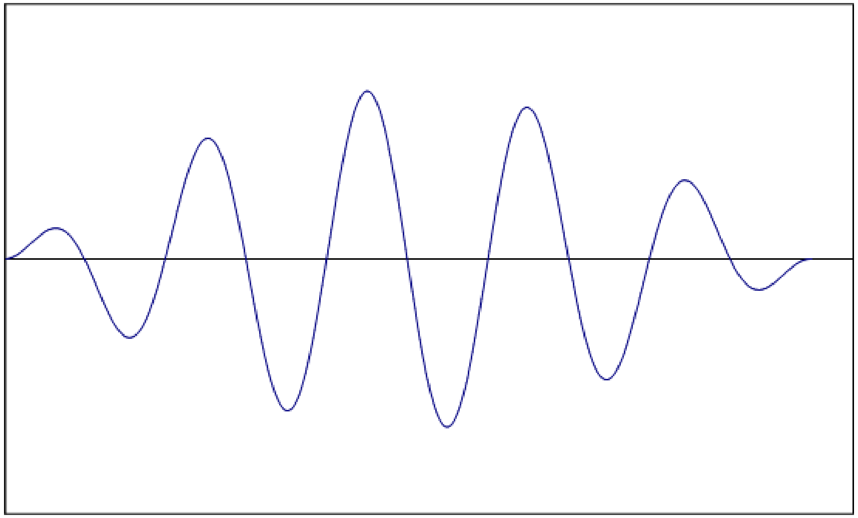
$$v_P = \sqrt{\frac{K + \frac{4}{3}G}{\rho}} = \sqrt{\frac{\lambda + 2G}{\rho}}$$

$$v_S = \sqrt{\frac{G}{\rho}}$$

The P-wave ("plane wave") modulus is equal to the uniaxial compaction modulus

Phase & Group Velocities





- Phase Velocity:

  Velocity of a moving wavefront = $\omega/q = f\lambda_w$

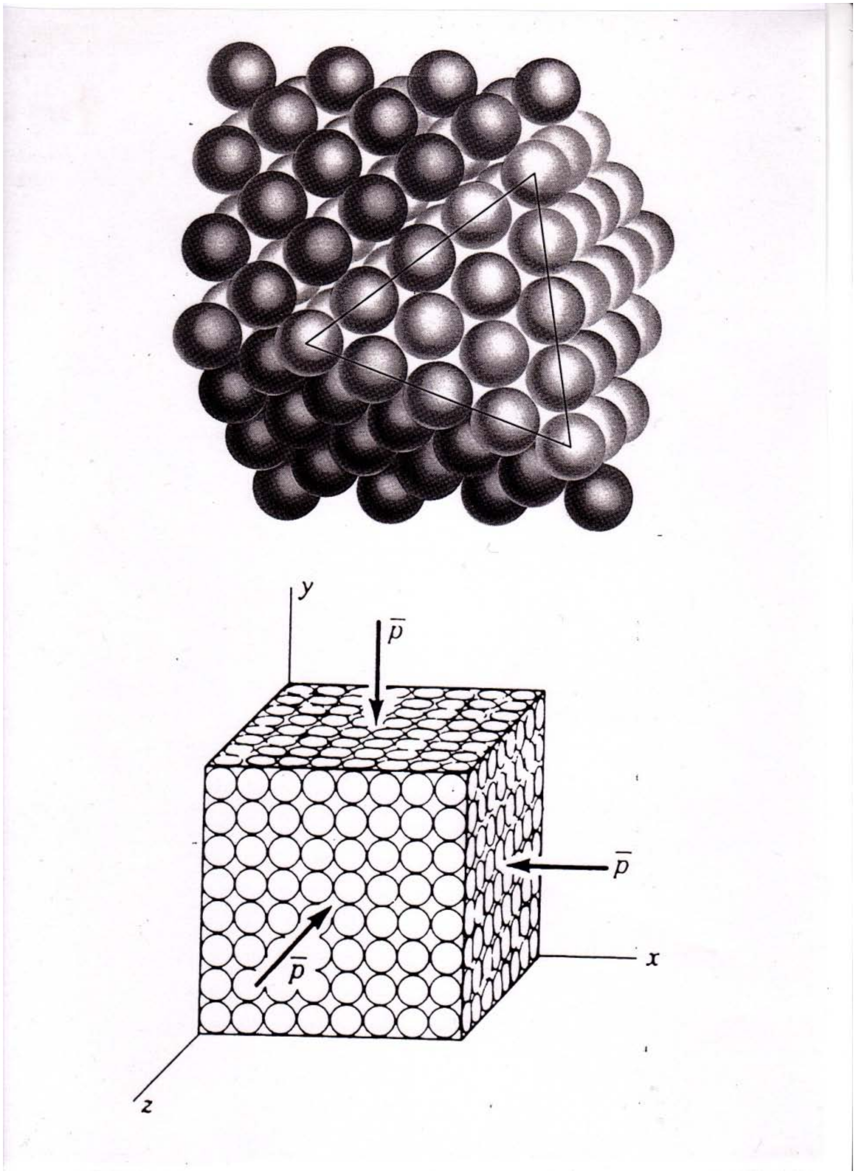
- Group Velocity:

  Velocity of carrier signal $= d\omega/dq$ ($\cong$energy velocity)

Anisotropy

Anisotropy

- Anisotropy is a result of structural order caused by heterogeneity at a length scale << wavelength of the probe.



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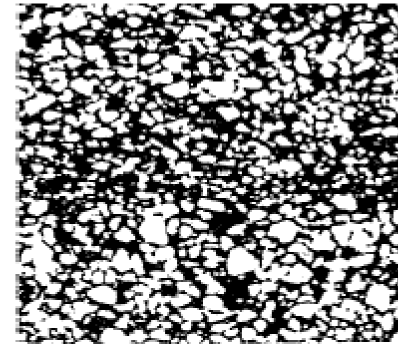
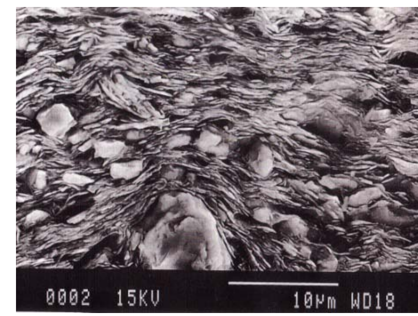
Sources of Anisotropy in Rocks

❑ Lithological (Intrinsic) anisotropy

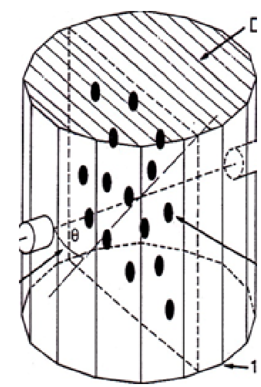
- ➢ Lamination / Bedding
- ➢ Oriented particles
  - ❖ Anisotropic (& oriented) particles





❑ Stress-Induced (Extrinsic) anisotropy

- ➢ Cracks & Fractures
- ➢ Directly stress-induced by elastic nonlinearity



Elasticity theory for Anisotropic Solids

- **Hooke's law:**

$$\sigma_{ij} = \sum_{kl} C_{ijkl}\varepsilon_{kl}$$

- **The 4th rank tensor C_{ijkl} has 3^4=81 components, but reduces directly to 21, because**
  $i \leftrightarrows j$; $k \leftrightarrows l$; $ij \leftrightarrows kl$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- **Permits reduced Voigt notation: $C_{ijkl} \rightarrow C_{IJ}$**

11	$\rightarrow$	1
22	$\rightarrow$	2
33	$\rightarrow$	3
23	$\rightarrow$	4
13	$\rightarrow$	5
12	$\rightarrow$	6

We tend to associate
$1 \rightarrow x$
$2 \rightarrow y$
$3 \rightarrow z$

Elasticity theory for Anisotropic Solids

- Hooke's law in Voigt notation:

$$\sigma_I = C_{IJ}\varepsilon_J$$

C is a 6x6 matrix; σ and ε now are 6-component vectors, indices 1-3 represent normal and 4-6 shear stresses or strains

- The number of components in the C-matrix reflects material symmetry:
  - Orthorhombic symmetry $\rightarrow$ 9
  - Transverse Isotropy (TI) $\rightarrow$ 5; with symmetry-axis z $C_{11}=C_{22}$, $C_{13}=C_{23}$, $C_{44}=C_{55}$, and $C_{66}=½(C_{11}-C_{12})$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\Gamma_{yz} \\ 2\Gamma_{xz} \\ 2\Gamma_{xy} \end{pmatrix}$$

Compliance:

$$S = C^{-1}$$

Hooke's law: $\sigma = C\varepsilon$

$$\varepsilon = S\sigma$$

On explicit form:

$$\begin{aligned} \sigma_x &= C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z \\ \sigma_y &= C_{12}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z \\ \sigma_z &= C_{13}\varepsilon_x + C_{23}\varepsilon_y + C_{33}\varepsilon_z \\ \tau_{yz} &= 2C_{44}\Gamma_{yz} \\ \tau_{xz} &= 2C_{55}\Gamma_{xz} \\ \tau_{xy} &= 2C_{66}\Gamma_{xy} \end{aligned}$$

$$\begin{aligned} \sigma_x &= (\lambda + 2G)\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z \\ \sigma_y &= \lambda\varepsilon_x + (\lambda + 2G)\varepsilon_y + \lambda\varepsilon_z \\ \sigma_z &= \lambda\varepsilon_x + \lambda\varepsilon_y + (\lambda + 2G)\varepsilon_z \\ \tau_{yz} &= 2G\Gamma_{yz} \\ \tau_{xz} &= 2G\Gamma_{xz} \\ \tau_{xy} &= 2G\Gamma_{xy} \end{aligned}$$

Christoffel Wave Equation

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} = C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l}$$

We insert a wave solution:

$$u_i = u_i^0 e^{j(\omega t - q_i x_i)}$$

where the directional cosine of the wave propagation direction is:

$$n_i = \frac{q_i}{\left|\vec{q}\right|}$$

⬇

$$\left[C_{ijkl} n_j n_l - \rho \mathrm{v}^2 \delta_{ik} \right] \bullet \left[u_k^0 \right] = 0$$

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Christoffel Wave Equation

$$\left[C_{ijkl} n_j n_l - \rho \mathrm{v}^2 \delta_{ik} \right] \cdot \left[u_k^0 \right] = 0$$

$$\begin{bmatrix} C_{11}n_1^2 + C_{66}n_2^2 + C_{44}n_3^2 - \rho v^2 & (C_{11} - C_{66})n_1 n_2 & (C_{13} + C_{44})n_1 n_3 \\ (C_{11} - C_{66})n_1 n_2 & C_{66}n_1^2 + C_{11}n_2^2 + C_{44}n_3^2 - \rho v^2 & (C_{13} + C_{44})n_2 n_3 \\ (C_{13} + C_{44})n_1 n_3 & (C_{13} + C_{44})n_2 n_3 & C_{44}(n_1^2 + n_2^2) + C_{33}n_3^2 - \rho v^2 \end{bmatrix} \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} = 0$$

Example:
Wave propagation along symmetry axis

- $n_1 = n_2 = 0; n_3 = 1$

$$\begin{bmatrix} C_{44} - \rho v^2 & 0 & 0 \\ 0 & C_{44} - \rho v^2 & 0 \\ 0 & 0 & C_{33} - \rho v^2 \end{bmatrix} \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} = 0$$

$$\mathrm{v}_{P;z} = \sqrt{\frac{C_{33}}{\rho}}$$

(u || z; l || z)

$$\mathrm{v}_{S;zx} = \mathrm{v}_{S;zy} = \sqrt{\frac{C_{44}}{\rho}}$$

(u || x or y; l || z)

Example:
Wave propagation in symmetry plane

- For example let $n_1 = 1$; $n_2 = n_3 = 0$

$$\begin{bmatrix} C_{11} - \rho v^2 & 0 & 0 \\ 0 & C_{66} - \rho v^2 & 0 \\ 0 & 0 & C_{44} - \rho v^2 \end{bmatrix} \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} = 0$$

Shear Wave Splitting

$$\mathrm{v}_{P;x} = \sqrt{\frac{C_{11}}{\rho}}$$

(u || x; l || x)

$$v_{s;xz} = \sqrt{\frac{C_{44}}{\rho}}$$

(u || z; l || x)

$$v_{s;xy} = \sqrt{\frac{C_{66}}{\rho}}$$

(u || y; l || x)

Wave propagation along a general direction

- Because of TI, we can look at e.g. the xz-plane only; choosing $n_1 = \sin\theta$; $n_2 = 0$; $n_3 = \cos\theta$ (θ is angle between wave propagation direction and z-axis)

$$v_S = \sqrt{\frac{C_{66}\sin^2\theta + C_{44}\cos^2\theta}{\rho}}$$

Particle motion || y; SH wave

$$v_{\text{qP or S}} = \sqrt{\frac{C_{11}\sin^2\theta + C_{33}\cos^2\theta + C_{44} \pm \sqrt{\Delta}}{2\rho}}$$

Particle motion in xz-plane; quasiP & quasi-SV wave

$$\Delta = \left[(C_{11} - C_{44})\sin^2\theta - (C_{33} - C_{44})\cos^2\theta\right]^2 + 4[C_{13} + C_{44}]^2 \sin^2\theta\cos^2\theta$$

Thomsen parameters

- Simplifying by introducing 3 anisotropy parameters which are small ($\rightarrow$ 0 for isotropy).

$$\varepsilon = \frac{C_{11} - C_{33}}{2C_{33}}$$

P-wave anisotropy

$$\gamma = \frac{C_{66} - C_{44}}{2C_{44}}$$

S-wave anisotropy

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})}$$

”moveout parameter”

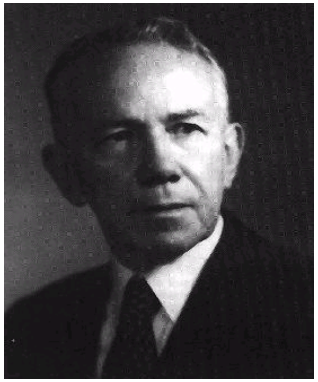
Angular dependence of wave velocities expressed by Thomsen's parameters

$$v_P(\theta) = v_P(0)\left[1+\delta\sin^2\theta\cos^2\theta+\varepsilon\sin^4\theta\right]$$

$$v_{SV} = v_S(0)[1+\frac{v_P^2(0)}{v_S^2(0)}(\varepsilon-\delta)\sin^2\theta\cos^2\theta]$$

$$v_{SH}(\theta) = v_S(0)\left[1+\gamma\sin^2\theta\right]$$

- In anisotropic media, the phase and group velocities will only be equal along symmetry directions (provided no dispersion)



Poroelasticity

Poroelasticity

2 stresses: The external (total) stress σ_{ij}
The pore pressure p_f

2 strains: The strain of a volume element attached to the rock's framework; ε_{ij}^s

Volumetric strain $\varepsilon_{vol} = -\frac{\Delta V}{V}$

The "increment of fluid content"; i.e.

$$\frac{displaced \quad fluid \quad volume}{total \quad volume} = \frac{\Delta V_p - \Delta V_f}{V} \Rightarrow$$

$$\zeta = -\phi \left(\frac{\Delta V_p}{V_p} + \frac{\Delta p_f}{K_f} \right)$$

Porosity:

$$\phi = \frac{V_p}{V} = \frac{V_f}{V}$$

Note: ε>0 for compaction, ΔV>0 for expansion

Biot-Hooke's law

Isotropic stress conditions

$$\Delta\sigma = K\varepsilon_{vol} - C\zeta$$
$$\Delta p_f = C\varepsilon_{vol} - M\zeta$$

***K*, *C* & *M* are poroelastic coefficients, related to the elastic coefficients of the ingredients:**

K_f:	Bulk modulus of pore fluid
K_s:	Bulk modulus of solid grains
ϕ:	Porosity
K_{fr}:	Bulk modulus of the drained rock (rock ”framework”)
G_{fr}:	Shear modulus of the rock framework

1st "thought experiment"

The poroelastic coefficients

K **is the bulk modulus in undrained isotropic loading (no fluid expelled)**

$$\Delta\sigma = K\varepsilon_{vol}$$ when $\zeta = 0$

In this case, pore pressure builds up:

$$\Delta p_f = \frac{C}{K}\Delta\sigma$$

C/K is named Skempton's B-parameter

2nd "thought experiment"

The poroelastic coefficients

K_{fr} is the bulk modulus measured in drained, isotropic loading (with constant (or zero) pore pressure):

$$\Delta\sigma = K_{fr}\varepsilon_{vol}$$

$$p_f = 0 \Rightarrow \quad K_{fr} = K - \frac{C^2}{M}$$

and:

$$\frac{\Delta V_p}{\Delta V} = \frac{C}{M} \equiv \alpha$$

defines the Biot coefficient α

If only pores (not grains) deform, then $\Delta V_p = \Delta V$ and $\alpha = 1$.

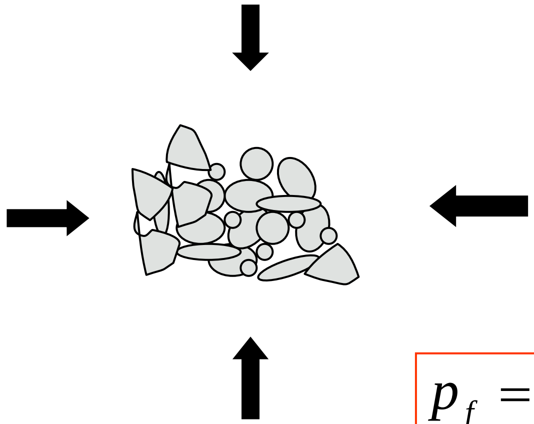
3rd "thought experiment"

The poroelastic coefficients

In an unjacketed test, external stress = pore pressure & porosity is constant =>

$$\varepsilon_{vol} = \frac{\Delta\sigma}{K_s}$$

$$\zeta = -\phi\left(\frac{1}{K_f} - \frac{1}{K_s}\right)\Delta p_f$$



$$p_f = \sigma$$

From these 3 experiment, the relations between the poroelastic coefficients and the elastic properties of the ingredients can be derived.

The poroelastic coefficients

$$\alpha = \frac{C}{M} = 1 - \frac{K_{fr}}{K_s}$$

$$\frac{1}{M} = \frac{\alpha - \phi}{K_s} + \frac{\phi}{K_f}$$

$$K = K_{fr} + \alpha^2 M = K_{fr} + \frac{K_f}{\phi} \frac{\alpha^2}{1 + \frac{K_f}{\phi K_s}(\alpha - \phi)}$$

Biot coefficient

Biot-Gassmann equation

+ $$G = G_{fr}$$

by hypothesis; no fluid effect on shear deformation

Biot-Gassmann equation

- Two equivalent expressions:

$$K = K_{fr} + \frac{K_f}{\phi} \frac{\alpha^2}{1 + \frac{K_f}{\phi K_s}(\alpha - \phi)}$$

$$\frac{K}{K_s - K} = \frac{K_{fr}}{K_s - K_{fr}} + \frac{K_f}{\phi(K_s - K_f)}$$

- Of great importance in seismic interpretation; contains fluid impact on P-wave velocity:

$$v_P = \sqrt{\frac{K + \frac{4}{3}G}{\rho}}$$

The effective stress principle

Volumetric deformation of a poroelastic material is controlled by an effective stress

$$\sigma' = \sigma - \alpha p_f$$

so that $\sigma' = K_{fr}\varepsilon_{vol}$ always

$\alpha \approx 1$ for soils and soft rocks (”Terzaghi’s effective stress”)

$\alpha < 1$ for hard rocks

In general, strain is controlled by:

$$\sigma'_{ij} = \sigma_{ij} - \alpha p_f \delta_{ij}$$

$\delta_{ij} = 0$ if $i \neq j$; no effect on shear

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Biot-Hooke's law

- Utilizing the effective stress principle, we can use Hooke's law as for solids – but with effective stresses replacing total stresses, and frame moduli replacing solid moduli (only normal stresses shown):

$$\varepsilon_x = \frac{1}{E_{fr}}\Delta\sigma_x' - \frac{\nu_{fr}}{E_{fr}}\Delta\sigma_y' - \frac{\nu_{fr}}{E_{fr}}\Delta\sigma_z'$$

$$\varepsilon_y = -\frac{\nu_{fr}}{E_{fr}}\Delta\sigma_x' + \frac{1}{E_{fr}}\Delta\sigma_y' - \frac{\nu_{fr}}{E_{fr}}\Delta\sigma_z'$$

$$\varepsilon_z = -\frac{\nu_{fr}}{E_{fr}}\Delta\sigma_x' - \frac{\nu_{fr}}{E_{fr}}\Delta\sigma_y' + \frac{1}{E_{fr}}\Delta\sigma_z'$$

Pore Compressibility

- **Two definitions of pore compressibility:**
  - External stress change (constant pore pressure) $\Rightarrow$

$$(\frac{1}{K_p}=)C_{pc}=-\frac{1}{V_p}\left(\frac{\Delta V_p}{\Delta\sigma}\right)_{p_f=const}=\frac{\alpha}{\phi K_{fr}}=\frac{1}{\phi}(\frac{1}{K_{fr}}-\frac{1}{K_s})$$

  - Pore pressure change (constant external stress) $\Rightarrow$

$$(\frac{1}{K_{pp}}=)C_{pp}=\frac{1}{V_p}\left(\frac{\Delta V_p}{\Delta p_f}\right)_{\sigma=const}=\frac{1}{\phi}(\frac{1}{K_{fr}}-\frac{1}{K_s})-\frac{1}{K_s}(=C_{pc}-\frac{1}{K_s})$$

Zimmerman, 1991

Porosity Change

$$\frac{\Delta\phi}{\phi} = \frac{\Delta V_p}{V_p} - \frac{\Delta V}{V}$$

$$\Downarrow$$

$$-\Delta\phi = \frac{\alpha - \phi}{K_{fr}}\left(\Delta\sigma - \Delta p_f\right)$$

So: The effective stress coefficient = 1 for porosity !

(may differ if solid matrix is not homogeneous)

The effective stress principle

M WARNING □

The effective stress principles above are valid only for the parameters for which they are derived and are based on linear poroelasticity.

For e.g. rock failure

$$\sigma_{ij}^{'} = \sigma_{ij} - \beta p_f \delta_{ij}$$

where $\beta \neq \alpha$ (evidence for $\beta \approx 1$).

For stress dependent wave velocities, permeability etc., different forms of effective stress principles may (or may not!) apply.

Time dependence in poroelasticity

The ζ - parameter is related to the volumetric flow rate Q [m^3/s] per unit area A [m^2]:

$$\frac{\partial \zeta}{\partial t} = \frac{\nabla \cdot \vec{Q}}{A}$$

Darcy's law

$$-\frac{\vec{Q}}{A} = \frac{k}{\eta} \nabla p_f$$

k: permeability
[1 Darcy $\approx$ 1 (μm)^2]
η: pore fluid viscosity
[1 cP = 10^{-3} Ns/m^2]

can be coupled to the poroelastic equations =>

Poroelastically coupled flow equations

Consolidation theory

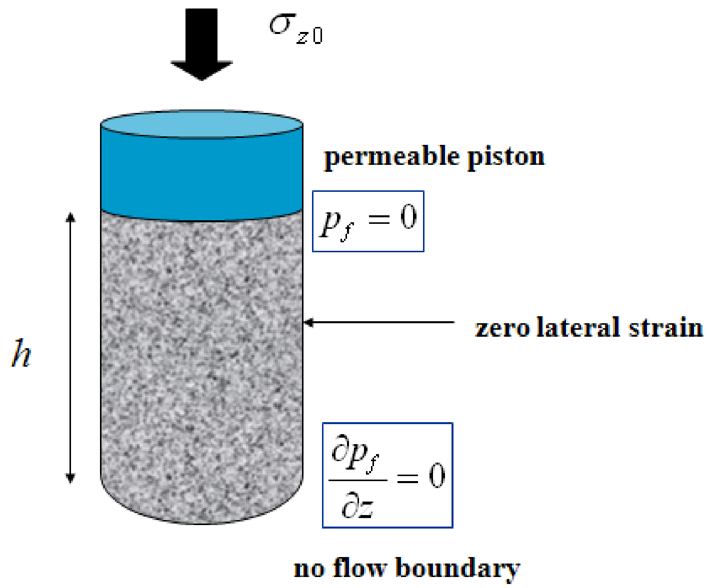
Consolidation theory – Time dependent poroelasticity

The evolution of pore pressure equilibrium in a uniaxially deforming rock sample after loaded with a stress σ_{z0} at t=0 is given by

$$\frac{\partial p_f}{\partial t} = C_D \frac{\partial^2 p_f}{\partial z^2}$$

$$C_D = \frac{k}{\eta}\left(M - \frac{C^2}{H}\right) \approx \frac{kK_f}{\eta\phi}\left[1 + \frac{K_f}{\phi(K_{fr} + \frac{4}{3}G_{fr})}\right]^{-1}$$

for K_{fr} & G_{fr} << Ks



Consolidation theory –
Time dependent poroelasticity

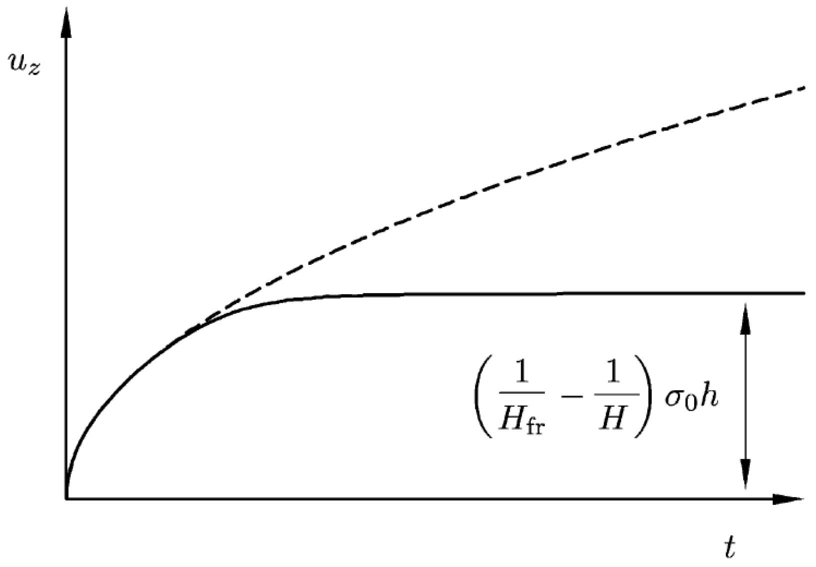
- The time (t_c)required to establish pore pressure equilibrium is given by the characteristic length scale (l_c) and the diffusion (or consolidation) coefficient C_D:

$$t_c \approx \frac{l_c^2}{C_D}$$

$$C_D \approx \frac{kK_f}{\eta\phi}\left[1+\frac{K_f}{\phi(K_{fr}+\frac{4}{3}G_{fr})}\right]^{-1}$$

- Note: Consolidation time is inversely proportional to permeability – makes a tremendeous difference between a Darcy-sand and a nanoDarcy shale!

Dashed curve: Infinitely high column



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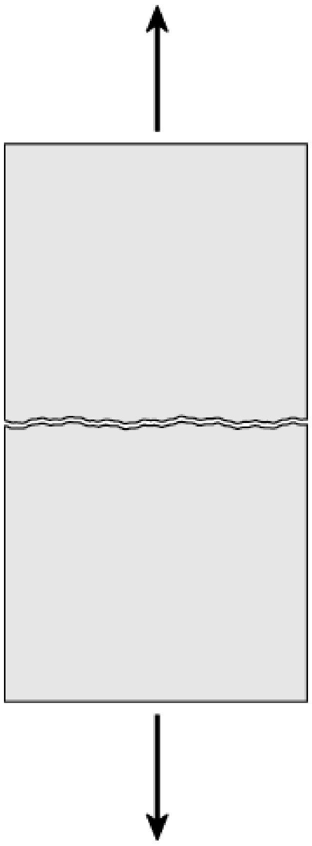
Rock failure

Tensile failure

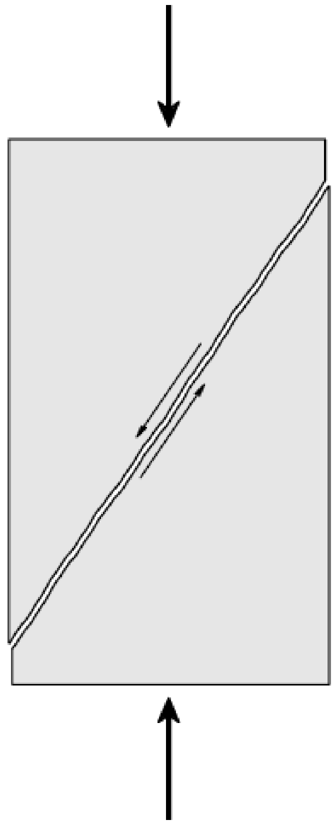


Failure criterion: $\sigma_3' = -T_0$

T_0 = tensile strength

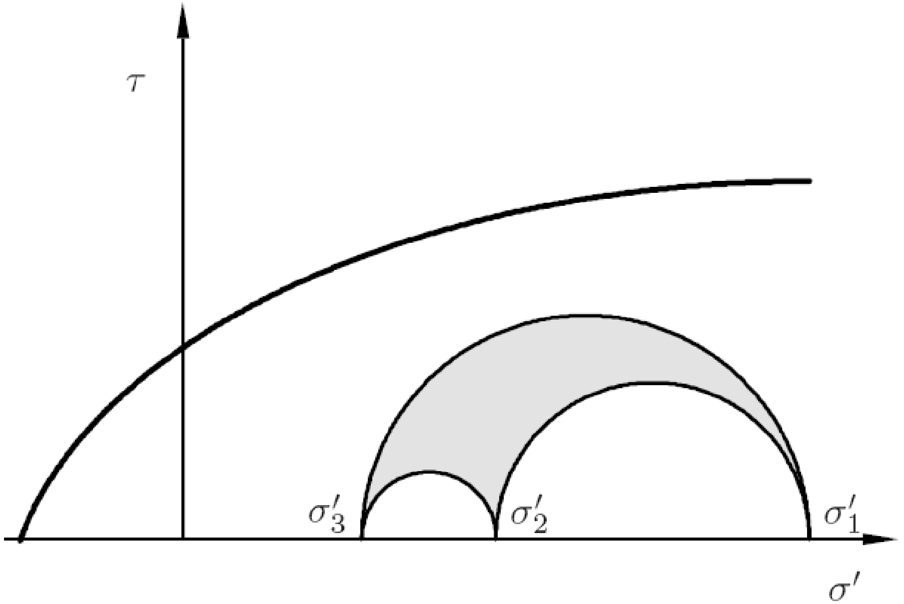
It is commonly assumed that the effective stress controlling rock failure is the net stress (effective stress coefficient =1)

Shear failure



Failure criterion:

$$|\tau_{\max}| = f(\sigma')$$



Shear failure

The Mohr-Coulomb criterion

$$|\tau| = S_0 + \mu\sigma'$$

S_0 = cohesion

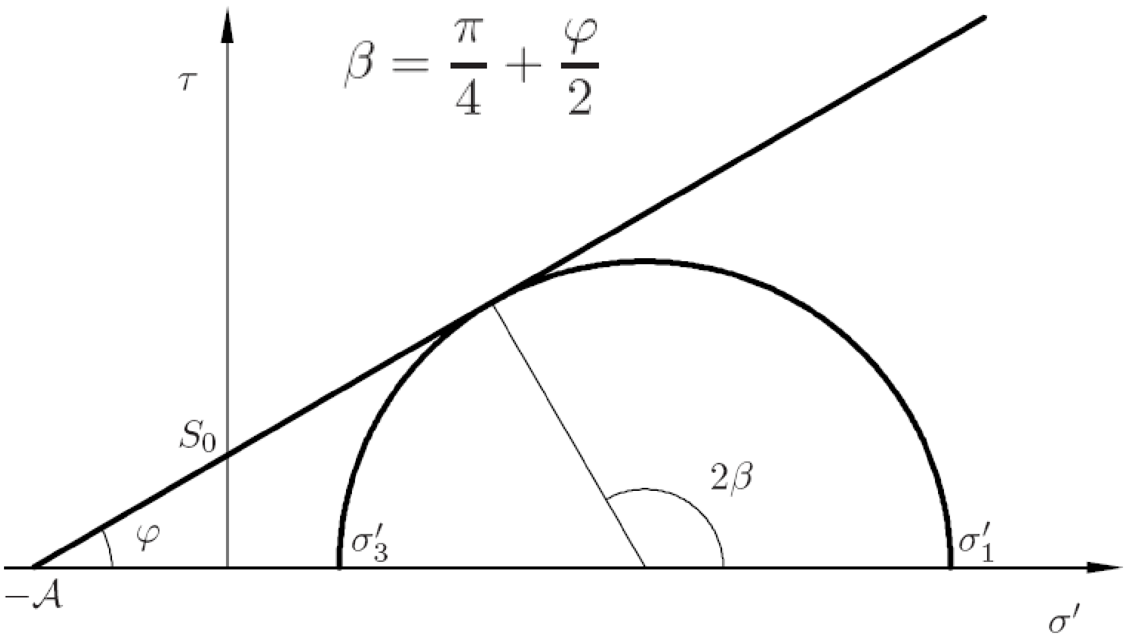
μ = coefficient of internal friction

$$\tan\varphi = \mu$$

φ = friction angle

β = failure angle

$$\beta = \frac{\pi}{4} + \frac{\varphi}{2}$$



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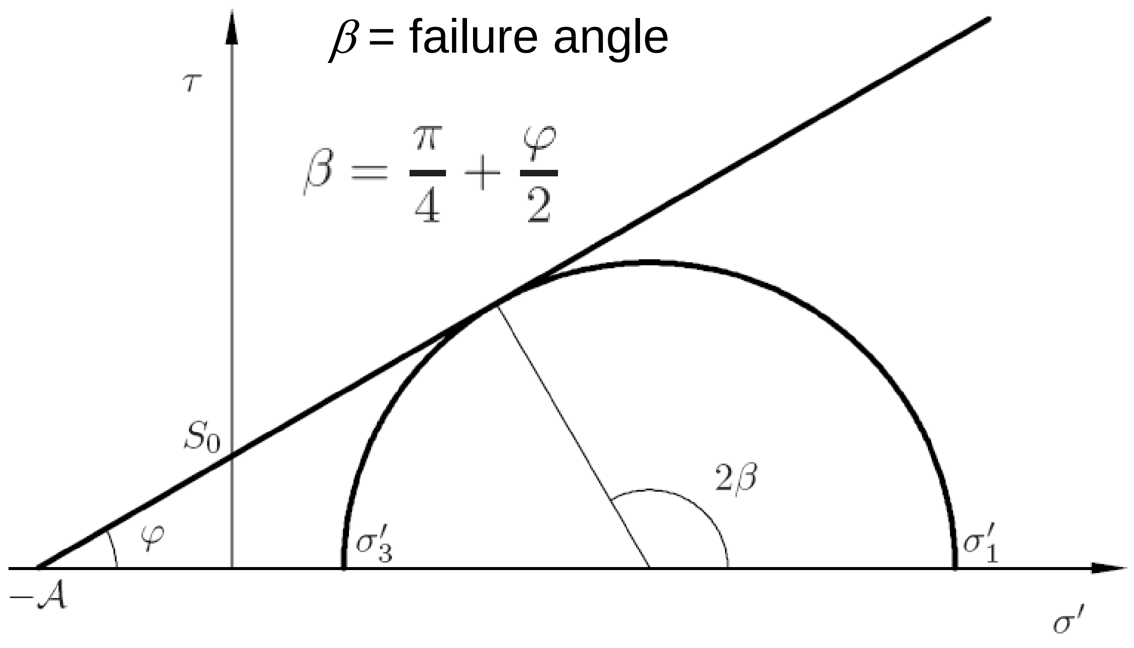
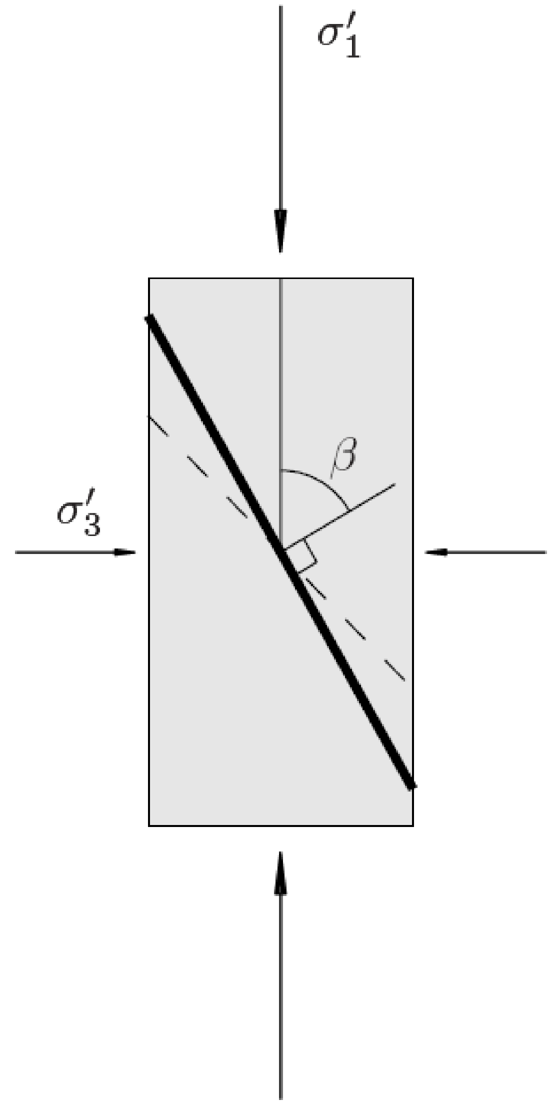


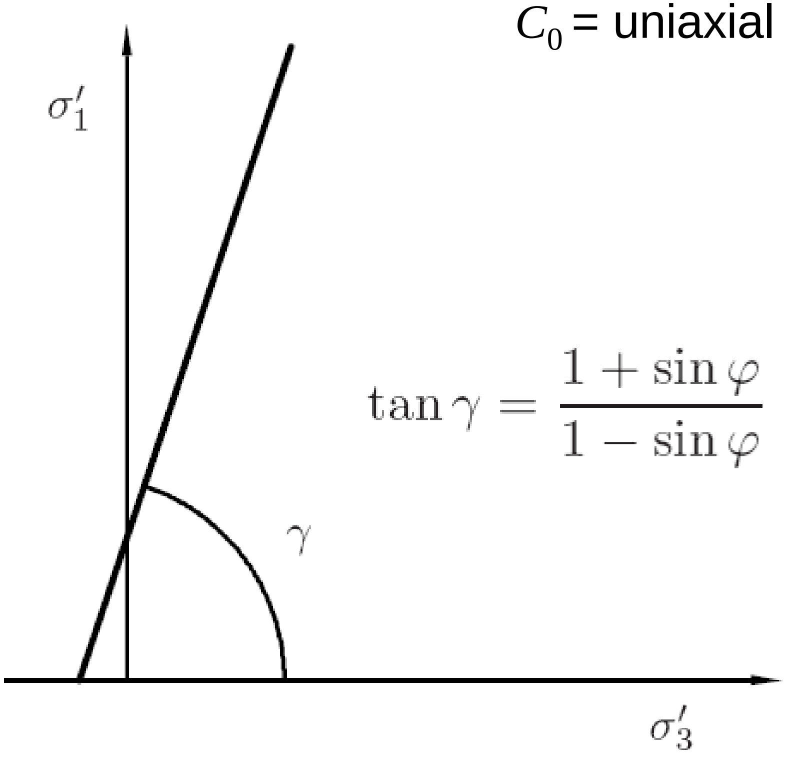


Shear failure

The Mohr-Coulomb criterion (in terms of principal stresses):

$$\sigma_1' = C_0 + \sigma_3' \tan^2 \beta$$

C_0 = uniaxial compressive strength



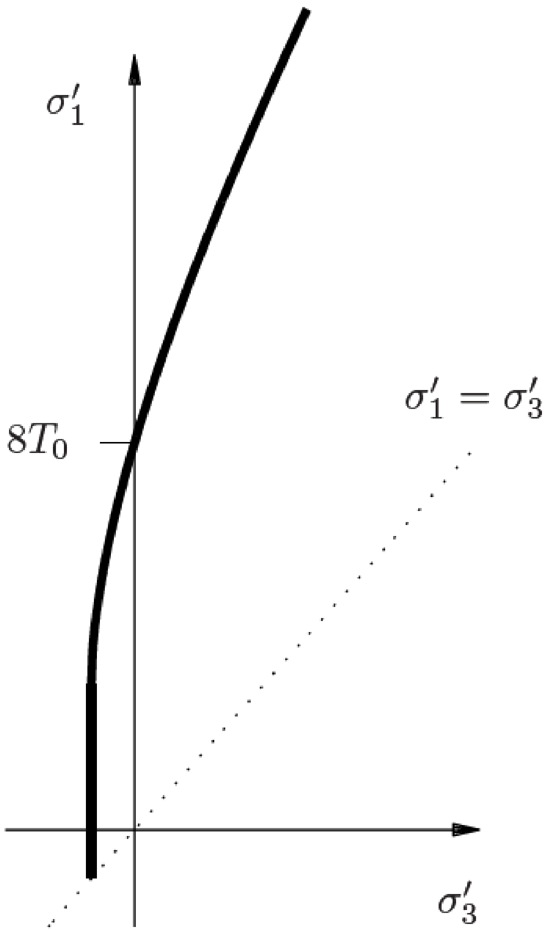
$$\tan \gamma = \frac{1 + \sin \varphi}{1 - \sin \varphi}$$

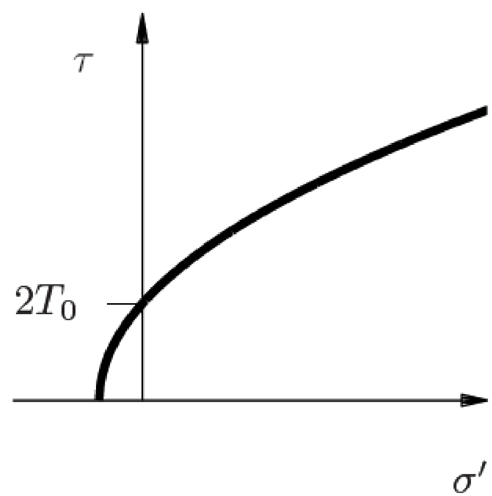
$$C_0 = 2S_0 \frac{\cos \varphi}{1 - \sin \varphi} = 2S_0 \tan \beta$$

Shear failure

The Griffith criterion: $(\sigma_1' - \sigma_3')^2 = 8T_0(\sigma_1' + \sigma_3') \qquad \text{if } \sigma_1' + 3\sigma_3' > 0$

$$\sigma_3' = -T_0 \qquad \text{if } \sigma_1' + 3\sigma_3' < 0$$





$$\tau^2 = 4T_0(\sigma' + T_0)$$

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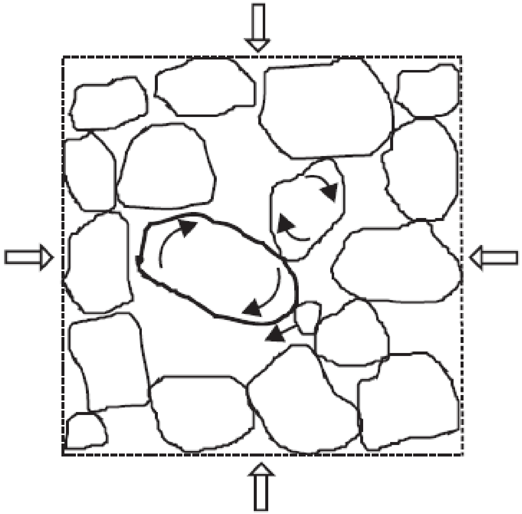
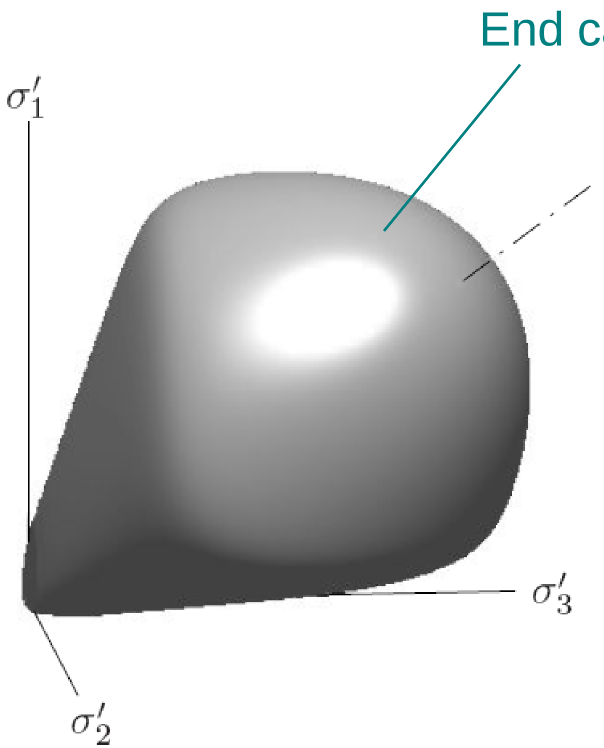
Compaction failure

Failure criterion: $p'^2 + q^2 = p*^2$ $p*$ = crushing pressure



Grain crushing

Pore collapse



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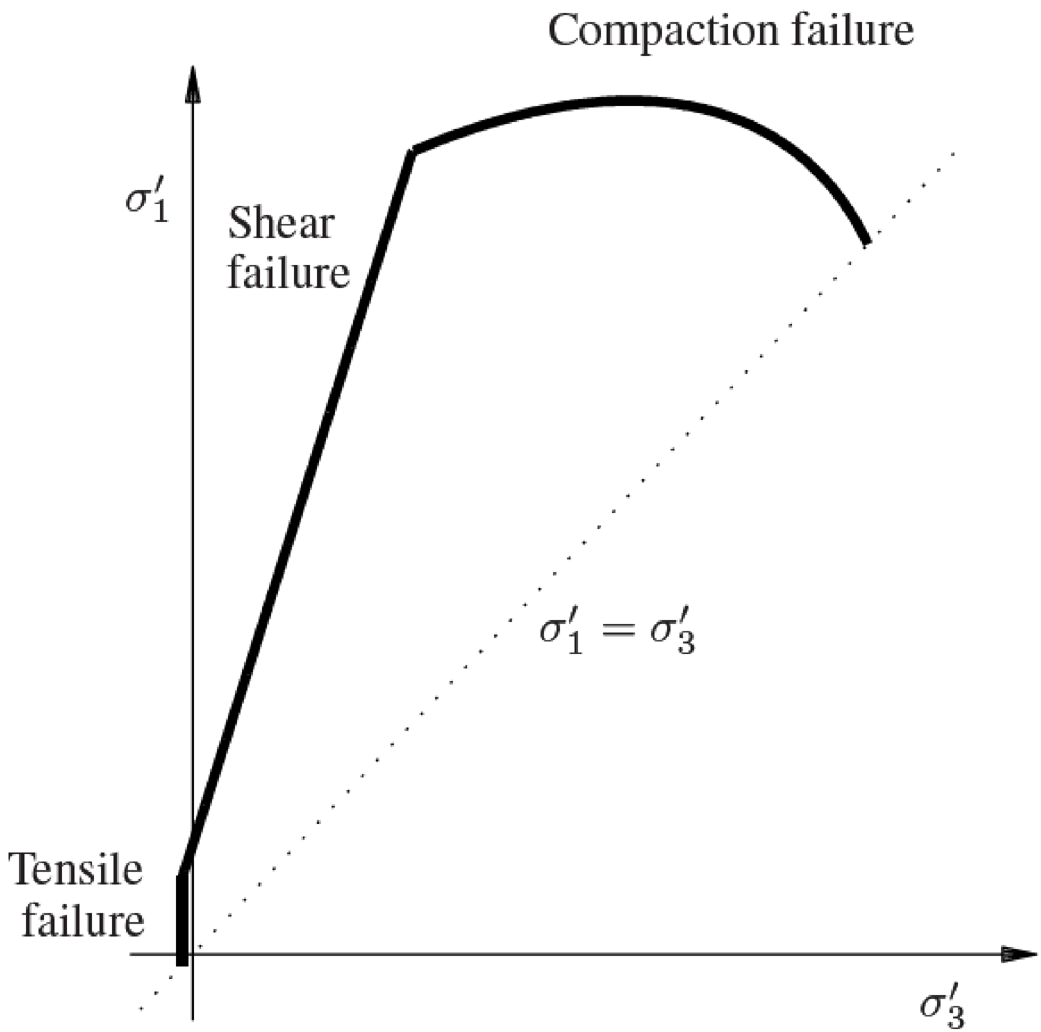
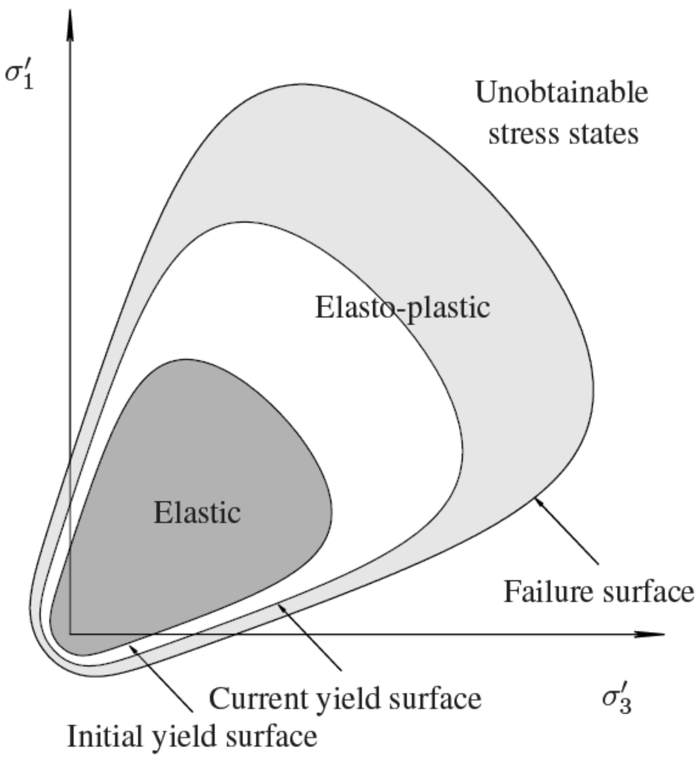

Compaction failure
σ_1'
Shear failure
$\sigma_1' = \sigma_3'$
Tensile failure
σ_3'

Plasticity & Hardening



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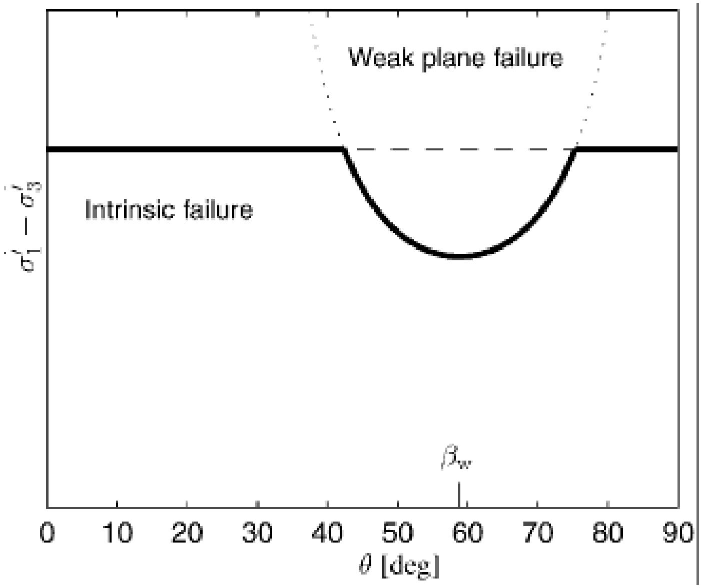
Anisotropic failure criterion



In layered rock, one may anticipate reduced cohesion (S_{0w}) and friction angle (φ_w) for a plane of a given orientation.

Shear failure along the weak plane occurs for a range of orientations of the plane with respect to the stress field.