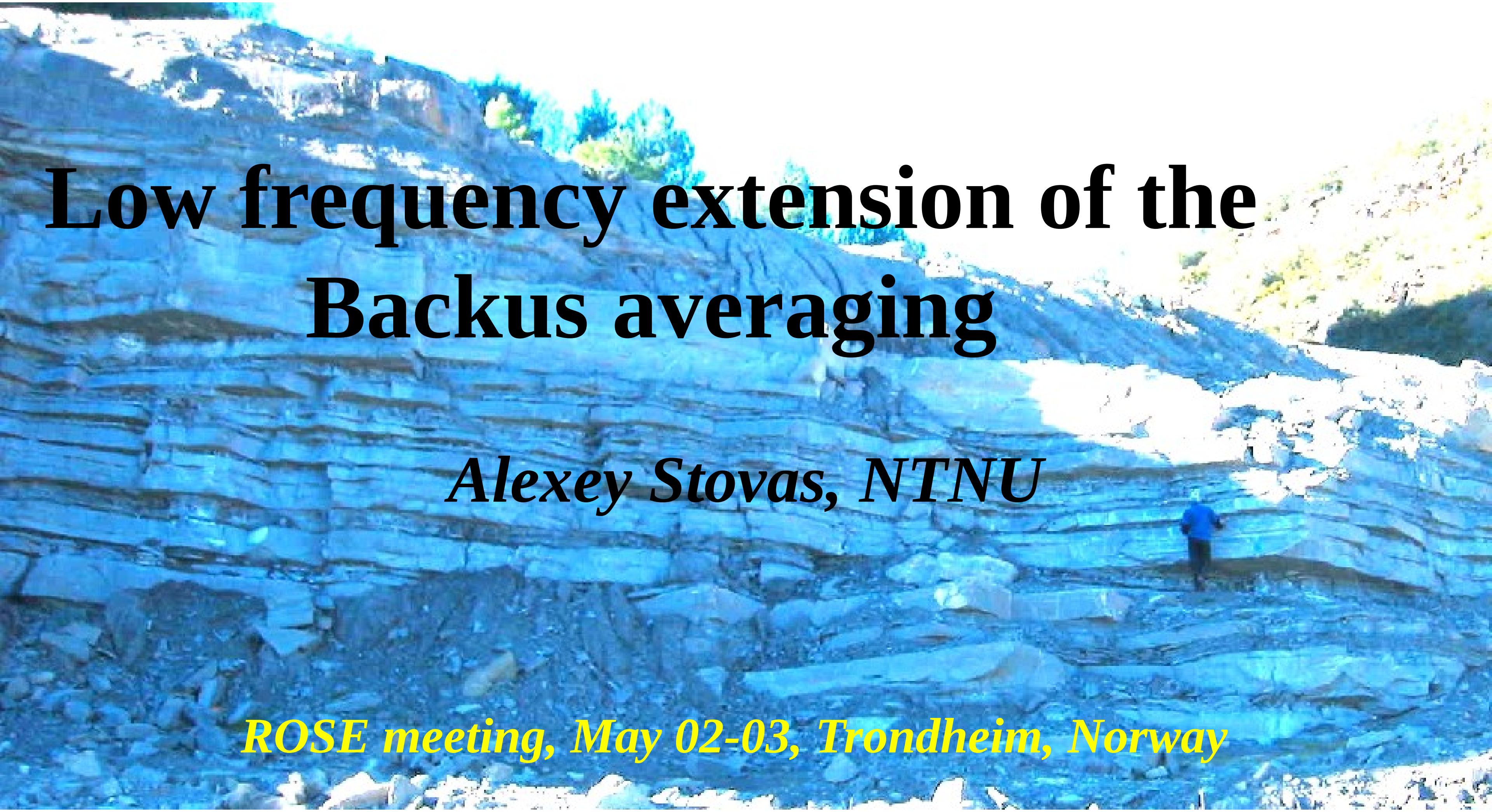
NTNU



Low frequency extension of the Backus averaging

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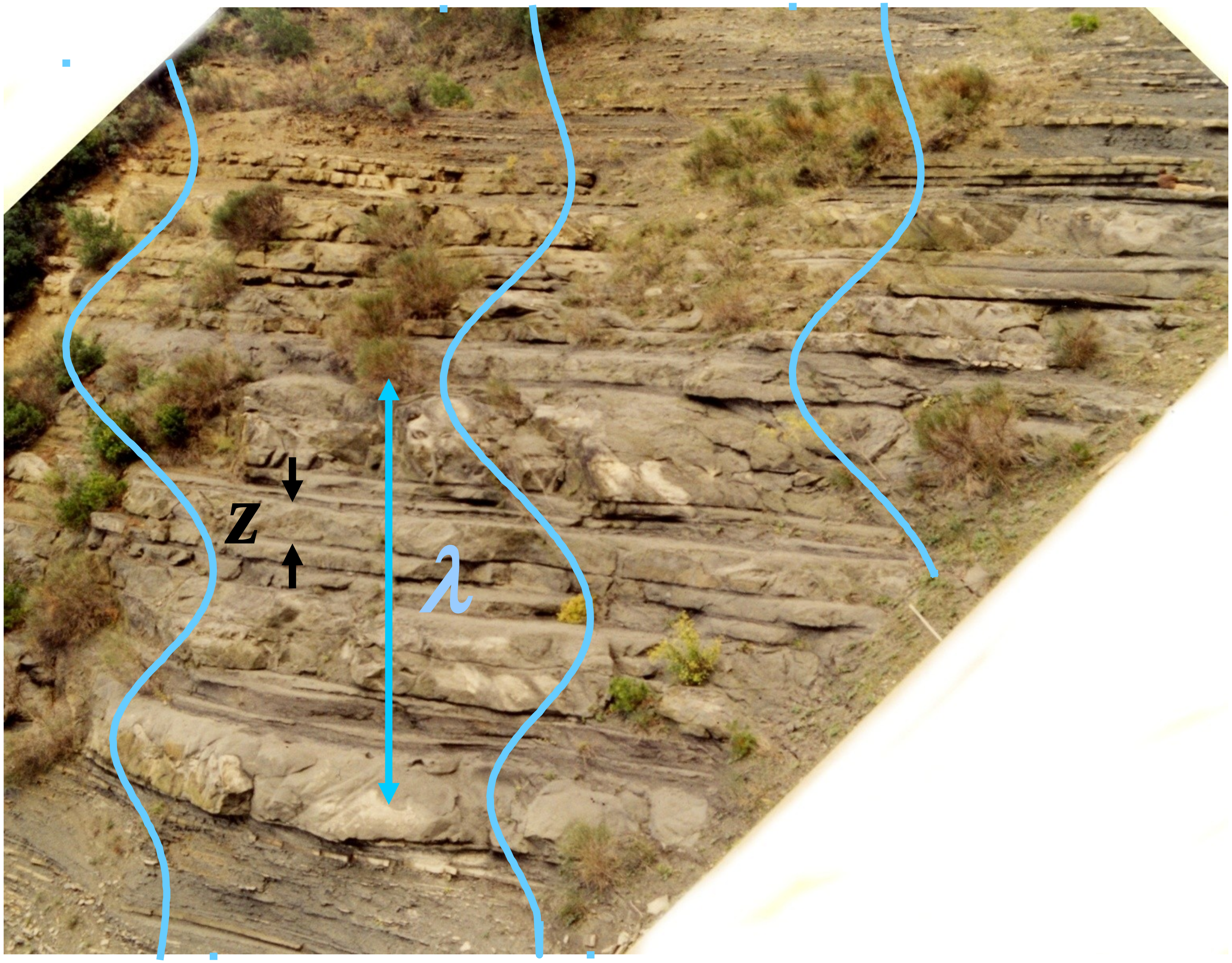
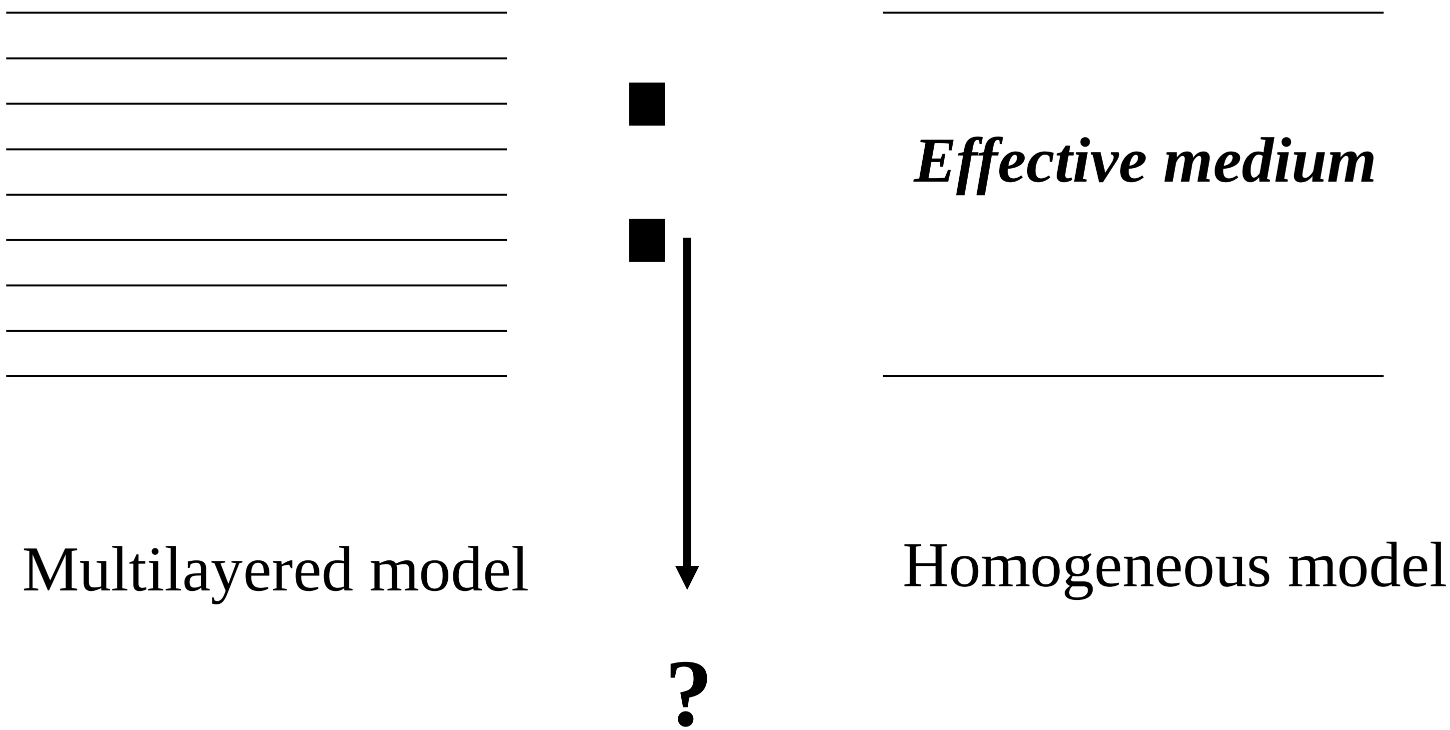
ROSE meeting, May 02-03, Trondheim, Norway

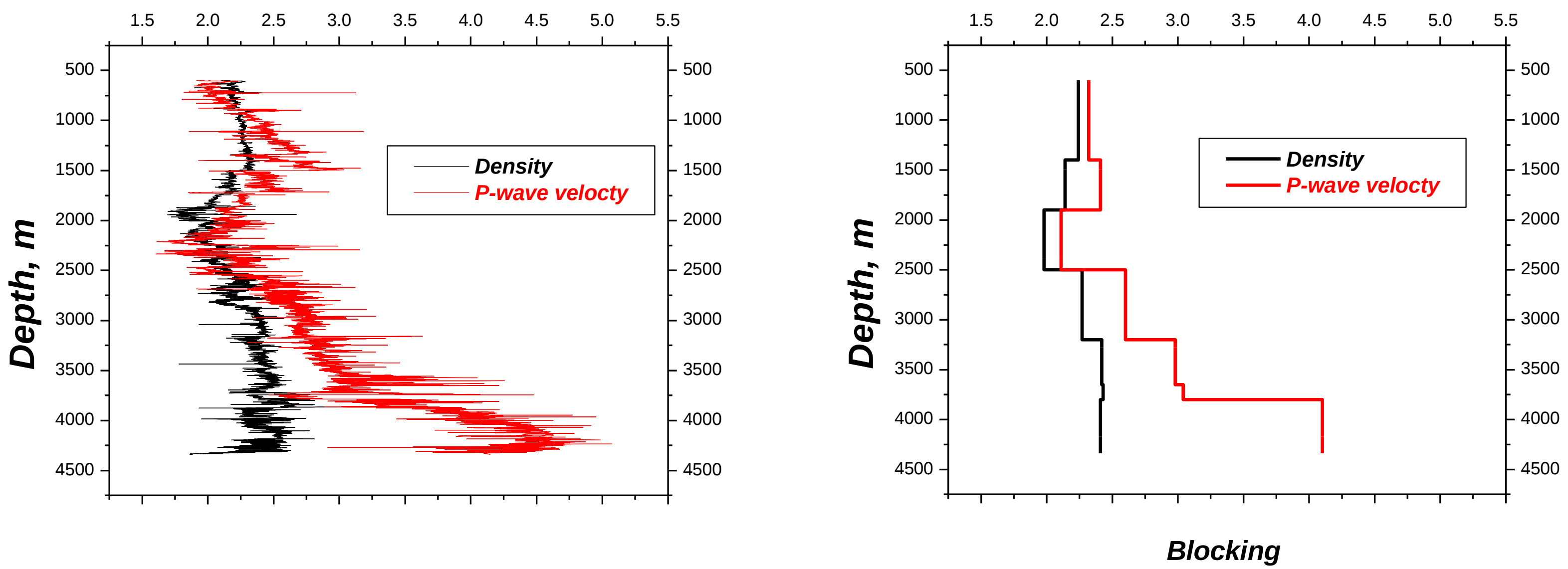
Upscaling



Upscaling problem



Well-log data



Key issue – number of model parameters

Backus averaging

Rytov, 1956; Backus, 1962; Schoenberg and Muir, 1989

$$\frac{d\mathbf{b}}{dz} = i\omega \mathbf{A}_j \mathbf{b}$$

$$\tilde{\mathbf{A}} = \left\langle \mathbf{A}_j \right\rangle = \frac{1}{H}\sum_{j=1}^{N} h_j \mathbf{A}_j$$

$$\mathbf{A}_j = \begin{pmatrix} 0 & \mathbf{M}_j \\ \mathbf{N}_j & 0 \end{pmatrix}$$

$$\tilde{c}_{33}^{-1} = \left\langle c_{33\,j}^{-1} \right\rangle$$

$$\tilde{c}_{13}\tilde{c}_{33}^{-1} = \left\langle c_{13\,j} c_{33\,j}^{-1} \right\rangle$$

$$\tilde{\rho} = \left\langle \rho_j \right\rangle$$

$$\tilde{c}_{11} - \tilde{c}_{13}^{2}\tilde{c}_{33}^{-1} = \left\langle c_{11\,j} - c_{13\,j}^{2} c_{33\,j}^{-1} \right\rangle$$

$$\tilde{c}_{44}^{-1} = \left\langle c_{44\,j}^{-1} \right\rangle$$

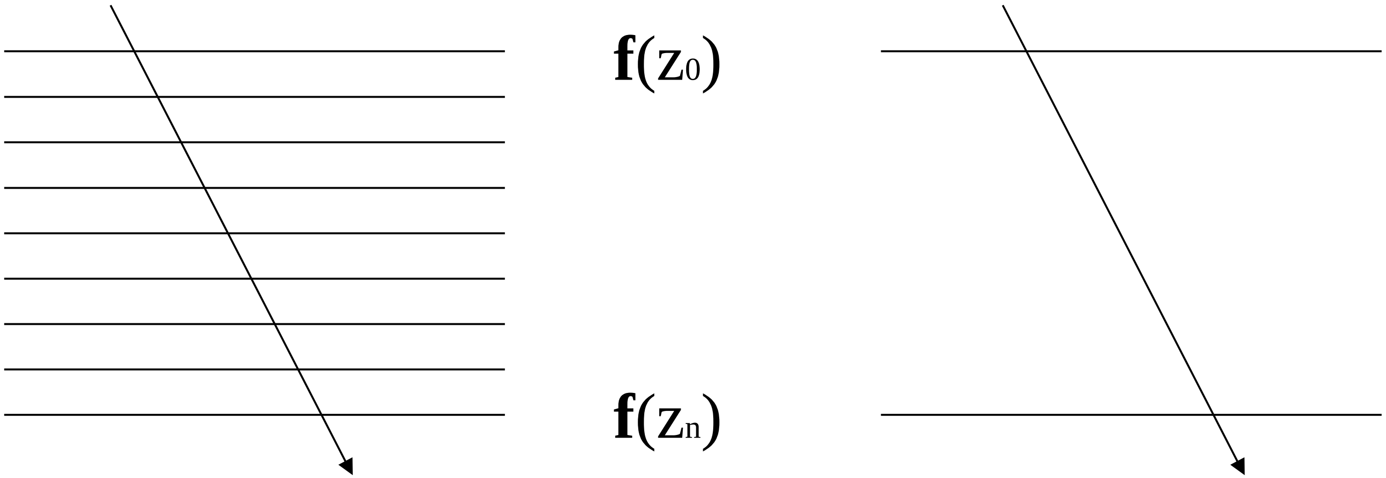
$$\mathbf{M}_j = \begin{pmatrix} c_{33\,j}^{-1} & p c_{13\,j} c_{33\,j}^{-1} \\ p c_{13\,j} c_{33\,j}^{-1} & \rho_j - p^2\left(c_{11\,j} - c_{13\,j}^{2} c_{33\,j}^{-1}\right) \end{pmatrix}$$

$$\mathbf{N}_j = \begin{pmatrix} \rho_j & p \\ p & c_{44\,j}^{-1} \end{pmatrix}$$

Backus from isotropic medium gives VTI medium

Propagator matrix

Thomson (1950), Haskell (1953), Gilbert and Backus (1966)



Multilayered model

Homogeneous model

$$\mathbf{f}\left(z_n\right) = \mathbf{P}\left(z_n, z_0\right)\mathbf{f}\left(z_0\right)$$

Baker–Campbell–Hausdorff formula

is the solution $\mathbf{Z} = \log\left[\exp(\mathbf{X})\exp(\mathbf{Y})\right]$

for noncommuting matrices **X** and **Y**

(Campbell, 1897; Poincare, 1899; Baker, 1902; Hausdorff,1906)

This formula links Lie groups to Lie algebras

$$\mathbf{Z} = \log\left(\exp(\mathbf{X})\exp(\mathbf{Y})\right) = \mathbf{X} + \mathbf{Y} + \frac{1}{2}[\mathbf{X},\mathbf{Y}] + \frac{1}{12}\left([\mathbf{X},[\mathbf{X},\mathbf{Y}]] - [\mathbf{Y},[\mathbf{X},\mathbf{Y}]]\right)$$

$$-\frac{1}{24}[\mathbf{Y},[\mathbf{X},[\mathbf{X},\mathbf{Y}]]]$$

$$-\frac{1}{720}\left([[[[\mathbf{X},\mathbf{Y}],\mathbf{Y}],\mathbf{Y}],\mathbf{Y}] + [[[[\mathbf{Y},\mathbf{X}],\mathbf{X}],\mathbf{X}],\mathbf{X}]\right)$$

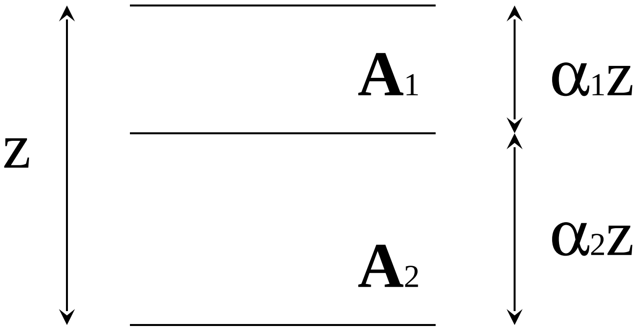
$$+\frac{1}{360}\left([[[[\mathbf{X},\mathbf{Y}],\mathbf{Y}],\mathbf{Y}],\mathbf{X}] + [[[[\mathbf{Y},\mathbf{X}],\mathbf{X}],\mathbf{X}],\mathbf{Y}]\right)$$

$$+\frac{1}{120}\left([[[[\mathbf{Y},\mathbf{X}],\mathbf{Y}],\mathbf{X}],\mathbf{Y}] + [[[[\mathbf{X},\mathbf{Y}],\mathbf{X}],\mathbf{Y}],\mathbf{X}]\right) + \ldots$$

Two layer example

1st order ODE:

$$\frac{d\mathbf{b}}{dz} = i\omega \mathbf{A}\mathbf{b}$$



Effective medium *Layered medium*

$$\mathbf{P} = \exp\left(i\omega \mathbf{A}\left(\omega\right) z\right) = \exp\left(i\omega \mathbf{A}_2 \alpha_2 z\right) \exp\left(i\omega \mathbf{A}_1 \alpha_1 z\right)$$

Matrix Taylor series

$$\mathbf{A}(\omega) = \mathbf{F_0} + (i\omega z)\,\mathbf{F_1} + (i\omega z)^2\,\mathbf{F_2} + (i\omega z)^3\,\mathbf{F_3} + \ldots$$

Effective medium

for the low-frequency band: $\omega z\,\mathrm{Re}(q(\omega)) \in [-\pi, \pi]$

Comment 1: We expanding not the propagator matrix but the logarithm of propagator matrix.

Comment 2: The same equations can be derived by using Magnus series and converting multiple integrals into multiple sums as it shown in Norris (1991).

Comment 3: The truncated Taylor series computed for any non-zero frequency does not correspond to any homogeneous elastic medium.

Matrix coefficients (two layers)

$$\mathbf{F_0} = \alpha_1 \mathbf{A_1} + \alpha_2 \mathbf{A_2} \qquad (\textit{Backus})$$

$$\mathbf{F_1} = \frac{1}{2}\alpha_1\alpha_2\left[\mathbf{A_2}, \mathbf{A_1}\right]$$

$$\mathbf{F_2} = \frac{1}{12}\alpha_1\alpha_2\left\{\alpha_2\left[\mathbf{A_2},\left[\mathbf{A_2}, \mathbf{A_1}\right]\right] + \alpha_1\left[\mathbf{A_1},\left[\mathbf{A_1}, \mathbf{A_2}\right]\right]\right\}$$

$$\mathbf{F_3} = -\frac{1}{24}\alpha_1^2\alpha_2^2\left[\mathbf{A_2}\left[\mathbf{A_1},\left[\mathbf{A_2}, \mathbf{A_1}\right]\right]\right]$$

...

$$\left[\mathbf{x}, \mathbf{y}\right] = \mathbf{xy} - \mathbf{yx} \qquad (\textit{the Lie bracket})$$

The case with vertical symmetry

$$\mathbf{A}_j = \begin{pmatrix} 0 & \mathbf{M}_j \\ \mathbf{N}_j & 0 \end{pmatrix}$$

$$[\mathbf{A}_2, \mathbf{A}_1] = \begin{pmatrix} \mathbf{M}_2\mathbf{N}_1 - \mathbf{M}_1\mathbf{N}_2 & 0 \\ 0 & \mathbf{N}_2\mathbf{M}_1 - \mathbf{N}_1\mathbf{M}_2 \end{pmatrix}$$

$$[\mathbf{A}_2, [\mathbf{A}_2, \mathbf{A}_1]] = 2\begin{pmatrix} 0 & \mathbf{M}_2\mathbf{M}_1\mathbf{N}_2 - \mathbf{M}_2\mathbf{M}_2\mathbf{N}_1 \\ \mathbf{N}_2\mathbf{M}_2\mathbf{N}_1 - \mathbf{N}_2\mathbf{M}_1\mathbf{N}_2 & 0 \end{pmatrix}$$

$$[\mathbf{A}_1, [\mathbf{A}_1, \mathbf{A}_2]] = 2\begin{pmatrix} 0 & \mathbf{M}_1\mathbf{M}_2\mathbf{N}_1 - \mathbf{M}_1\mathbf{M}_1\mathbf{N}_2 \\ \mathbf{N}_1\mathbf{M}_1\mathbf{N}_2 - \mathbf{N}_1\mathbf{M}_2\mathbf{N}_1 & 0 \end{pmatrix}$$

$$[\mathbf{A}_1, [\mathbf{A}_2, \mathbf{A}_1]] = -[\mathbf{A}_1, [\mathbf{A}_1, \mathbf{A}_2]]$$

The case with vertical symmetry

$$\tilde{\mathbf{A}}(\omega) = \begin{pmatrix} \tilde{\mathbf{P}}(\omega) & \tilde{\mathbf{M}}(\omega) \\ \tilde{\mathbf{N}}(\omega) & \tilde{\mathbf{Q}}(\omega) \end{pmatrix}$$

$$\tilde{\mathbf{M}}(\omega) = \alpha_1\mathbf{M_1} + \alpha_2\mathbf{M_2} + (i\omega z)^2\frac{1}{6}\alpha_1\alpha_2\left\{\alpha_2\left(\mathbf{M}_2\mathbf{M}_1\mathbf{N}_2 - \mathbf{M}_2\mathbf{M}_2\mathbf{N}_1\right) + \alpha_1\left(\mathbf{M}_1\mathbf{M}_2\mathbf{N}_1 - \mathbf{M}_1\mathbf{M}_1\mathbf{N}_2\right)\right\} + \ldots$$

$$\tilde{\mathbf{N}}(\omega) = \alpha_1\mathbf{N_1} + \alpha_2\mathbf{N_2} + (i\omega z)^2\frac{1}{6}\alpha_1\alpha_2\left\{\alpha_2\left(\mathbf{N}_2\mathbf{M}_2\mathbf{N}_1 - \mathbf{N}_2\mathbf{M}_1\mathbf{N}_2\right) + \alpha_1\left(\mathbf{N}_1\mathbf{M}_1\mathbf{N}_2 - \mathbf{N}_1\mathbf{M}_2\mathbf{N}_1\right)\right\} + \ldots$$

$$\tilde{\mathbf{P}}(\omega) = \frac{1}{2}\alpha_1\alpha_2(i\omega z)\left(\mathbf{M}_2\mathbf{N}_1 - \mathbf{M}_1\mathbf{N}_2\right) - (i\omega z)^3\frac{1}{12}\alpha_1^2\alpha_2^2\left(\mathbf{N}_2\mathbf{M}_1\mathbf{M}_1\mathbf{N}_2 - \mathbf{N}_2\mathbf{M}_1\mathbf{M}_2\mathbf{N}_1 + \mathbf{M}_2\mathbf{N}_1\mathbf{M}_2\mathbf{N}_1 - \mathbf{M}_2\mathbf{N}_1\mathbf{M}_1\mathbf{N}_2\right) + \ldots$$

$$\tilde{\mathbf{Q}}(\omega) = -\frac{1}{2}\alpha_1\alpha_2(i\omega z)\left(\mathbf{M}_2\mathbf{N}_1 - \mathbf{M}_1\mathbf{N}_2\right) - (i\omega z)^3\frac{1}{12}\alpha_1^2\alpha_2^2\left(\mathbf{N}_2\mathbf{M}_1\mathbf{M}_1\mathbf{N}_2 - \mathbf{N}_2\mathbf{M}_1\mathbf{M}_2\mathbf{N}_1 + \mathbf{M}_2\mathbf{N}_1\mathbf{M}_2\mathbf{N}_1 - \mathbf{M}_2\mathbf{N}_1\mathbf{M}_1\mathbf{N}_2\right) + \ldots$$

Note, that complex matrices P(w) and Q(w) do not affect the wave propagation since the their traces are zero. They result in complex eigen vectors (the angle between stress and strain is frequency dependent). Matrix series for M(w) and N(w) contain the even order terms in frequency, while matrix series for P(w) and Q(w) – odd order terms.

Periodically layered medium

$$\mathbf{A}_1(\omega) = \mathbf{F_0} + (i\omega z)\mathbf{F_1} + (i\omega z)^2\mathbf{F_2} + (i\omega z)^3\mathbf{F_3} + \ldots$$

$$\mathbf{A}_N(\omega) = \mathbf{F_0} + \left(\frac{i\omega z}{N}\right)\mathbf{F_1} + \left(\frac{i\omega z}{N}\right)^2\mathbf{F_2} + \left(\frac{i\omega z}{N}\right)^3\mathbf{F_3} + \ldots$$

$$= \mathbf{F_0} + \frac{1}{N}(i\omega z)\mathbf{F_1} + \frac{1}{N^2}(i\omega z)^2\mathbf{F_2} + \frac{1}{N^3}(i\omega z)^3\mathbf{F_3} + \ldots$$

N – number of periods

$$\mathbf{A}_{N\to\infty}(\omega) = \mathbf{A}_N(\omega \to 0) = \mathbf{F_0}$$

Single mode P-wave vertical propagation

$$\mathbf{A}_j = \begin{pmatrix} 0 & c_{33(j)}^{-1} \\ \rho_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\rho_j V_j^2} \\ \rho_j & 0 \end{pmatrix}$$

Single mode vertical propagation

$$\mathbf{F}_0 = \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 = \begin{pmatrix} 0 & \dfrac{\alpha_1}{\rho_1 V_1^2} + \dfrac{\alpha_2}{\rho_2 V_2^2} \\ \alpha_1 \rho_1 + \alpha_2 \rho_2 & 0 \end{pmatrix}$$

$$\mathbf{F}_1 = \frac{1}{2} \alpha_1 \alpha_2 \rho_1 \rho_2 \left(\frac{1}{\rho_1^2 V_1^2} - \frac{1}{\rho_2^2 V_2^2} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Comment: *For medium with vertical symmetry axis, the matrix* **A** *is anti-diagonal, it follows that all matrices* $\mathbf{F}_{2k}$ *are also anti-diagonal, while all matrices* $\mathbf{F}_{2k+1}$ *are diagonal.*

Single mode vertical propagation

$$\mathbf{F}_2 = \frac{1}{6}\alpha_1\alpha_2\rho_1\rho_2\left(\frac{1}{\rho_1^2 V_1^2} - \frac{1}{\rho_2^2 V_2^2}\right)\begin{pmatrix} 0 & \frac{\alpha_1}{\rho_1 V_1^2} - \frac{\alpha_2}{\rho_2 V_2^2} \\ \alpha_1\rho_1 - \alpha_2\rho_2 & 0 \end{pmatrix}$$

$$\mathbf{F}_3 = -\frac{1}{12}\alpha_1^2\alpha_2^2\rho_1^2\rho_2^2\left(\frac{1}{\rho_1^4 V_1^4} - \frac{1}{\rho_2^4 V_2^4}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Velocity dispersion

Time-average model

Zero-order scattering *Second-order scattering* *Fourth-order scattering*

$$\frac{1}{V^2(\omega)} = \left(\frac{\alpha_1}{V_1}+\frac{\alpha_2}{V_2}\right)^2 + \frac{4r^2}{1-r^2}\frac{\alpha_2\alpha_1}{V_2V_1} + \omega^2 H^2 \frac{4\alpha_1^2\alpha_2^2 r^2}{3V_1^2V_2^2\left(1-r^2\right)^2} + \omega^4 H^4 \frac{4\alpha_1^2\alpha_2^2 r^2}{45V_1^2V_2^2\left(1-r^2\right)^2}\left[\left(\frac{\alpha_1}{V_1}+\frac{\alpha_2}{V_2}\right)^2 + \frac{2\alpha_1\alpha_2\left(1+3r^2\right)}{V_1V_2\left(1-r^2\right)}\right] + O\left(\omega^5\right)$$

Backus model

Dispersive Backus model

Time-average model: ω*->infinity or r=0*

Backus model: ω*=0*

Dispersive Backus model: ω *is small*

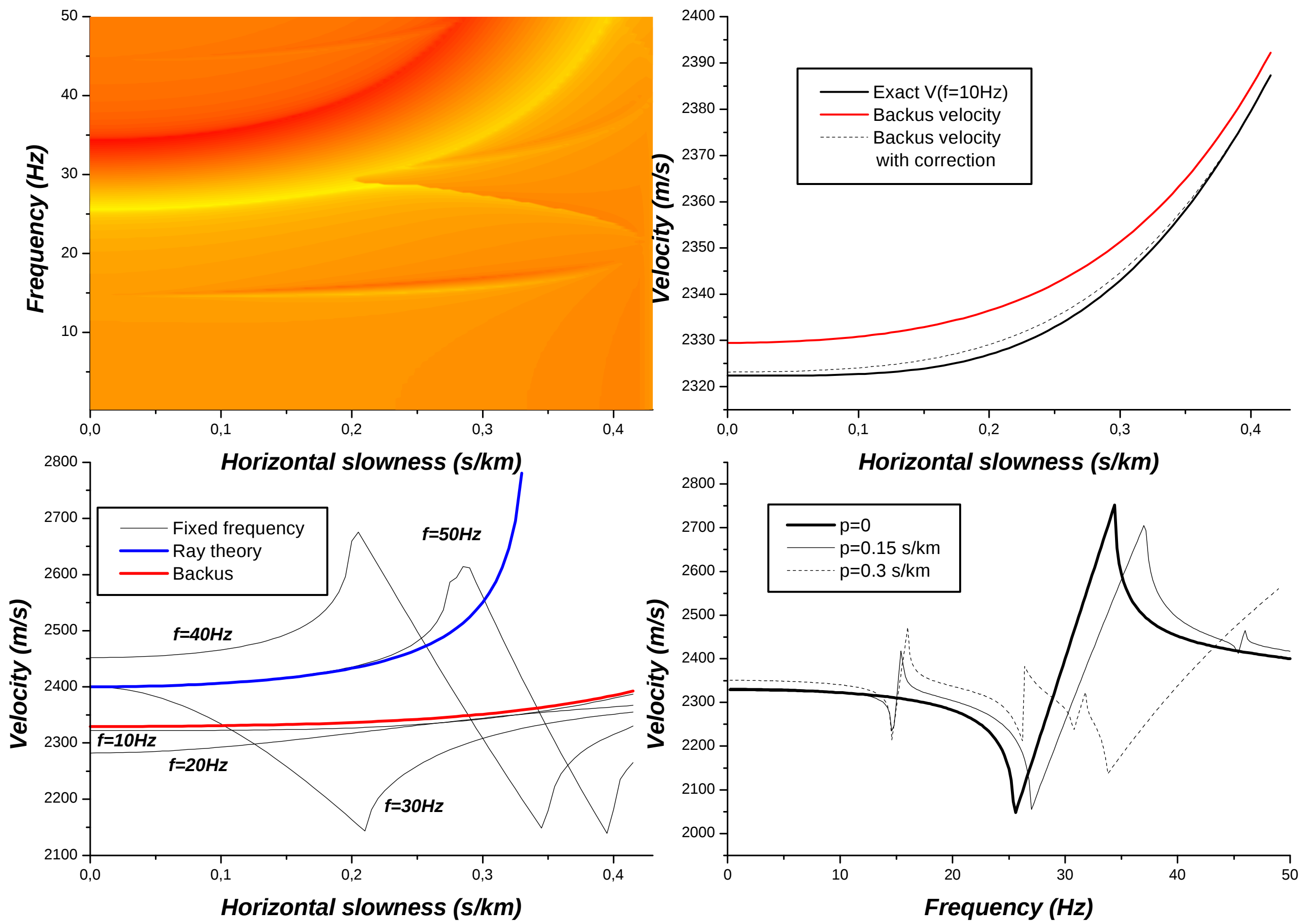
Velocity dispersion
(traveltime vs impedance)

$$T^2(\omega) = (t_1 + t_2)^2 + t_1 t_2 \left(\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2} - 2 \right) + \omega^2 \frac{t_1^2 t_2^2}{12} \left(\frac{Z_2}{Z_1} - \frac{Z_1}{Z_2} \right)^2$$

$$+ \omega^4 \frac{t_1^2 t_2^2}{180} \left(\frac{Z_2}{Z_1} - \frac{Z_1}{Z_2} \right)^2 \left[(t_1 + t_2)^2 + 2 t_1 t_2 \left(\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2} - 1 \right) \right] + O\left(\omega^5\right)$$

t_j are the traveltimes within each layer and Z_j are the impedances

Two layers isotropic medium



Extension for 3 layers*

$$\mathbf{P} = \exp\left(i\omega\mathbf{A}(\omega)\,z\right) = \prod_{j=N}^{1} \exp\left(i\omega\mathbf{A}_j \alpha_j z\right)$$

Note the reverse way of composing matrix exponents

$$\mathbf{F_0} = \sum_{j=1}^{N} \alpha_j \mathbf{A}_j \quad (Backus)$$

$$\mathbf{F_1} = \frac{1}{2}\sum_{j>k} \alpha_j \alpha_k \left[\mathbf{A}_j, \mathbf{A}_k\right]$$

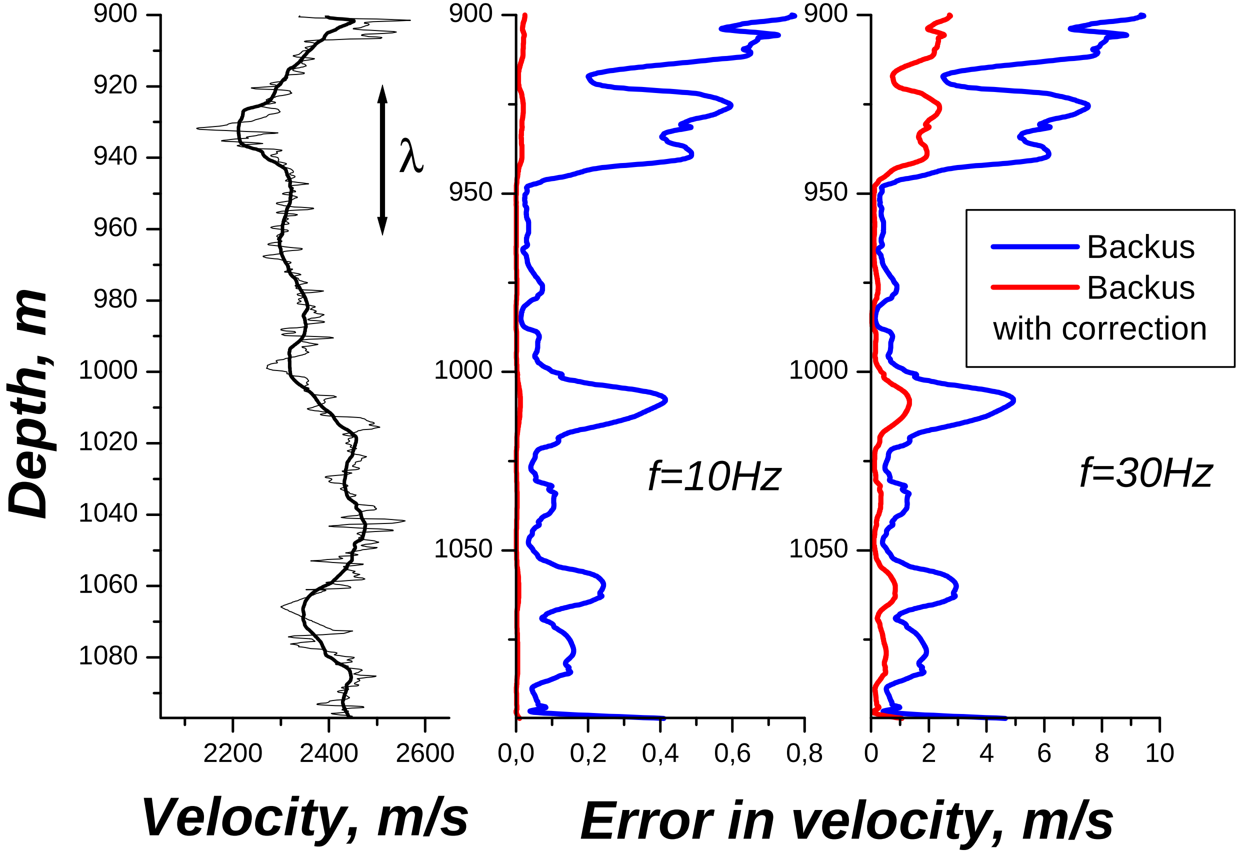
Contribution from all possible pairs of layers

$$\mathbf{F_2} = \frac{1}{12}\sum_{j>k>l}\left(\alpha_j^2\alpha_k\left[\mathbf{A}_j,\left[\mathbf{A}_j,\mathbf{A}_k\right]\right] + \alpha_j\alpha_k^2\left[\mathbf{A}_k,\left[\mathbf{A}_k,\mathbf{A}_j\right]\right] + 2\alpha_j\alpha_k\alpha_l\left(\left[\mathbf{A}_l,\left[\mathbf{A}_k,\mathbf{A}_j\right]\right] + \left[\mathbf{A}_j,\left[\mathbf{A}_k,\mathbf{A}_l\right]\right]\right)\right)$$

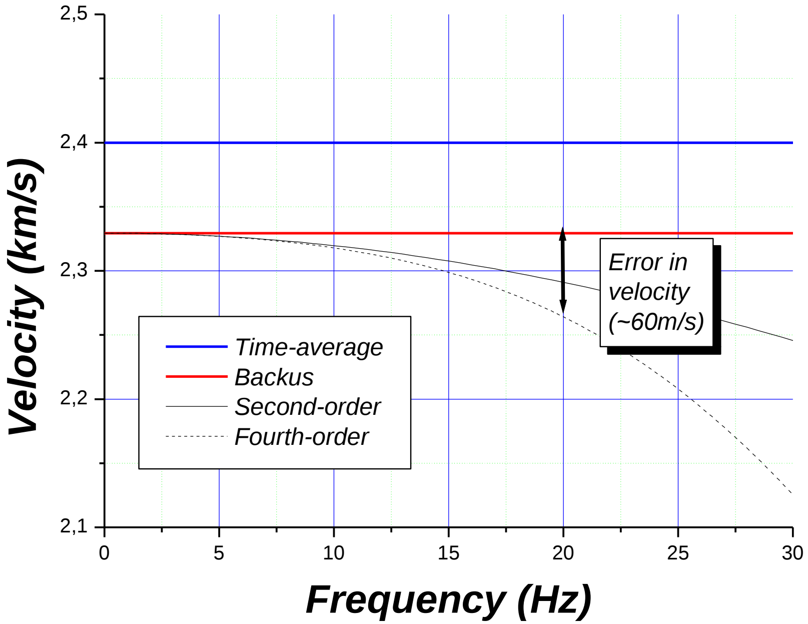
...

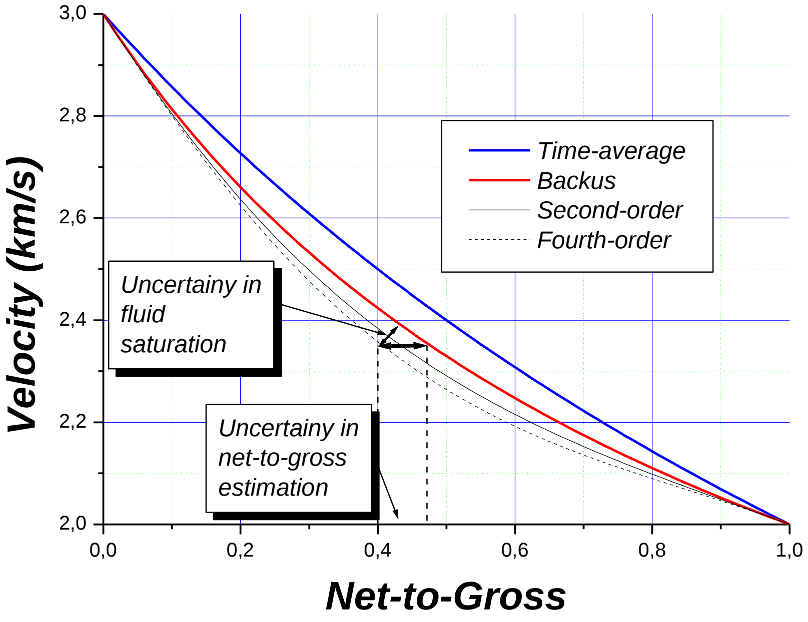
Contribution from all possible three layers

Smoothing of log data



Phase velocity for large contrast





Upscaling of reservoir properties

Conclusion&Discussion

- Velocity dispersion is very important issue for contrast velocity models. In this case the Backus averaging could not be accurate enough.
- The method is an extension of the standard Backus averaging for low-frequency case (correction term)
- Method is based on expansion of the logarithm of propagator matrix in frequency
- The frequency-dependent term can be computed as a sum of contributions from all the layers composed by combinations of three layers

Acknowledgement

- I would like to acknowledge NFR via the ROSE project