Mindlin’s friction term and implication for shear modulus and anisotropy in granular media

By

Kenneth Duffaut, Martin Landrø, Roger Sollie and Ørjan Pedersen

ROSE meeting, Trondheim

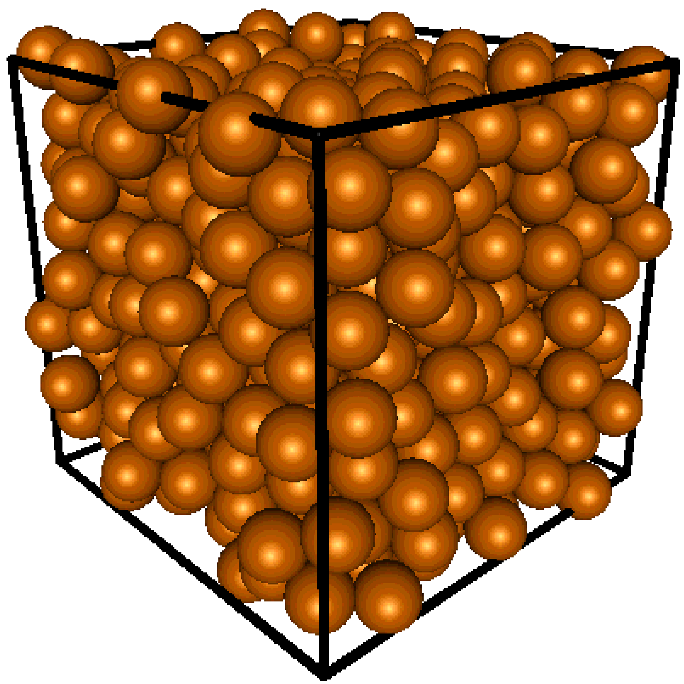
2. May, 2011

Statoil

NTNU – Trondheim
Norwegian University of Science and Technology

Objective

- Model effective elastic properties (e.g. shear rigidity) of a random packing of identical spheres with intergrain contact friction undergoing either hydrostatic or uniaxial loading

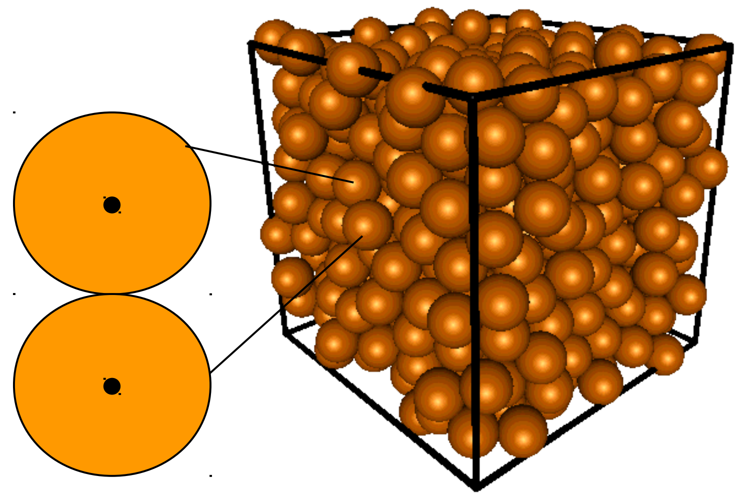


Sphere assembly

Assumptions – Grain contact model

- Constant porosity or pore volume (~36%)
  - Random dense packing configuration
- Isotropic homogenous spherical grains
  - No variation with either position or direction
- Small strain (grain deformation << grain radius)
- Hydrostatic loading (stress equal in all directions)
- Mean strain-field

Sphere assembly



A pair of grains in contact

Effective moduli – existing models

Effective bulk modulus

$$K_{dry} = \frac{C_p(1-\varphi)}{12\pi R} S_N$$

Normal contact stiffness

$$S_N = \frac{\partial F_N}{\partial \delta_N} = \frac{4Ga}{(1-\nu)}$$ (Hertz, 1882)

S_T = 0: Effective shear modulus - ~small contact friction

$$G_{dry} = \frac{3}{5} K_{dry}$$

Tangential contact stiffness

$$S_T = \left(\frac{\partial F_T}{\partial \delta_T}\right) = 0, \quad \mu \approx 0$$

S_T > 0: Effective shear modulus - ~infinite contact friction

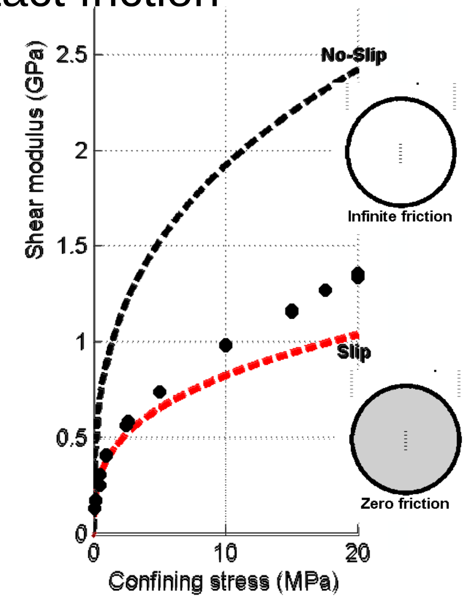
$$G_{dry} = \frac{3}{5}\left(1 + \frac{3}{2}\frac{S_T}{S_N}\right) K_{dry} = \frac{3}{5}\left(\frac{5-4\nu}{2-\nu}\right) K_{dry}$$

Tangential contact stiffness

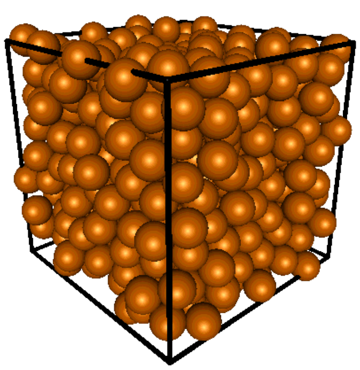
$$S_T = \left(\frac{\partial F_T}{\partial \delta_T}\right) = \frac{8Ga}{(2-\nu)}, \quad \mu = \infty$$

(Mindlin, 1949)

F_N = Normal force
δ_N = Normal displacement
a = Grain contact radius
G= Grain shear modulus
ν= Grain Poisson's ratio
R = grain radius
C_p = coordination number
φ = Porosity
F_T = Tangential force
δ_T = Tangential displacement
μ = Contact friction



Dry sphere assembly

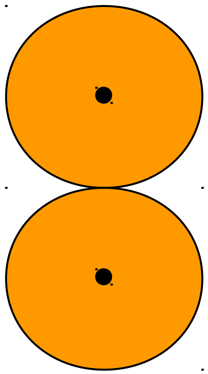


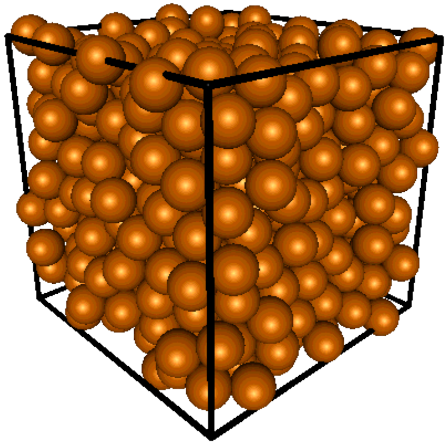
(Digby, 1981, Walton, 1987)

Friction (μ)

Definition

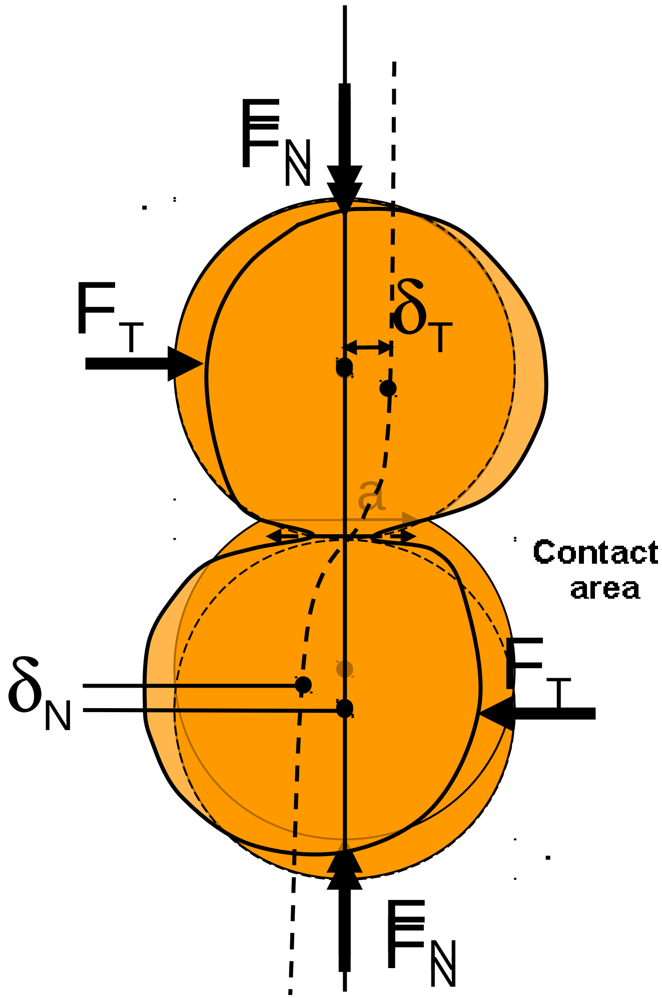
- Friction is the force resisting the relative motion of two solid surfaces in contact
  - **Internal friction** is the force resisting motion between elements making up a solid while it undergoes deformation





Contact stiffnesses - New model

A pair of identical grains with arbitrary contact friction



Schematic diagram

Normal contact stiffness

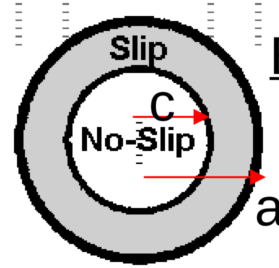
$$S_N = \frac{\partial F_N}{\partial \delta_N} = \frac{4Ga}{(1-\nu)}$$

Tangential contact stiffness

$$S_T = \left(\frac{\partial F_T}{\partial \delta_T}\right) = \frac{8Ga}{(2-\nu)}\left(1-\frac{F_T}{\mu F_N}\right)^{\frac{1}{3}} = \frac{8Ga}{(2-\nu)} f(\mu)$$

(Mindlin, 1949)

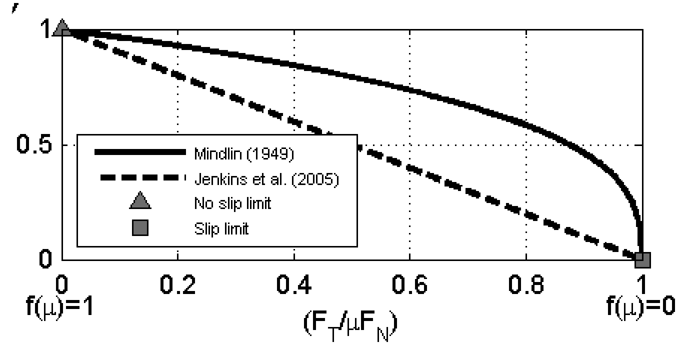
Partial slip



Bird view

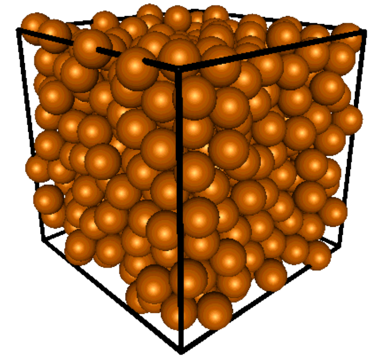
Non-zero friction

$$f(\mu) = \left(1-\frac{F_T}{\mu F_N}\right)^{\frac{1}{3}}$$



Effective moduli – New model

Sphere assembly



S_T = 0: Effective bulk and shear modulus - ~zero contact friction

$$K_{dry} = \frac{C_p(1-\varphi)}{12\pi R} S_N$$

$$G_{dry} = \frac{3}{5} K_{dry}$$

S_T > 0: Effective shear modulus - infinite contact friction

$$G_{dry} = \frac{3}{5}\left(1 + \frac{3}{2}\frac{S_T}{S_N}\right) K_{dry} = \frac{3}{5}\left(\frac{5-4\nu}{2-\nu}\right) K_{dry}$$

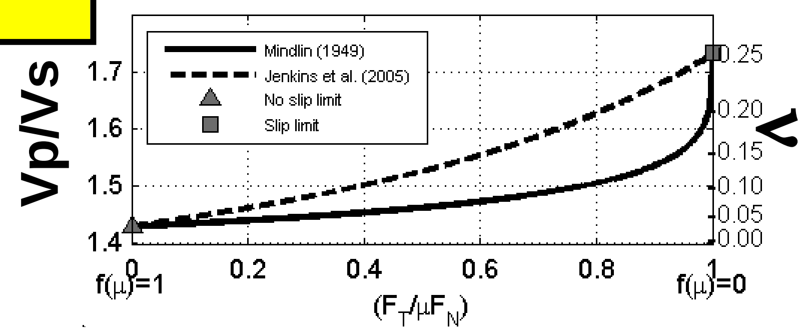
(Digby, 1981, Walton, 1987)

S_N = Normal contact stiffness
S_T = Normal contact stiffness
$f(\mu)$ = Mindlin's friction term

f(μ) [0,1]: Effective shear modulus – arbitrary contact friction

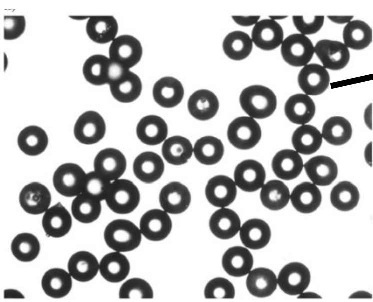
$$G_{dry} = \frac{3}{5}\left[1 + \frac{3}{2}\frac{S_T f(\mu)}{S_N}\right] K_{dry} = \frac{3}{5}\left[1 + \frac{3(1-\nu)}{2-\nu} f(\mu)\right] K_{dry}$$

$$f(\mu) = \left(1 - \frac{F_T}{\mu F_N}\right)^{\frac{1}{3}}$$

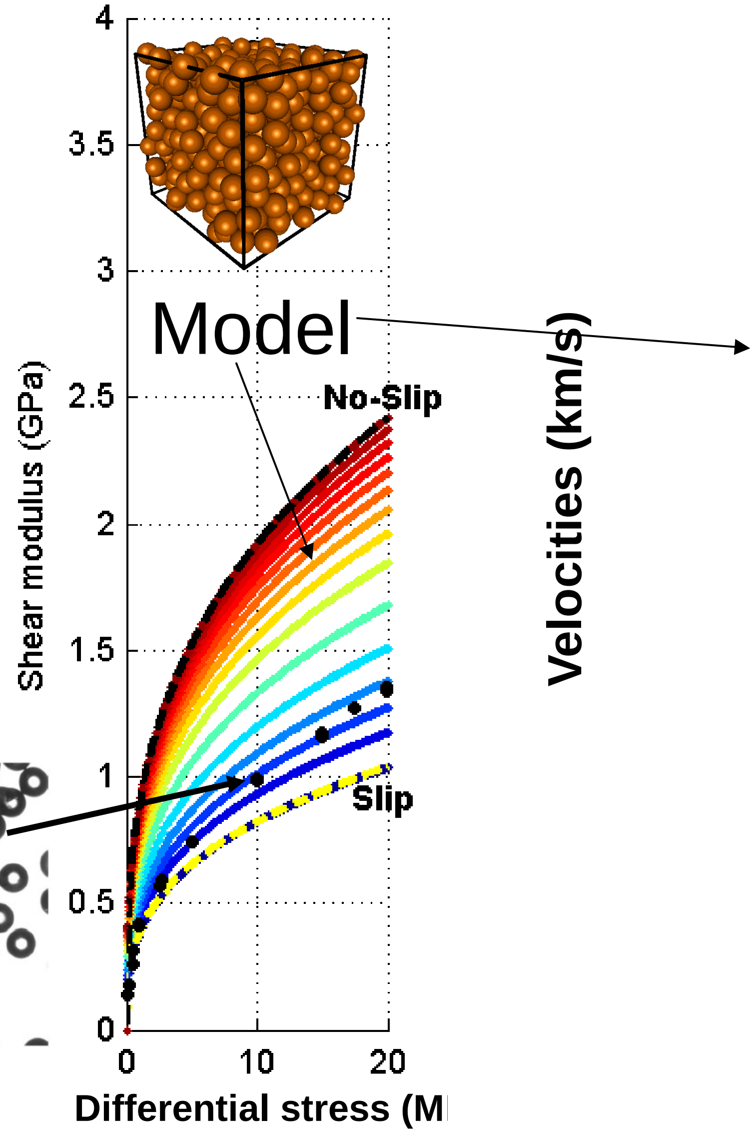


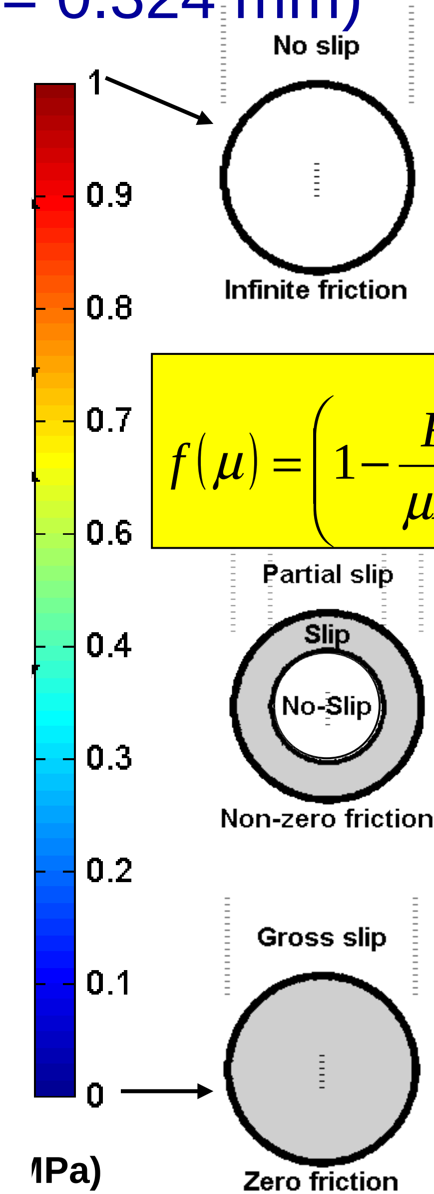
Model vs. dry core measurements

Loose glass beads (Spherical grains of size = 0.324 mm)



(Zimmer, 2007)



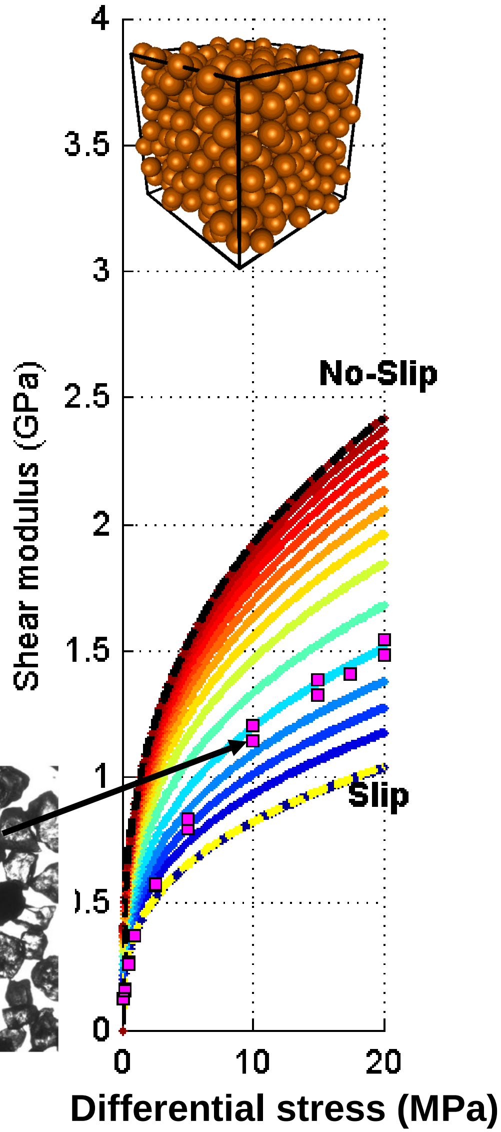


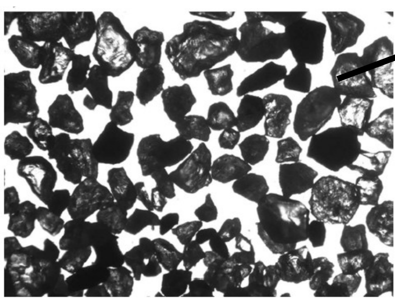
$$f(\mu) = \left(1 - \frac{F_T}{\mu F_N}\right)^{\frac{1}{3}}$$

Porosity range: 37.6% 36.2%

Model vs. dry core measurements

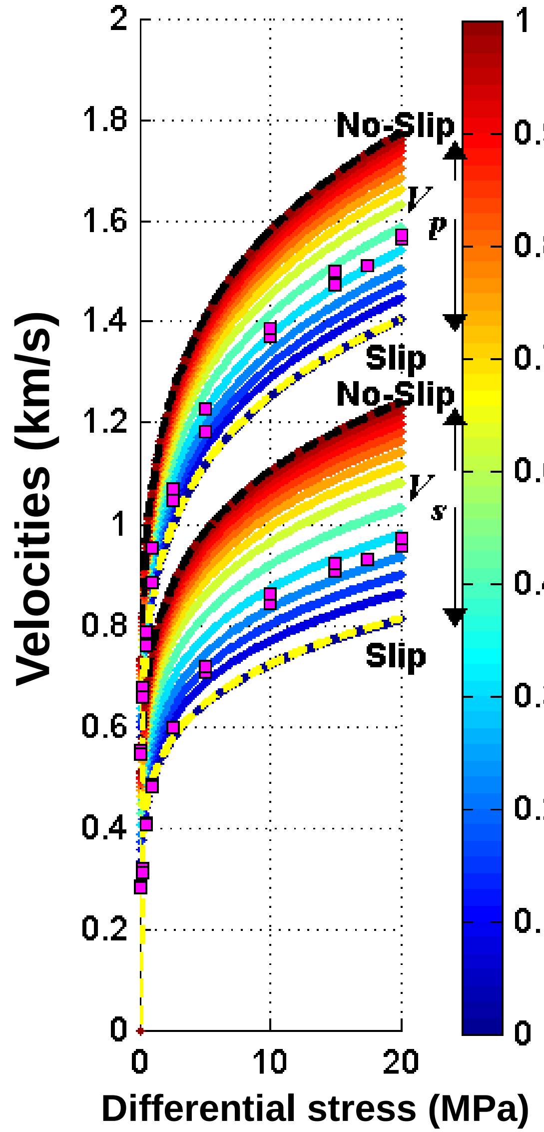
Loose sand (angular grains of size = 0.324 mm)

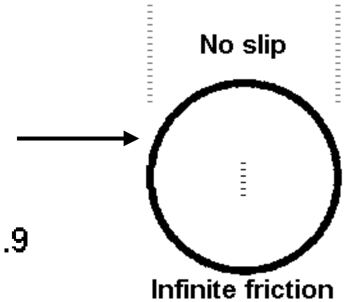




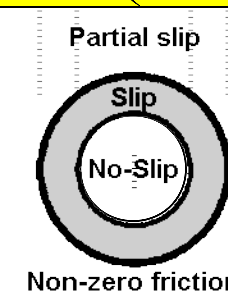
(Zimmer, 2007)

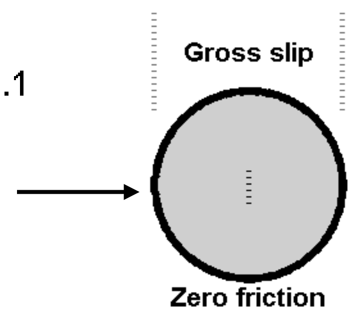
Porosity range: 39.5% 37.2%





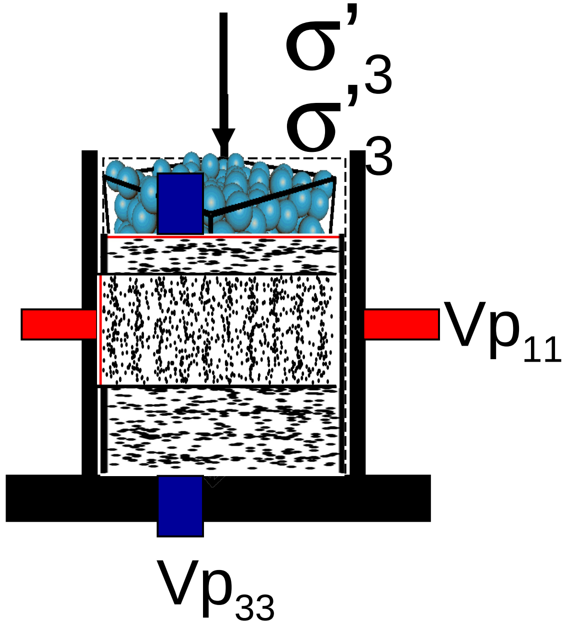
$$f(\mu) = \left(1 - \frac{F_T}{\mu F_N}\right)^{\frac{1}{3}}$$

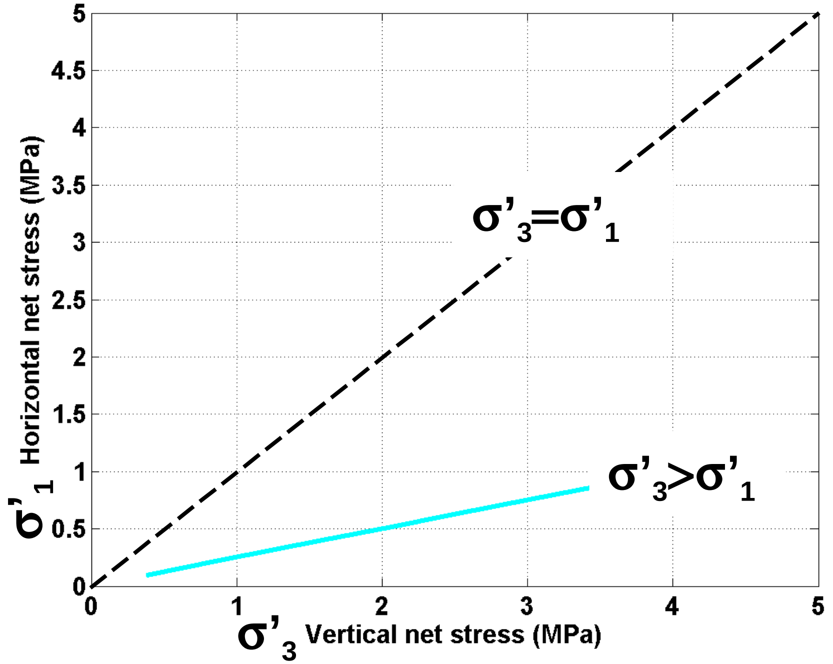




Uniaxial strain loading ($e_3 \neq 0$, $e_1=e_2=0$)

Definition

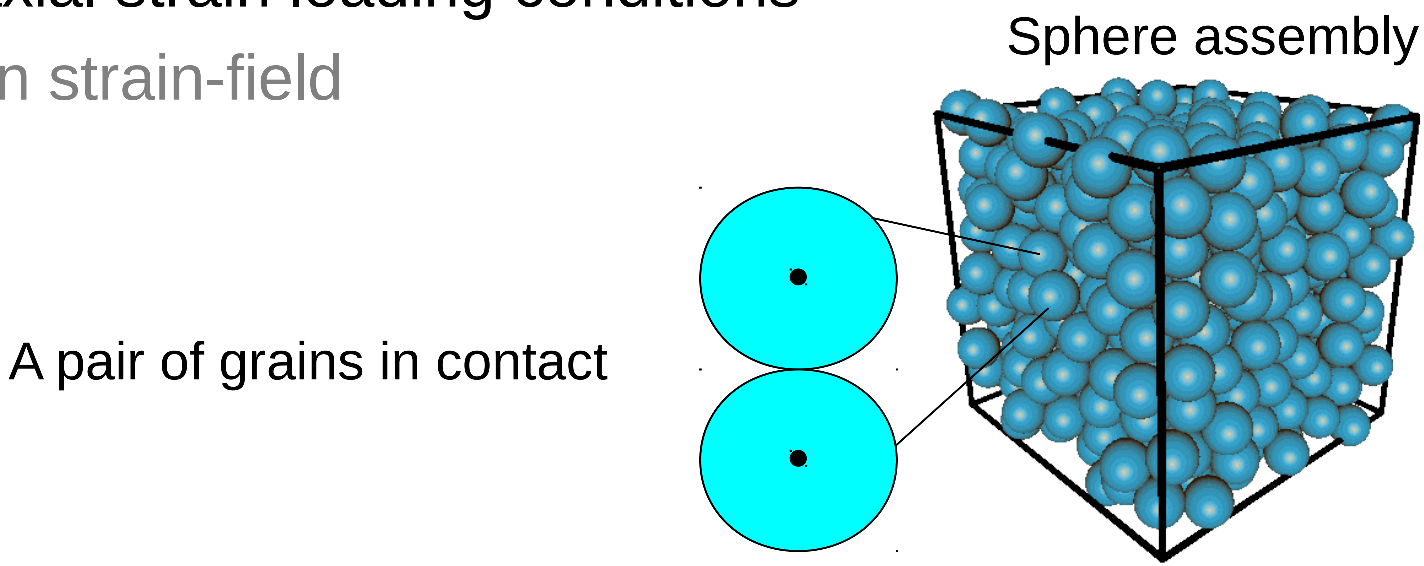




Directional dependent velocity variation induced by stress differences

Assumptions – Grain contact model

- Constant porosity or pore volume (~36%)
  - Random dense packing configuration
- Isotropic homogenous spherical grains
  - No variation with either position or direction
- Small strain (grain deformation << grain radius)
- Uniaxial strain loading conditions
- Mean strain-field



Walton model – infinite contact friction

Uniaxial strain loading – The sphere assembly is elastically anisotropic

Effective elastic stiffness constants:

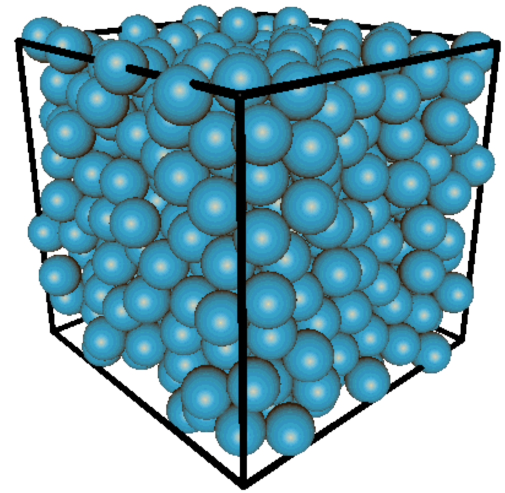
$$C_{11}^{dry} = 3\alpha + 6\beta$$

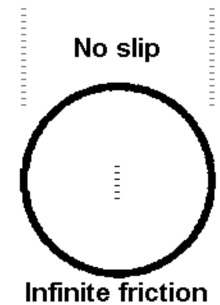
$$C_{13}^{dry} = 2\alpha - 4\beta$$

$$C_{33}^{dry} = 8\alpha + 8\beta$$

$$C_{44}^{dry} = C_{55}^{dry} = 2\alpha + 5\beta$$

$$C_{66}^{dry} = \alpha + 4\beta$$





where

$$\alpha = \frac{1}{4}\left(\frac{3(1-\varphi)^2 C_p^2 G^2 (2-\nu)\sigma_3^{'}}{16\pi^2(1-\nu)^2\nu}\right)^{\frac{1}{3}}$$

$$\beta = \frac{1}{4}\left(\frac{3(1-\varphi)^2 C_p^2 G^2 (1-\nu)\sigma_3^{'}}{16\pi^2(2-\nu)^2\nu}\right)^{\frac{1}{3}}$$

□ = Pore volume or Porosity
□ = Grain Poisson's ratio
G = Grain shear modulus
σ_3' = Vertical differential stress
C_p = # grain contact points

(Walton, 1987)

Walton model – ~ zero contact friction

Uniaxial strain loading – The sphere assembly is elastically anisotropic

Effective elastic stiffness constants:

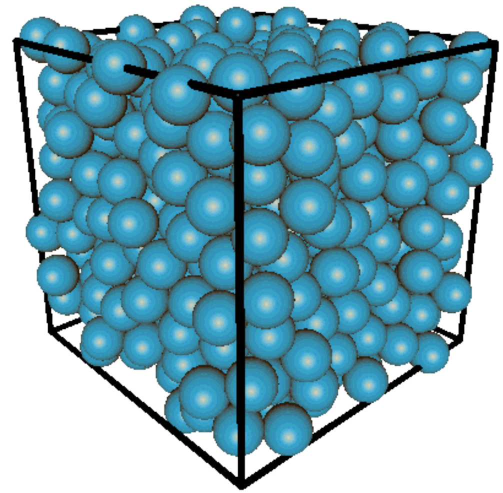
$$C_{11}^{dry} = 3\alpha$$

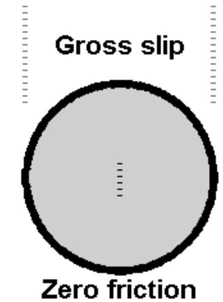
$$C_{13}^{dry} = 2\alpha$$

$$C_{33}^{dry} = 8\alpha$$

$$C_{44}^{dry} = C_{55}^{dry} = 2\alpha$$

$$C_{66}^{dry} = \alpha$$





where

$$\alpha = \frac{1}{4}\left(\frac{3(1-\varphi)^2 C_p^2 G^2 (2-\nu)\sigma_3'}{16\pi^2(1-\nu)^2\nu}\right)^{\frac{1}{3}}$$

□ = Pore volume or Porosity
□ = Grain Poisson's ratio
G = Grain shear modulus
σ_3' = Vertical differential stress
C_p = # grain contact points

(Walton, 1987)

New model – Arbitrary contact friction

Uniaxial strain loading - Elastic stiffness constants as a function of S_N,S_T and f(μ)

Effective elastic stiffness constants:

$$C_{11}^{dry} = \frac{3(1-\varphi)C_p}{64\pi R}\left[S_N + S_T f(\mu)\right]$$

$$C_{13}^{dry} = \frac{(1-\varphi)C_p}{32\pi R}\left[S_N - S_T f(\mu)\right]$$

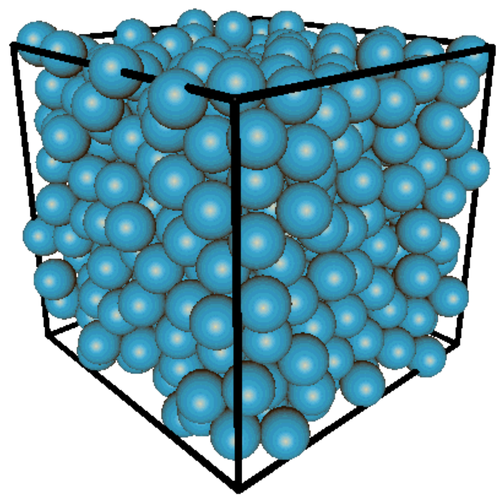
$$C_{33}^{dry} = \frac{(1-\varphi)C_p}{16\pi R}\left[2S_N + S_T f(\mu)\right]$$

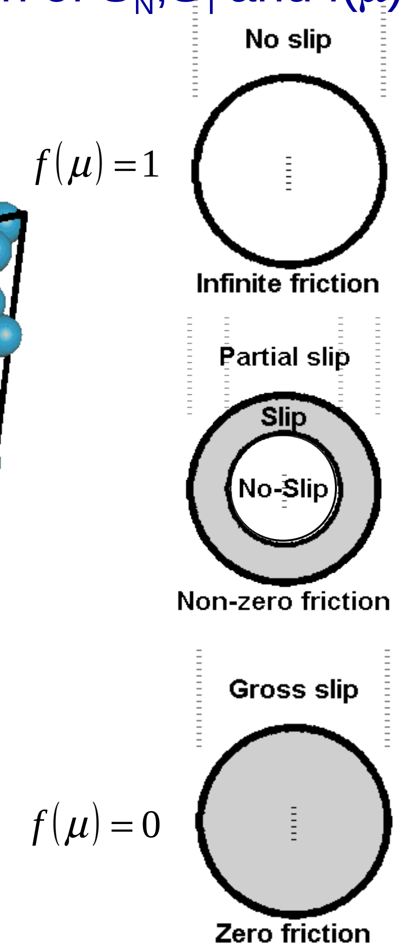
$$C_{44}^{dry} = \frac{(1-\varphi)C_p}{128\pi R}\left[4S_N + 5S_T f(\mu)\right]$$

$$C_{66}^{dry} = \frac{(1-\varphi)C_p}{64\pi R}\left[S_N + 2S_T f(\mu)\right]$$

where

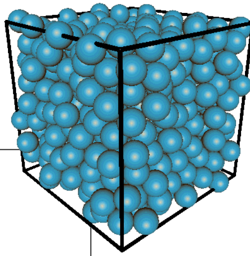
$$f(\mu) = \left(1 - \frac{F_T}{\mu F_N}\right)^{\frac{1}{3}}$$





▯ = Pore volume or Porosity
R = Grain radius
C_p = # grain contact points
S_N = Normal contact stiffness
S_T = Tangential contact stiffness
f(μ) = Mindlin's friction term

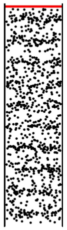
Model vs. dry loose sand measurements



P-wave velocity vs. vertical net stress

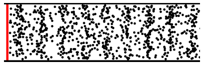
Vertical P-wave velocity:

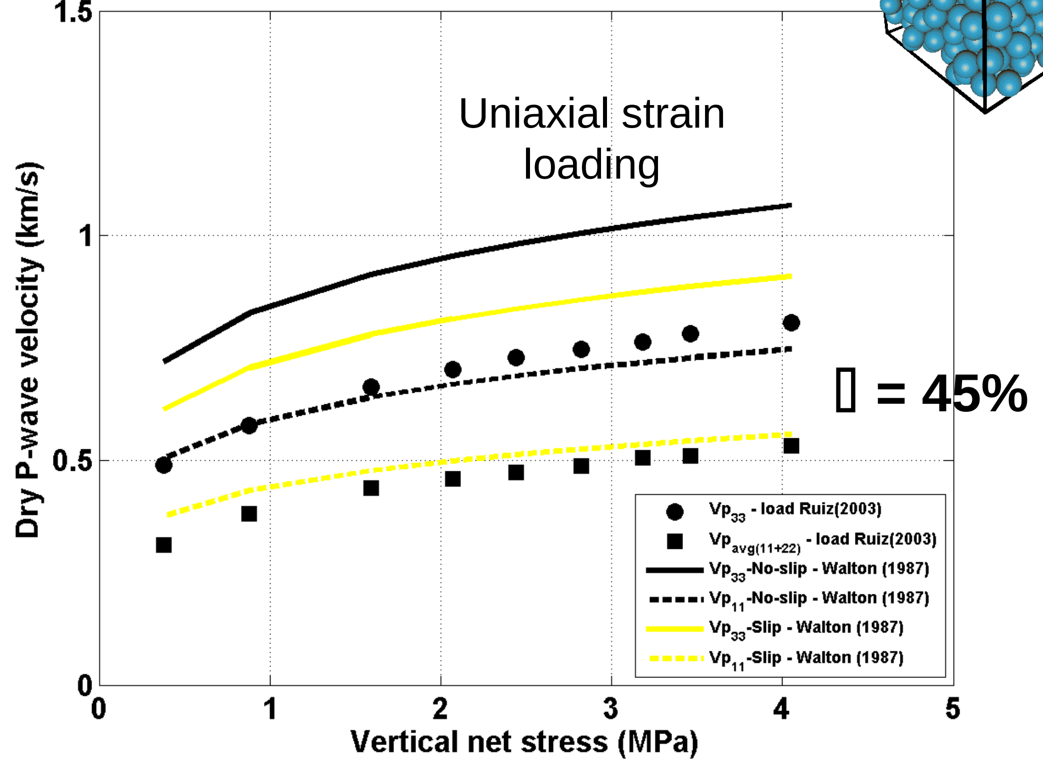
● $$Vp_{33}^{dry} = \sqrt{\frac{C_{33}^{dry}}{(1-\varphi)\rho_{ma}}}$$



Horizontal P-wave velocity:

■ $$Vp_{11}^{dry} = \sqrt{\frac{C_{11}^{dry}}{(1-\varphi)\rho_{ma}}}$$



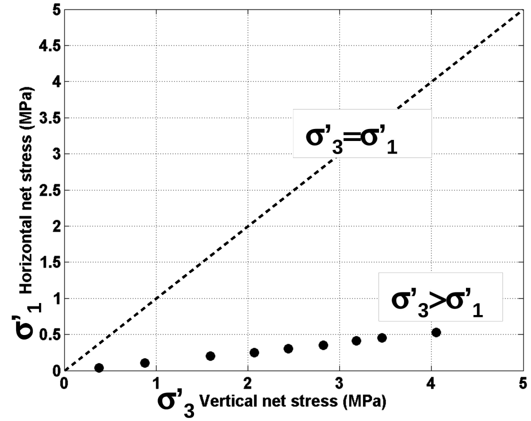


φ = Pore volume or Porosity

ρ_{ma} = Grain density

C_{33}^{dry} = Vertical P-wave modulus

C_{11}^{dry} = Horizontal P-wave modulus



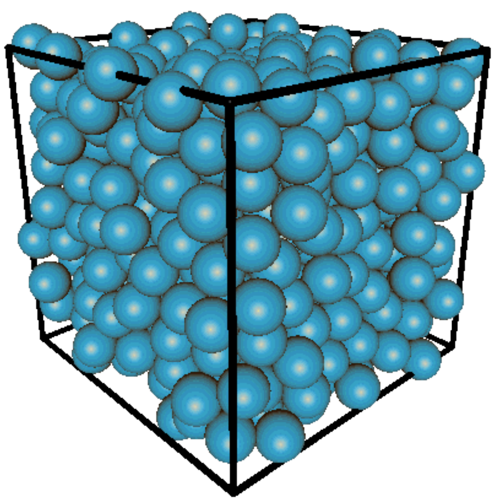
Stress-induced velocity anisotropy

New model

Thomsen(1986) anisotropy parameters as function of f(μ) and ν

P-wave anisotropy:

$$\varepsilon = \frac{C_{11}^{dry} - C_{33}^{dry}}{2\,C_{33}^{dry}} = -\frac{\left[5 + 2\left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]}{16\left[1 + \left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]}$$



S-wave anisotropy:

$$\gamma = \frac{C_{66}^{dry} - C_{44}^{dry}}{2\,C_{44}^{dry}} = -\frac{\left[1 + \left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]}{2\left[2 + 5\left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]}$$

$$\delta = \frac{\left(C_{13}^{dry} + C_{44}^{dry}\right)^2 - \left(C_{33}^{dry} - C_{44}^{dry}\right)^2}{2\,C_{33}^{dry}\left(C_{33}^{dry} - C_{44}^{dry}\right)} = -\frac{9\left[2 + \left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]^2 - \left[4 + \left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]^2}{48\left[1 + \left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]\left[2 + \left(\frac{1-\nu}{2-\nu}\right) f(\mu)\right]}$$

ν = Grain Poisson's ratio

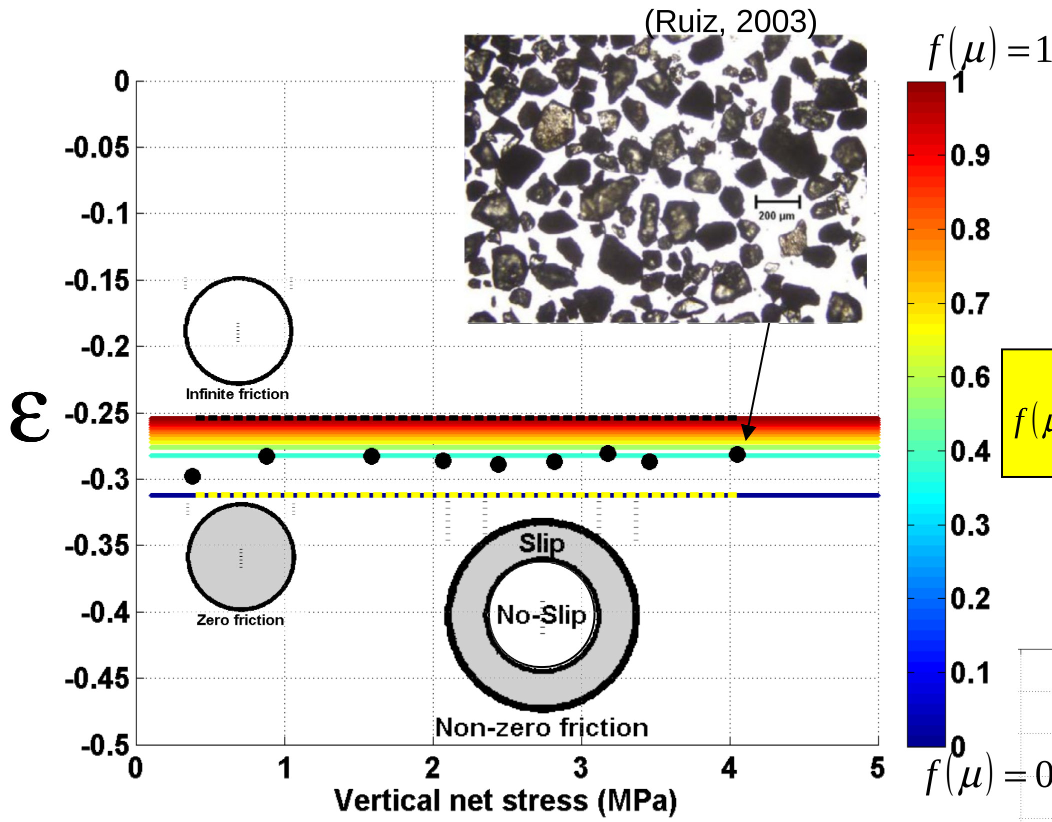
f(μ) = Mindlin's friction term

Model vs. dry loose sand measurements

Stress- and friction-induced elastic anisotropy

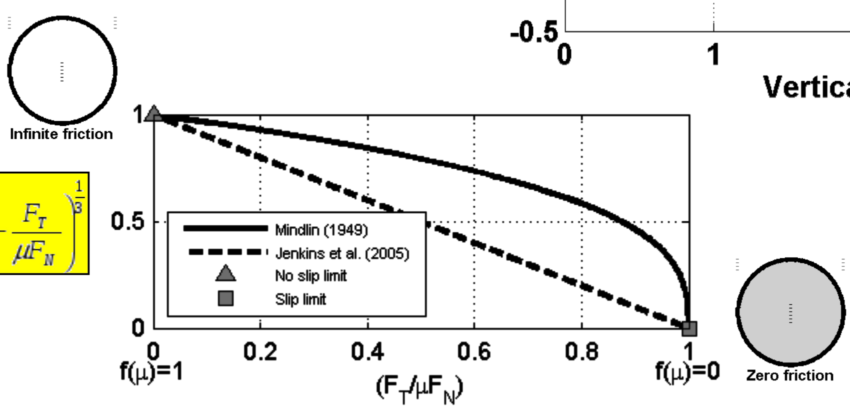
P-wave anisotropy:

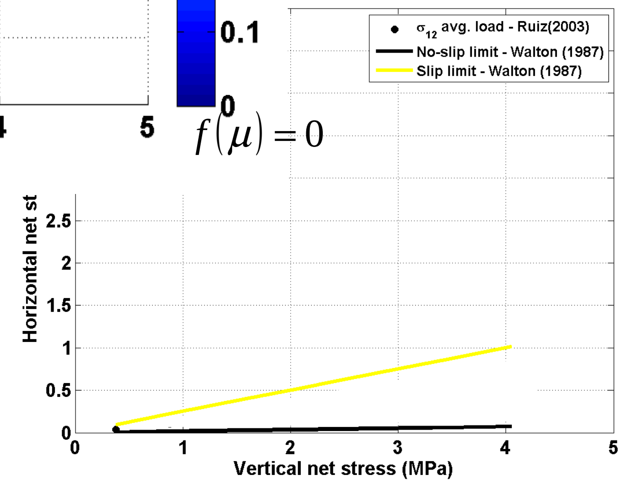
$$\varepsilon = -\frac{\left[5+2\left(\frac{1-\nu}{2-\nu}\right)f(\mu)\right]}{16\left[1+\left(\frac{1-\nu}{2-\nu}\right)f(\mu)\right]}$$



$$f(\mu) = \left(1-\frac{F_T}{\mu F_N}\right)^{\frac{1}{3}}$$

$$f(\mu) = \left(1-\frac{F_T}{\mu F_N}\right)^{\frac{1}{3}}$$

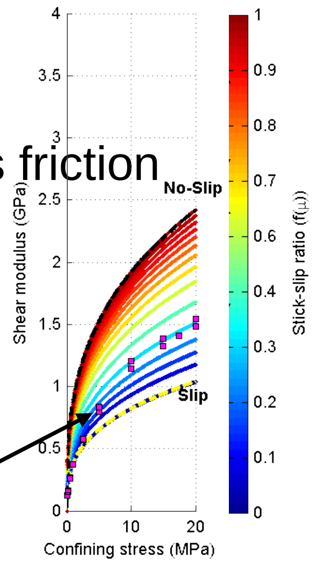




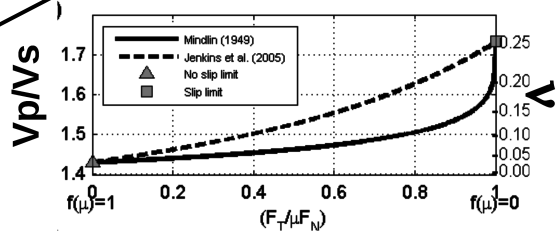
Conclusions

- The shear modulus is made friction-dependent by use of Mindlin's friction theory

$$G_{dry} = \frac{3}{5}\left[1 + \frac{3(1-\nu)}{2-\nu} f(\mu)\right] K_{dry}$$



- As intergrain friction increases
  - Larger effective moduli and velocities and lower Vp/Vs
  - Stress-sensitivity
    - increases for moduli (e.g. dG/dσ') and velocities (dV/dσ')
    - decreases for the Vp/Vs



- Ultrasonic measurements on sand of angular grains show higher dynamic shear rigidity than the perfect slip model – <u>Increasing internal friction due to grain interlocking</u>

- Grain contact conditions are controlling velocities and their stress-sensitivity in loose sands

Conclusions cont'

- Mindlin's friction theory revisited for uniaxial loading conditions – The friction-dependent model predicts
  1. instant large negative stress-induced elastic anisotropy (>20%)
  2. 20% difference in stress-induced elastic anisotropy between small and infinite contact friction
     - Extrapolating observation to cemented sandstone -> less stress-induced elastic anisotropy
  3. The model is only valid for small stress ratios $\leq$¼
- Ultrasonic measurements on a set of loose sand are in accordance with model-predictions
  - The sand with angular grains has higher internal friction than that of perfectly smooth spheres

Thank you