NTNU

Caustics in a periodically layered VTI medium

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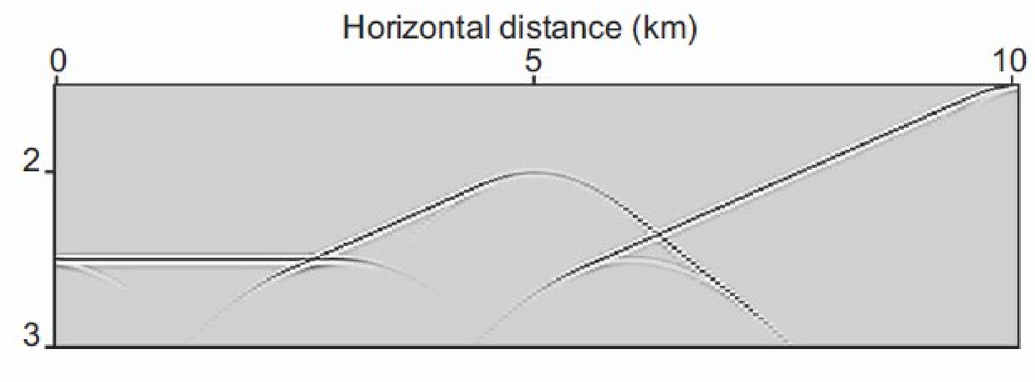
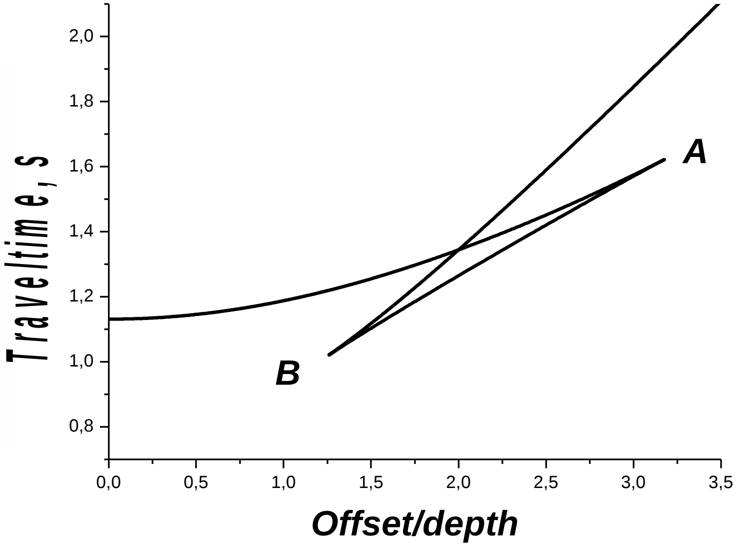
Caustics in seismic

P-wave structural caustics



qSV-wave caustics in VTI



Outline

- Wave propagation in a periodically layered VTI medium
- Low- and high-frequency limits
- Dispersion equation analysis
- Computational aspects
- Caustic asymptotical analysis
- Numerics
- Conclusions

Wave propagation in a periodically layered VTI medium

$$\frac{d\mathbf{b}}{dz} = i\omega \mathbf{M}\mathbf{b} \qquad \mathbf{b} = \left(u_z, \sigma_{xz}, \sigma_{zz}, u_x \right)^T$$

$$\mathbf{M} = \begin{pmatrix} 0 & \mathbf{A} \\ \mathbf{B} & 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} c_{33}^{-1} & pc_{13}c_{33}^{-1} \\ pc_{13}c_{33}^{-1} & \rho - p^2\left(c_{11} - c_{13}^2 c_{33}^{-1} \right) \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \rho & p \\ p & c_{44}^{-1} \end{pmatrix}$$

Wave propagation in a periodically layered VTI medium

Dispersion equation

$$\det\left(\mathbf{P} - \exp\left(i\omega H\theta\right)\mathbf{I}\right) = 0$$

Propagator matrix

$$\mathbf{P} = \exp(i\omega h_N \mathbf{M}_N)\ldots\exp(i\omega h_1 \mathbf{M}_1)$$

Effective vertical slowness

$$\theta = \theta\left(p, \omega\right)$$

$\operatorname{Re}\theta = q$ Vertical slowness of the envelope

$\operatorname{Im}\theta = \gamma$ Attenuation due to scattering

Low-frequency limit

Low-frequency limit

$$\mathbf{P} = \exp\left(i\omega H \tilde{\mathbf{M}}(\omega) \right)$$

$$\tilde{\mathbf{M}}(\omega) = \frac{1}{H}\sum_k h_k \mathbf{M}_k + \frac{i\omega}{2H^2}\sum_{k>l} h_k h_l \left(\mathbf{M}_k \mathbf{M}_l - \mathbf{M}_l \mathbf{M}_k \right) + o(\omega)$$

$$\tilde{\mathbf{M}}(0) = \frac{1}{H}\sum_k h_k \mathbf{M}_k$$

(Backus averaging)

High-frequency limit

Single mode propagator matrix

$$\mathbf{Q} = \exp(i\omega h_N \mathbf{F}_N)\ldots\exp(i\omega h_1 \mathbf{F}_1)$$

$$\mathbf{F}_j = q_j^{(\alpha_j)} \mathbf{n}_j^{(\alpha_j)} \mathbf{m}_j^{(\alpha_j)T}$$

$\mathbf{m}_j^{(\alpha_j)T}$ and $\mathbf{n}_j^{(\alpha_j)}$ are the left- and right-hand-side eigen-vectors of matrix $\mathbf{M}_j$ with eigen-value $q_j^{(\alpha_j)}$

$$\mathbf{Q} = \exp(i\omega \sum_j h_j q_j^{(\alpha_j)}) \left(\mathbf{n}_N^{(\alpha_N)} \mathbf{m}_N^{(\alpha_N)T} \ldots \mathbf{n}_1^{(\alpha_1)} \mathbf{m}_1^{(\alpha_1)T} \right)$$

$$= \exp(i\omega \sum_j h_j q_j^{(\alpha_j)} + \beta) \left(\mathbf{n}_N^{(\alpha_N)} \mathbf{m}_1^{(\alpha_1)T} \right)$$

$$\beta = \ln \left(\mathbf{m}_N^{(\alpha_N)T} \ldots \mathbf{n}_1^{(\alpha_1)} \right)$$

High-frequency limit

Root of dispersion equation

$$\theta = \frac{1}{H}\sum_j h_j q_j^{(\alpha_j)} - \frac{i\beta}{\omega H}$$

$$\downarrow$$

$$\theta = \frac{1}{H}\sum_j h_j q_j^{(\alpha_j)}$$

Direct single mode wave, no multiples

Dispersion equation analysis

Dispersion equation

$$\det\left(\mathbf{P} - x\mathbf{I}\right) = 0$$

Propagator blocking

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{11} & i\mathbf{P}_{12} \\ i\mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} \qquad \det\left(\mathbf{P}\right) = 1$$

Dispersion equation analysis

From Schoenberg (1983)

$$\left(x+x^{-1}\right)^2-a_1\left(x+x^{-1}\right)+a_2-2=0$$

$$\cos\left(\omega H\theta_P\right)+\cos\left(\omega H\theta_S\right)=\frac{1}{2}a_1\left(p,\omega\right)$$

$$\cos\left(\omega H\theta_P\right)\cos\left(\omega H\theta_S\right)=\frac{1}{4}a_2\left(p,\omega\right)-\frac{1}{2}$$

$a_1\left(p,\omega\right)$ $a_2\left(p,\omega\right)$ trace and the sum of principal second minors of the matrix **P**

$$\pm\mathrm{Re}\,\theta_P=\pm q_P\left(p,\omega\right) \qquad \pm\mathrm{Re}\,\theta_S=\pm q_S\left(p,\omega\right)$$

Dispersion equation analysis

$$b_1(p,\omega) = a_1(p,\omega)/4$$

$$b_2(p,\omega) = a_2(p,\omega)/4 - 1/2$$

$$\downarrow$$

$$y^2 - 2b_1 y + b_2 = 0$$

$$|y_1| \le 1 \quad |y_2| \le 1$$

Propagating envelopes

$$b_2 = -1 \pm 2b_1$$

$$y = \pm 1 \qquad D(p,\omega) = 0$$

$$b_2 = b_1^2$$

Dispersion equation analysis

Evanescent envelopes

$$|y| > 1$$

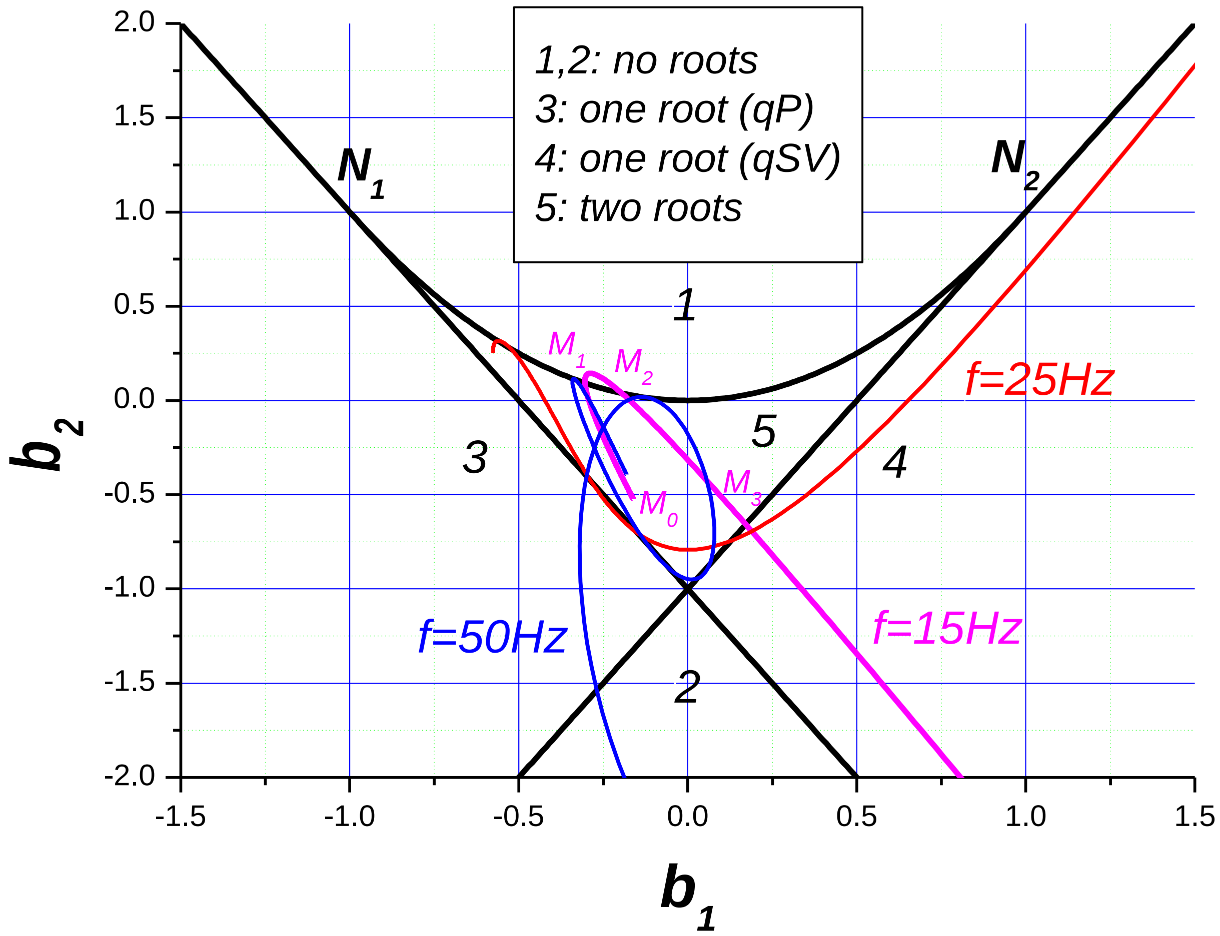
$$\cos(\omega H \theta) = y$$

$$\theta = \pm \frac{1}{\omega H}\left[2\pi n + i\ln\left(y + \sqrt{y^2 - 1}\right)\right],\ n \in \mathbf{Z},\ y > 1$$

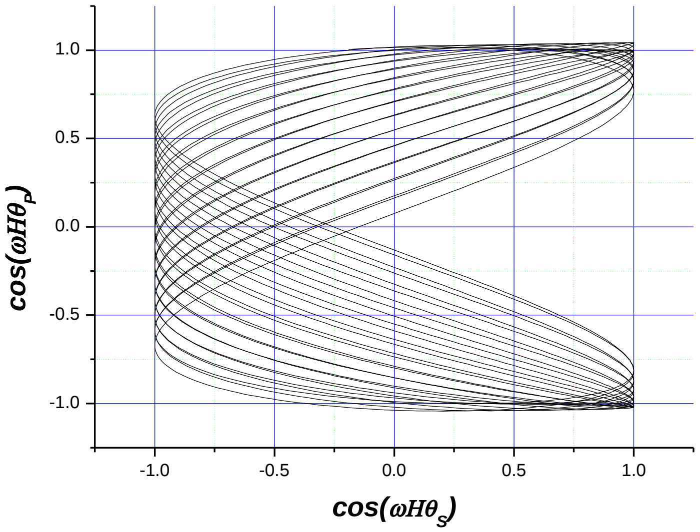
$$\theta = \pm \frac{1}{\omega H}\left[(2n+1)\pi + i\ln\left(-y + \sqrt{y^2 - 1}\right)\right],\ n \in \mathbf{Z},\ y < -1$$

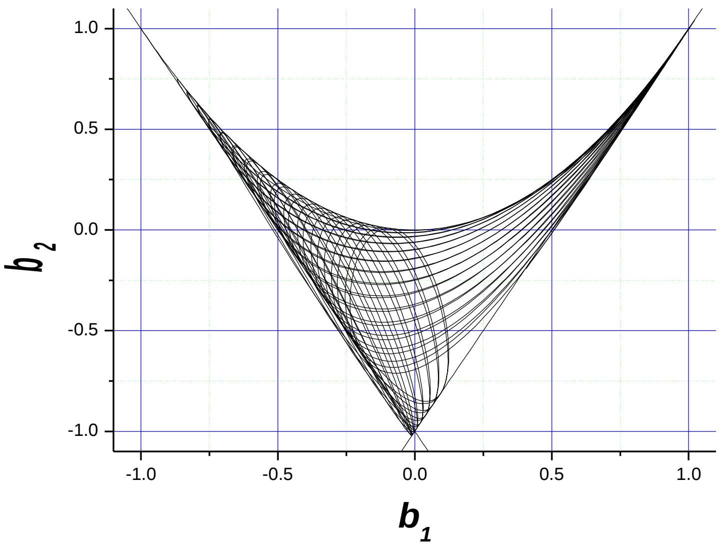
$$q = \operatorname{Re}\theta = const$$

Dispersion equation analysis



Dispersion equation analysis





Normal incidence case: a Lissajous curve

Computational aspects

Vertical energy flux

$$E = -\frac{1}{2}\mathrm{Re}\left(u_x \sigma_{xz}^* + u_z \sigma_{zz}^*\right)$$

$$\mathbf{b} = \left(u_z, \sigma_{xz}, \sigma_{zz}, u_x\right)^T$$

$\left|\exp\left(i\omega H\theta\right)\right| > 1$ Up-going enevelope

$\left|\exp\left(i\omega H\theta\right)\right| < 1$ Down-going enevelope

Amplitude propagator

$\mathbf{R} = \mathbf{EPE}^{-1}$ (**E** contains of the eigen-vector-columns of matrix **M**)

Caustics asymptotical analysis

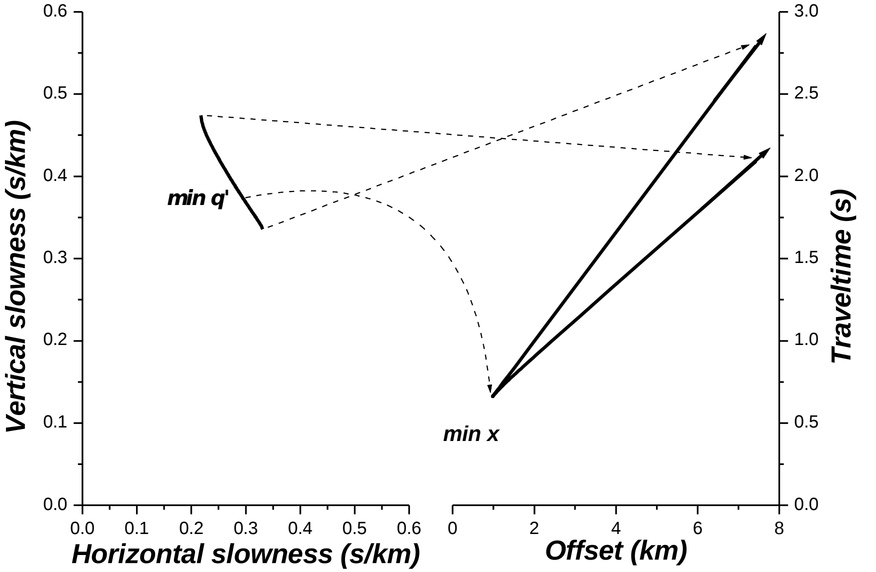
$$y\left(p^{(0)}\right)=1 \qquad dy/dp\left(p^{(0)}\right)=\alpha\neq 0$$

In the neighborhood of $p^{(0)}$

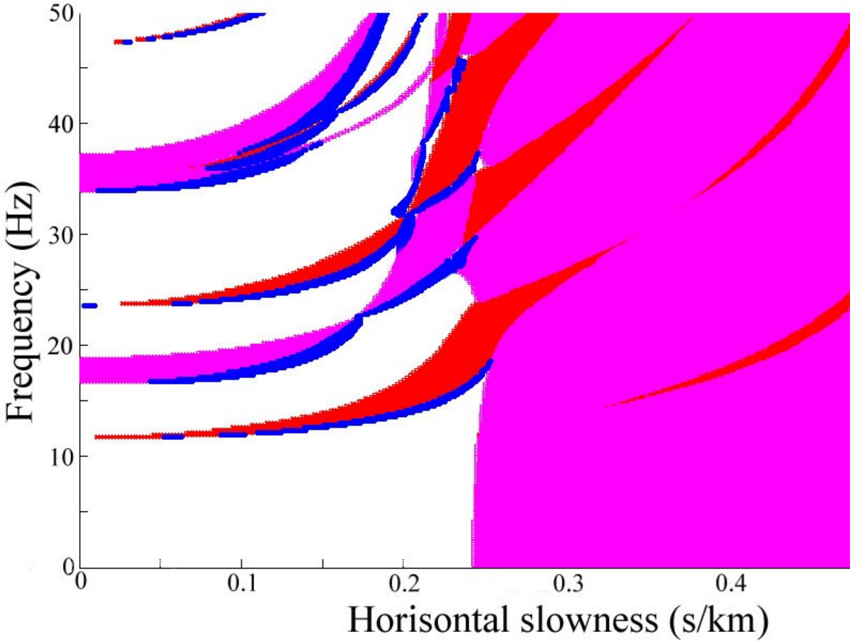
$$1-\frac{\omega^2 H^2 dq^2}{2}\approx 1+\alpha dp$$

$$dq=O\left(\sqrt{dp}\right) \qquad dq/dp=O\left(1/\sqrt{dp}\right)$$

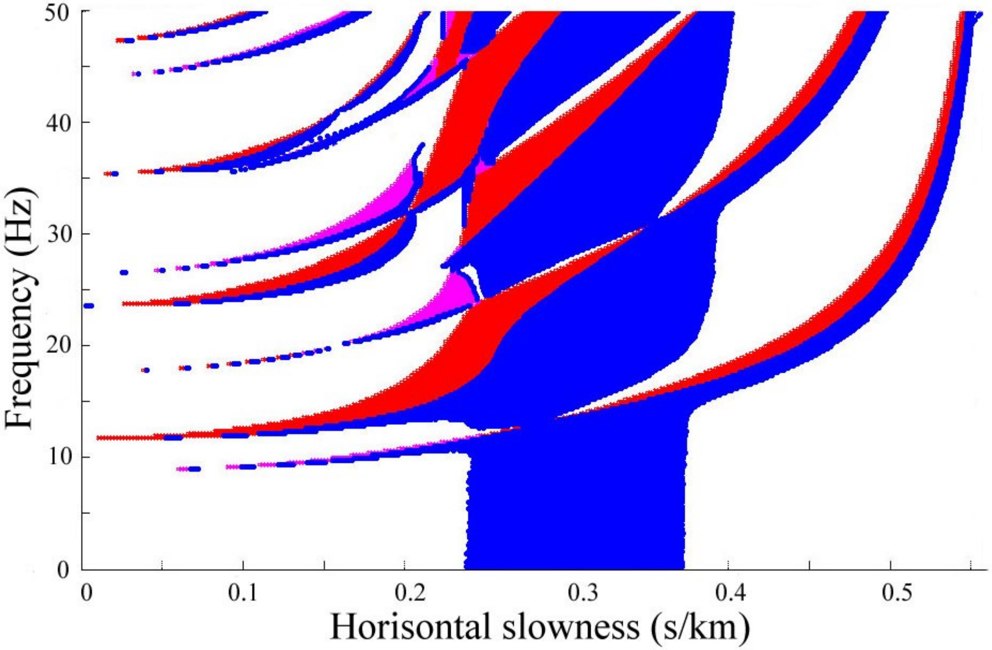
$$\lim_{p\to p^{(0)}}\left(dq/dp\right)=-\infty$$



Caustic regions



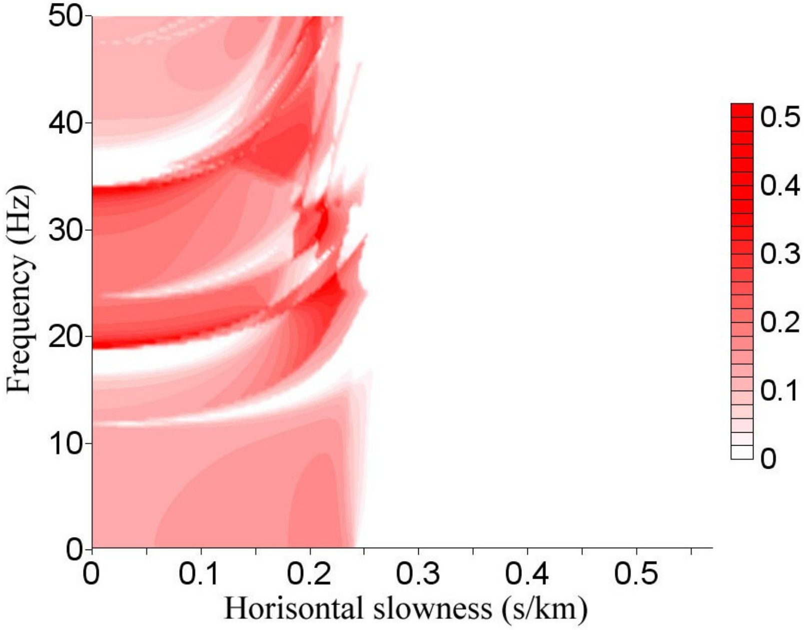
qP-wave



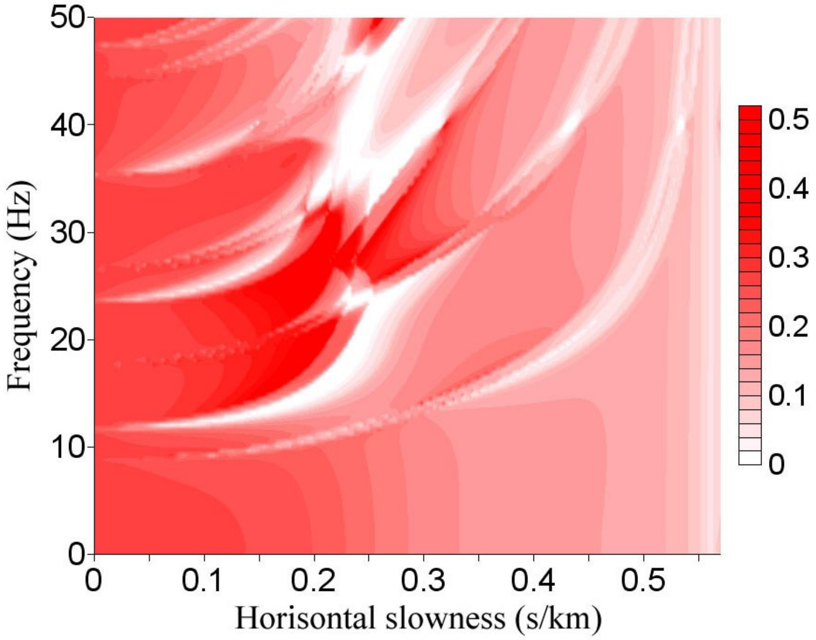
qSV-wave

The propagating, evanescent and caustic regions for qP-wave (to the top) and qSV-wave (to the bottom) shown in (p-f) domain. The regions are indicated by colors: red – no waves, white – both waves, pink – only qSV-wave and blue – caustic.

Vertical energy flux

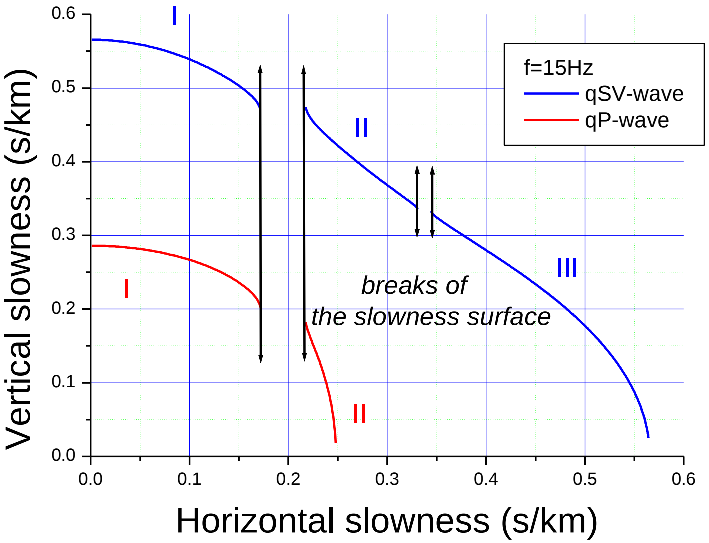


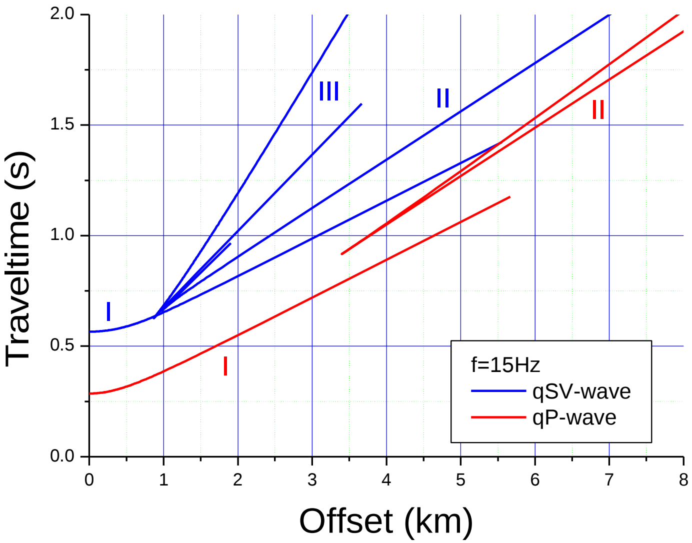
qP-wave



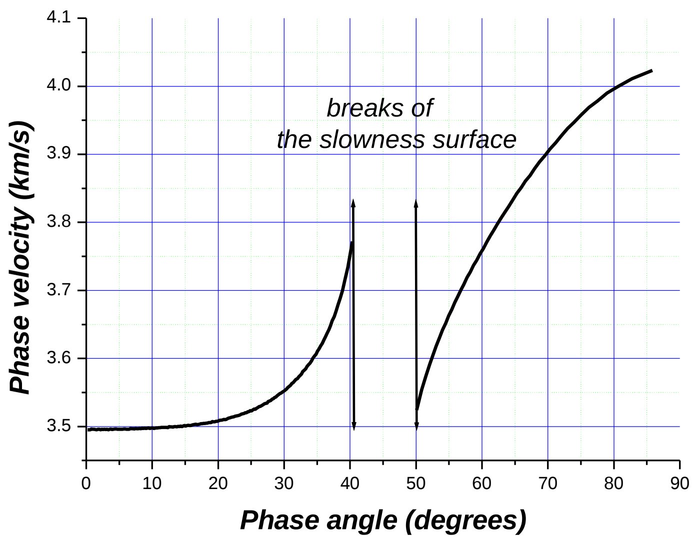
qSV-wave

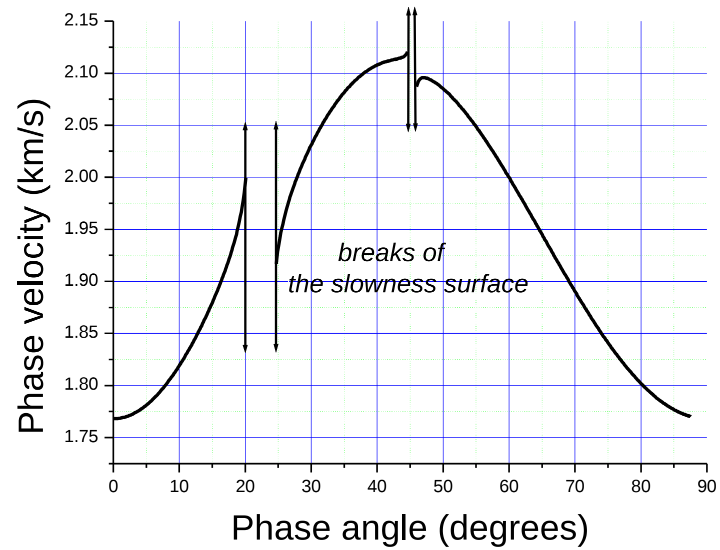
Discontinuous slowness surface versus traveltime





Phase velocities

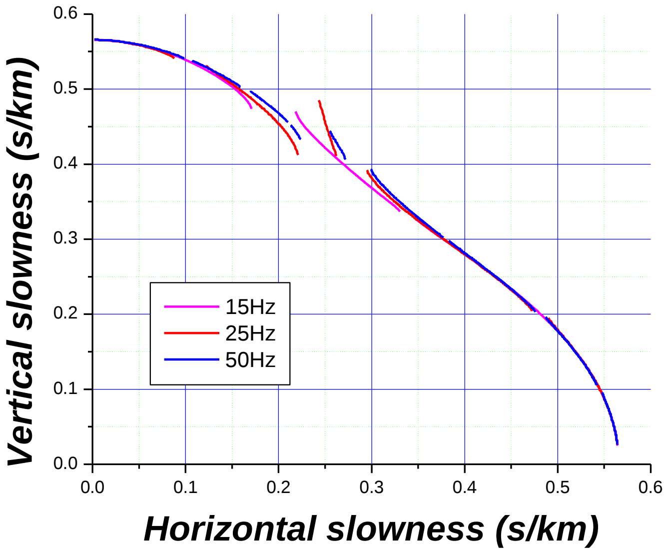


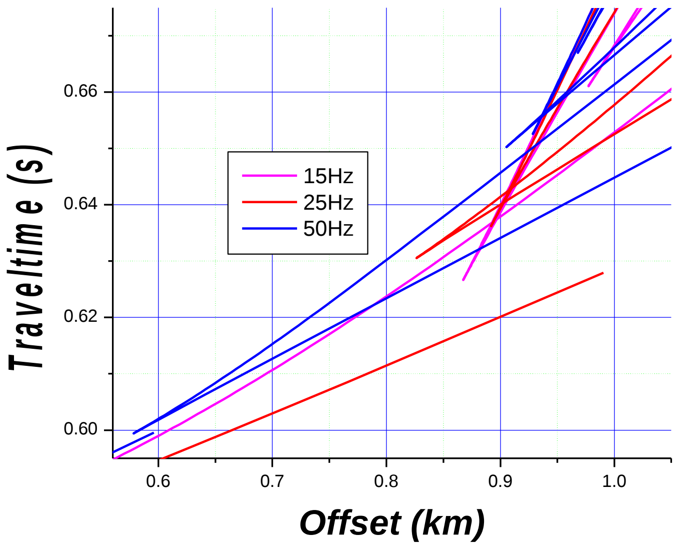


qP-wave

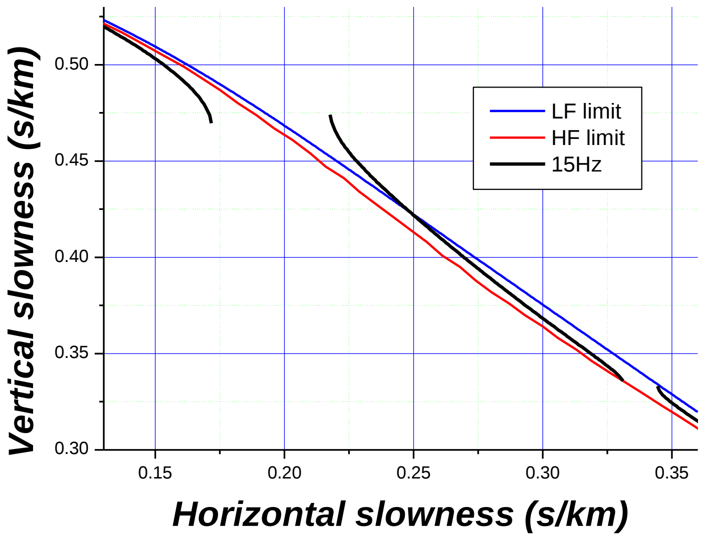
qSV-wave

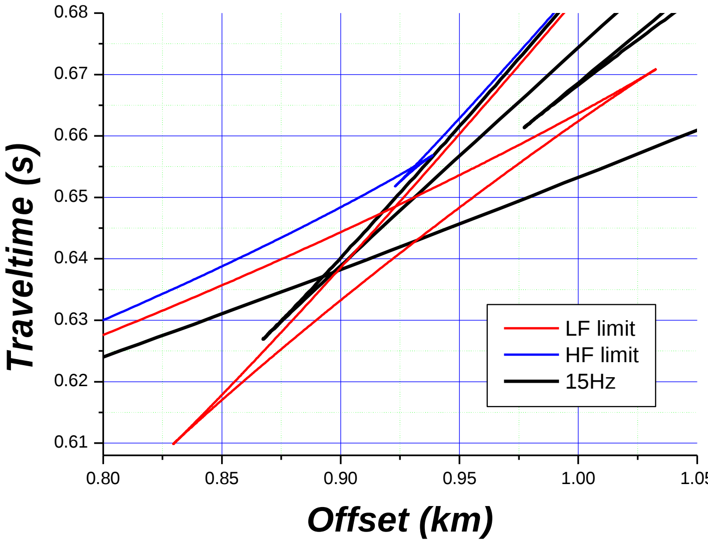
Single frequency caustics



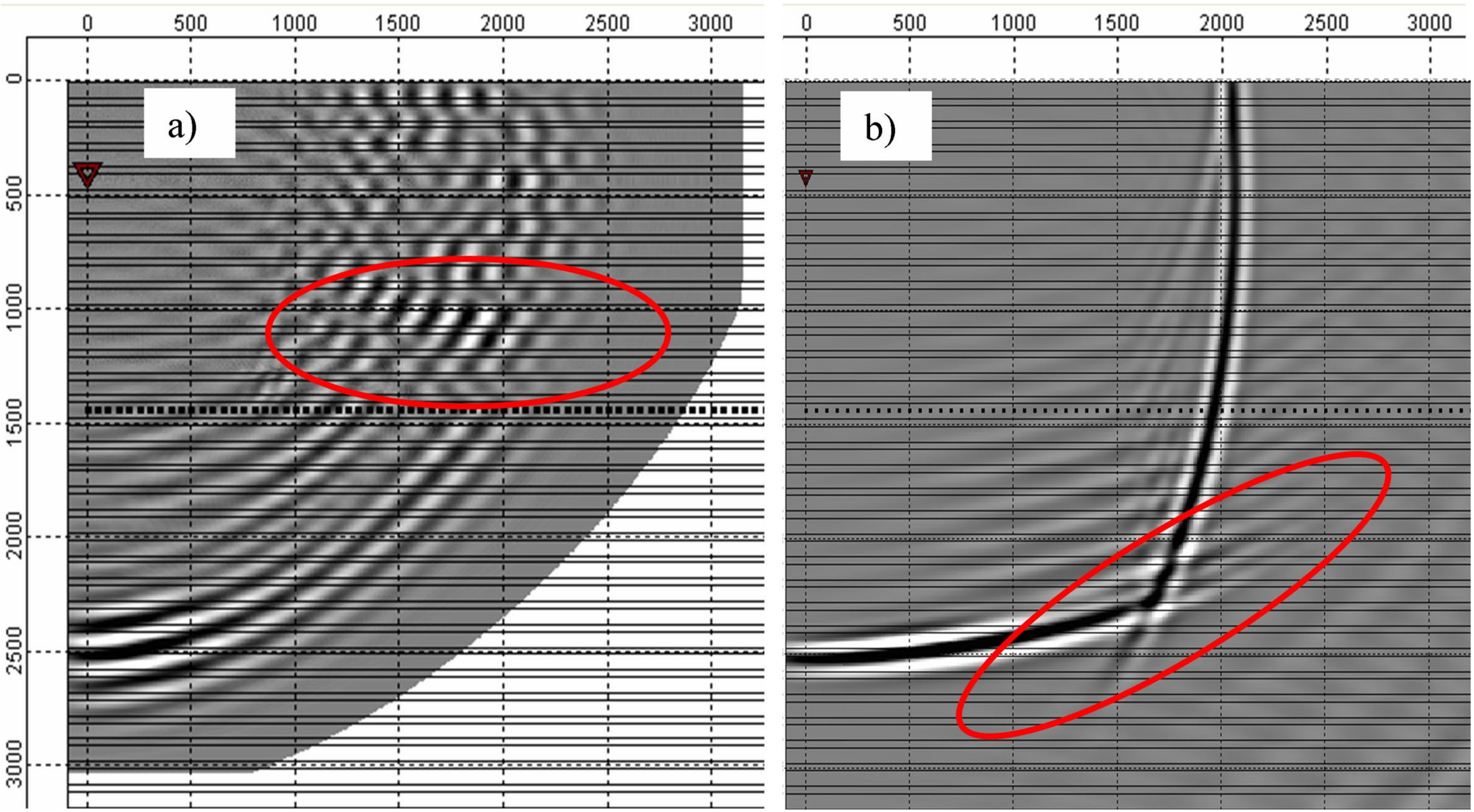


Single frequency versus frequency limits





Snapshots



qP-wave

qSV-wave

Conclusions

- From analysis of wave propagation in a periodically layered VTI medium, I define the discontinuous effective slowness surface for qP- and qSV-wave envelopes for given frequencies.
- The discontinuties result in the caustics in group domain.
- The qP- and qSV-wave caustics give different pictures on snaphots.

Acknowledgements

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