AVO THEORY

Introduction

The AVO program allows the user to perform three separate types of AVO analysis:

1. AVO reconnaissance analysis
2. Forward AVO modelling
3. AVO inversion

In this section on AVO theory, we will be considering all three of these analysis options, and the theory behind the calculations. Fundamental to all of this analysis are two input data streams: pre-stack seismic data and rock parameter measurements. Let us start by considering the basic rock physics that is so crucial to an understanding of seismic lithology and AVO.

The input of rock parameter measurements into AVO is done by reading well log data using the File/Open/Log View option. Ideally, these are digitized files from logs recorded in a well bore. However, any one (or all) of the logs may be either created from scratch or derived from one of the other logs via a transform relationship.

The AVO program allows the user to read in up to nine logs, seven of which are specified by the program, and two of which are defined by the user and can include any log curves not defined by the previous seven. (For information on input formats and specific menus, see the Reading Data and Modelling Menus parts of the manual).

The seven defined curves are:
- P-wave sonic
- Density
- S-wave sonic
- Poisson’s ratio
- Resistivity
- Gamma Ray
- SP

Of these seven curves, the first four are mandatory since they are used in the Zoeppritz calculations, whereas the last three curves represent non-essential measurements that may be used either to define lithologic zones or in the transform relationships.

In the next section, we will consider each of these measurements in more detail.
**Basic Rock Physics**

**AVO** has what seems like an extremely simple goal: determine the density, P-wave velocity, and S-wave velocity of the earth (Poisson’s ratio can be derived from P-wave and S-wave velocity) and then infer the lithology and fluid content from these parameters. However, this apparent simplicity is complicated by two rather difficult problems:

1. How do we unambiguously determine these three parameters?
2. How do we infer lithology from the physical parameters?

We will defer answering question (1) until later. However, let us first try to answer question (2) by looking at a model of the earth's lithology.

Figure 1 is a simplified rock cross-section, and shows that density and velocity can be affected by:

1) the number of minerals and their percentages, as well as the shape of the grains (the rock matrix)
2) the porosity of the rock
3) the types of fluids filling the pore space

![Figure 1](image)

**Figure 1** Overall rock properties are determined by matrix type, porosity, and fluid type.
If we assume that there is a single mineral type, or that we know the average value of the overall rock matrix, and that there are two fluids filling the pores (water and a hydrocarbon), Wyllie’s equation can be used to determine the density and velocity. For density, we can write

$$\rho_b = \rho_m (1 - \phi) + \rho_w S_w \phi + \rho_{hc} (1 - S_w) \phi$$

(1)

where:

- $\rho_b$ = bulk density of the rock
- $\rho_m$ = density of the rock matrix
- $\rho_f$ = density of the fluid
- $\phi$ = porosity of the rock
- $S_w$ = water saturation
- $\rho_w$ = density of water (close to 1 g/cm$^3$)
- $\rho_{hc}$ = density of hydrocarbon.

Figure 2 is a plot of density versus water saturation in both a gas reservoir and an oil reservoir with a porosity of 25%.

Figure 2. Wyllie’s equation applied to an oil and gas reservoir.
In Figure 2, notice that, as expected, density drops much more rapidly in a gas reservoir than an oil reservoir. As we shall find out in subsequent sections, density also enters into the equations of the compressional and shear-wave velocities, and of acoustic impedance, all of which affect the response of seismic waves to the subsurface. The dramatic difference seen in the density of gas and oil filled reservoirs will therefore play an important role in the seismic interpretation of these reservoirs.

Again, the most straightforward relationship between porosity and velocity is given by the Wyllie time-average formula where:

\[
\frac{1}{V_b} = \frac{(1-\phi)}{V_m} + \frac{S_w \phi}{V_w} + \frac{(1-S_w) \phi}{V_{hc}}
\]

where:
- \(V_b\) = bulk velocity,
- \(V_{hc}\) = velocity of hydrocarbon,
- \(V_m\) = matrix velocity, and
- \(V_w\) = water velocity

A plot of Wyllie’s equation for a porous gas sand and a porous oil sand of differing water saturation is given in Figure 3.

![Figure 3. P-wave velocity versus water saturation for a gas sand and an oil sand.](image)
It turns out that Figure 3 shows a good fit in oil sands, but a poor fit in gas sands. To understand why this is, let us consider velocity in more detail.

**P and S-wave Velocity Theory**

There are two types of waves that are of great interest to us when analyzing seismic data, the compressional wave, or P-wave, and the transverse wave, or S-wave. The equation for both of these waves can be written:

\[
V = \left( \frac{M}{\rho} \right)^{1/2}
\]

where:  
\[M = \text{elastic modulus}\]  
\[\rho = \text{density}\]

To understand the concept of elastic modulus, we will first review the concepts of stress and strain.

Figure 4 shows three ways in which a rock can be deformed: by pushing from above (compression), by pulling from above (tension), or by pushing from the side (shear). Compression and tension can be seen to be related, since one results in a change that is simply the negative of the other. In both tension and compression, notice that the volume (or, in the case shown, the area) of the rock changes, but its shape does not. In the case of a shear deformation, the shape of the rock has changed, but its volume has not.

**Figure 4.** A rock may be deformed by (a) compression, (b) tension, and (c) shear, where \(F = \text{force applied.}\)
Figure 4 also illustrates the concepts of stress and strain. The force per unit area required to produce a deformation is called stress. The deformation resulting from a stress is called a strain. Figure 4 shows three types of strains, two related to a compressive or tensional stress, and one related to a shear stress. These are:

\[ e_L = \frac{\Delta L}{L} \]  \hspace{2cm} (4)

\[ e_w = \frac{\Delta W}{W} \]  \hspace{2cm} (5)

\[ e_s = \frac{\Delta Y}{X} = \tan \theta \]  \hspace{2cm} (6)

Figure 4 shows only a two-dimensional cross-section of a rock cube. A complete description of strain involves the three-dimensional cube of rock itself, and is termed volumetric strain. This is illustrated in Figure 5, which shows a cubical volume before and after a stress. In this case, the strain is written:

\[ \Theta = \frac{\Delta V}{V} \]  \hspace{2cm} (7)

where:
\[ \Theta = \text{volumetric strain} \]
\[ \Delta V = \text{change in volume} \]
\[ V = \text{initial volume} \]

Figure 5. Volumetric stress, or cubical dilatation
Stress is given as force per unit area, and this can be written as:

$$p = \frac{F}{A} \quad (8)$$

where:

- $p$ = stress
- $F$ = force
- $A$ = area

In a perfectly elastic medium, stress and strain can be related to each other by Hooke’s Law, which states that:

$$\text{STRESS} = \text{CONSTANT} \times \text{STRAIN} \quad (9)$$

Strain is nondimensional, since it simply describes the proportional deformation of a material. The constant that relates stress and strain (which was introduced in equation 1) is called the modulus and has the same units as stress (force over area, or dynes/cm²).

For a purely **longitudinal strain**, as illustrated in Figure 3(a), the modulus is called Young’s modulus and the stress/strain relationship can be written:

$$p_L = \frac{\Delta L}{L} E \quad (10)$$

where: $E$ = Young’s modulus

For a **shear strain**, the modulus is called the shear modulus, or the rigidity, and can be written

$$p_S = \mu \varepsilon_S \quad (11)$$

where:

- $p_S$ = shear stress
- $\mu$ = shear modulus
- $\varepsilon_S$ = shear strain

For a **volumetric strain** the constant is called the bulk modulus, or incompressibility. In symbols:

$$p_H = K \frac{\Delta V}{V} \quad (12)$$

where:

- $p_H$ = hydrostatic stress
- $\Delta V/V$ = volumetric strain or dilatation
- $K$ = bulk modulus
Note also that incompressibility is the inverse of compressibility or:

\[
K = \frac{1}{C}
\]  
where: \( C = \text{compressibility} \)  

Typical values of bulk modulus, in units of \(10^{10} \text{ dynes/cm}^2\), are:

- Limestone (matrix): 60
- Sandstone (matrix): 40
- Sandstone pore volume: 0.9
- Water: 2.38
- Oil: 1.0
- Gas: 0.021

Without going into a mathematical derivation, P-wave velocity may be written:

\[
\alpha = \left[ \frac{K + \frac{4}{3}\mu}{\rho} \right]^{1/2}
\]  
where: \( \alpha = \text{P-wave velocity} \).

For S-waves, the equation is written:

\[
\beta = \left( \frac{\mu}{\rho} \right)^{1/2}
\]  
where: \( \beta = \text{S-wave velocity} \).

An important diagnostic in seismic lithologic determination is the ratio of P-wave velocity to S-wave velocity. We can derive from the preceding equations that

\[
\gamma = \left( \frac{\alpha}{\beta} \right)^2 = \frac{K}{\mu} + \frac{4}{3}
\]  

Another important parameter is the Poisson’s ratio, which can be given in terms of \( \alpha \) to \( \beta \) ratio:

\[
\sigma = \frac{\gamma - 2}{2(\gamma - 1)}
\]  
where: \( \sigma = \text{Poisson’s ratio} \), \( \gamma = \left( \frac{\alpha}{\beta} \right)^2 \).
Figure 6 shows a plot of Poisson’s ratio versus $\alpha$ to $\beta$ ratio. Even though we normally do not expect negative Poisson’s ratios, there is no theoretical reason why they can’t become negative.

![Plot of Sigma vs Vp/Vs](image)

**Figure 6. Poisson's ratio as a function of P-wave to S-wave ratio.**
**The Biot-Gassmann Theory**

The equations used in the *Biot-Gassmann* modelling option of the *AVO* program are those given by Gregory (1977 on pages 33 and 37 (equations 21 through 30)).

The basic problem can be formulated as follows:

> Given the P-wave velocity (and optional density) of a rock for known porosity and water saturation, derive the P-wave and S-wave velocities (and, therefore Poisson’s ratio) at different porosities and water saturations. Other parameters that are needed are the densities and bulk modulus of the water, hydrocarbon, and matrix solid, and also the dry rock Poisson’s ratio.

Let us start by defining all the initial parameters that are known or assumed concerning the rock we are studying:

- \( \phi_o \) = known porosity
- \( S_{wo} \) = known water saturation
- \( V_o \) = P-wave velocity for \( \phi_o \) and \( S_{wo} \)
- \( \sigma \) = dry rock Poisson’s Ratio (assumed to be 0.12)
- \( \rho_w \) = density of water
- \( \rho_s \) = density of solid matrix
- \( \rho_h \) = density of hydrocarbon (gas or oil)
- \( K_s \) = bulk modulus of solid
- \( K_w \) = bulk modulus of water
- \( K_h \) = bulk modulus of hydrocarbon

From the above known values, we can calculate density and bulk modulus values for the fluid filled rock, using the following equations (note that if the density is supplied as a known parameter, the problem becomes one of solving for the matrix density):

\[
\begin{align*}
\rho_f &= \text{density of fluid} \\
&= \rho_w S_{wo} + \rho_h (1-S_{wo}) \quad (18) \\
\rho_o &= \text{density of fluid-filled rock} \\
&= \rho_f \phi_o + \rho_s (1-\phi_o) \quad (19) \\
K_f &= \text{bulk modulus of fluid} \\
&= 1 / (S_{wo}/K_w + (1-S_{wo})/K_h) \quad (20)
\end{align*}
\]
The other values that will be needed are \( K_{bo} \), the initial bulk modulus of the dry rock, and \( \mu_{bo} \), the initial shear modulus of the dry rock. Note that these values are a function of porosity only.

The value of \( K_{bo} \) can be solved using the quadratic equation.

\[
ay^2 + by + c = 0
\]

where:
\[
y = 1 - \left( \frac{K_{bo}}{K_s} \right)
\]
\[
a = S - 1
\]
\[
b = \phi_o S \left( \frac{K_s}{K_f} - 1 \right) - S + \left( \frac{M}{K_s} \right)
\]
\[
c = -\phi_o \left( S - \left( \frac{M}{K_s} \right) \right) \left( \frac{K_s}{K_f} - 1 \right)
\]
\[
S = \frac{3(1 - \sigma)}{1 + \sigma}
\]
\[
M = V_o^2 \rho_o
\]

We can therefore solve for \( K_{bo} \) by inverting equation (21) to get:

\[
y = \left[ -b + \sqrt{b^2 - 4ac} \right] / 2a
\]

\[
K_{bo} = (1 - y)K_s
\]

We will also need a way of calculating \( K_b \) at new porosity values. Therefore, we introduce the pore bulk modulus \( K_p \), given by:

\[
K_p = \phi_o / \left( 1/K_{bo} - (1/K_s) \right)
\]

Now that all the initial parameters have been defined, values of \( \rho, K_f, K_b \) and \( \mu_b \) can be found at each new porosity and water saturation value, using the equations:

\[
\rho = \rho_w S_w \phi + \rho_h (1 - S_w) \phi + \rho_s (1 - \phi)
\]

\[
K_f = 1/(S_w/K_w + (1 - S_w)/K_h)
\]

\[
K_b = 1/(\phi/K_p + 1/K_s)
\]

\[
\mu_b = \frac{3(1 - 2\sigma)}{2(1 + 2\sigma)} K_b
\]

where:
\( \phi = \) new value of porosity
\( S_w = \) new value of water saturation.
Finally, new values of P and S-wave velocity can be computed using the formulas:

\[
V_p^2 = \left( \frac{(1 - K_b/K_s)^2}{1 - \phi - K_b/K_s} \right) \frac{1/K_s + \phi/K_f}{\mu_b/\rho} \\
Vs^2 = \mu_b/\rho
\]

**(Empirical Relationships Among the Various Parameters)**

In the previous sections, we have looked at theoretical relationships between physical parameters such as P-wave velocity, S-wave velocity, Poisson’s ratio, and density, and the constituent parts of the rock itself: the rock matrix, porosity, and fluid content. Often, we do not know the details of the rocks themselves, but wish to derive an empirical log curve by transforming one of the other curves. This has been the subject of many papers over the years and will be discussed in this section. The actual application of this theory in the AVO program is in the Edit Logs/Replace item on the menu.

**Relationship Between P-wave Velocity and Resistivity**

In many older fields, the only logs that are available are resistivity logs. It has been observed that, in wet clastic rocks, the resistivity log and the P-wave sonic tend to track each other. A number of empirical relationships have therefore been derived to allow the geophysicist to derive a P-wave sonic from a resistivity log. The oldest relationship is from Faust (Geophysics, V18):

\[
\alpha = a(Rd)^c
\]

where: \(\alpha = \) P-wave velocity \\
\(a, c = \) constants \\
\(R = \) resistivity value \\
\(d = \) depth

The values derived by Faust for \(c\) and \(a\) are the default values in the Replace option.

A more recent formulation does not involve using depth in the equation. The most general expression for this equation is:

\[
\Delta t = a + bR^c
\]

where: \(\Delta t = \) transit-time \\
\(a, b, c = \) constants.
This equation, in various forms, has been published by Kim (1964), Rudman et al (1975), and McCoy and Smith (1979). Specifically, McCoy and Smith propose setting $a$ to zero, leading to the equation:

\[ \Delta t = bR^c \]  

(33)

In the Replace/P-wave option of AVO, the values derived by McCoy and Smith are the defaults. It is also important to note that the AVO program gives the user the ability to derive regression coefficients for this equation. This is found in the Display/Cross-plot option.

**Relationship Between P-wave Velocity and Density**

In the AVO program, there are two ways of deriving P-wave velocity from density (or, in an inverse fashion, deriving density from P-wave velocity). These equations are often referred to by the names of the individuals who first published them: Gardner’s equation and Lindseth’s equation. Gardner’s equation is the better known of the two equations, and is written:

\[ \rho = aV^b, \]  

(34)

where:

\[ a = 0.23 \]

\[ b = 0.25 \]

are empirically derived values from a wide range of sedimentary rocks. Again, the values of $a$ and $b$ can be determined by using the regression fit analysis under Display/Cross-plot.

The second equation, Lindseth’s, is a linear fit between velocity and acoustic impedance, and is written:

\[ V = a(\rho V) + b \]  

(35)

where:

\[ a = 0.308 \]

\[ b = 3400 \text{ ft/s} \]

were empirically derived values from Lindseth (1979). Notice that we can write the above equation as a functional relationship between $V$ and $\rho$ in the following way:

\[ \Delta t = c - d\rho \]  

(36)

where:

\[ \Delta t = 1/V \]

\[ c = 1/b \]

\[ d = a/b \]

Using equation (36), it is possible to derive regression coefficients from the Display/Cross-plot option. (i.e. plot $\rho$ vs. transit-time and perform a linear fit).
Relationships Between P and S Velocities

Castagna’s Relationship

As we found in a previous section, the Biot-Gassmann model is mathematically complex. Also, the theory falls down when applied to small grained clastic rocks, such as mudstones. In this case, Castagna et al (1985) derived a much simpler empirical relationship between P-wave and S-wave velocity, which can be written:

\[ \alpha = 1.16\beta + 1.36 \quad (37) \]

where velocity is in km/s.

This is simply the equation for a straight line. A plot of this line, and the observations that fit it from previous work, is given in Figure 7. This line is also called the mudrock line.

![Figure 7](Castagna et al, 1985)
There are several other ways in which this relationship can be plotted, and we will consider two: Poisson’s ratio versus P-wave velocity and $\alpha/\beta$ ratio versus P-wave velocity. These plots are shown in Figures 8 and 9, respectively. Notice that the plot of Poisson’s ratio against P-wave velocity (in Figure 8) shows that the lowest Poisson’s ratio is 0.1. On the $\alpha/\beta$ ratio against P-wave velocity plot (in Figure 9) the curve approaches the value 1.5 asymptotically. These values ($S = 0.1$ and $\alpha/\beta = 1.5$) represent the so-called “dry rock” value for a dry, porous sandstone. Thus, the “mudrock” line approaches the “dry rock” line as the P-wave velocity increases.

Finally, Castagna also shows that Gassmann’s equations predict velocities that fall approximately on the mudrock line in the water-saturated case.

Figure 8

Figure 9
The Krief Relationship

A more recent paper by Krief et al (The Log Analyst, November-December, 1990) proposes that an excellent linear fit can be found between P and S-wave velocity if the square of the two quantities is cross-plotted. Their equation reads:

\[ \alpha^2 = a\beta^2 + b \]  

(38)

If we let \( \alpha \) and \( \beta \) be measured in km/s, the regression coefficients determined by Krief et al can be summarized:

<table>
<thead>
<tr>
<th>Lithology</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sandstone (Wet)</td>
<td>2.213</td>
<td>3.857</td>
</tr>
<tr>
<td>Sandstone (Gas)</td>
<td>2.282</td>
<td>0.902</td>
</tr>
<tr>
<td>Sandstone (Shaly)</td>
<td>2.033</td>
<td>4.894</td>
</tr>
<tr>
<td>Limestone</td>
<td>2.872</td>
<td>2.755</td>
</tr>
</tbody>
</table>
AVO GRADIENT ANALYSIS

Introduction

A large part of current industry practice in AVO analysis involves what is referred to as intercept/gradient analysis. To understand the basics of this procedure we will discuss the following three topics:

1. The basic approximations of the Zoeppritz equations
2. Transforming our data from the constant offset to the constant angle domain
3. Various display options

Approximations of the Zoeppritz Equations

The Aki, Richards and Frasier Approximation

The Zoeppritz equations allow us to derive the exact plane wave amplitudes of a reflected P-wave as a function of angle, but do not give us an intuitive understanding of how these amplitudes relate to the various physical parameters. Over the years, a number of approximations to the Zoeppritz equations have been made. The first was by Bortfield in 1961 (Geophysical Prospecting, V. 9, p. 485-502). His formula was further refined by Richards and Frasier (Geophysics, June 1976), and by Aki and Richards (Quantitative Seismology, 1980). The Aki, Richards and Frasier approximation is appealing because it is written as three terms, the first involving P-wave velocity, the second involving density, and the third involving S-wave velocity. Their formula can be written:

\[
R(\theta) = \frac{\Delta\alpha}{\alpha} + \frac{\Delta\rho}{\rho} + \frac{\Delta\beta}{\beta} \tag{1}
\]

where:

- \(a = \frac{1}{2 \cos^2 \theta} = (1 + \tan^2 \theta)/2\)
- \(b = 0.5 - [(2 \beta^2 / \alpha^2) \sin^2 \theta]\)
- \(c = -(4 \beta^2 / \alpha^2) \sin^2 \theta\)
- \(\alpha = (\alpha_1 + \alpha_2)/2\)
- \(\beta = (\beta_1 + \beta_2)/2\)
- \(\rho = (\rho_1 + \rho_2)/2\)
- \(\Delta\alpha = \alpha_2 - \alpha_1\)
- \(\Delta\beta = \beta_2 - \beta_1\)
- \(\Delta\rho = \rho_2 - \rho_1\)
- \(\theta = (\theta_i + \theta_t)/2\), where \(\theta_t = \arcsin\left[(\alpha_2 / \alpha_1) \sin \theta_i\right]\)
Equation (1) can be rearranged to give:

\[
R(\theta) = \frac{1}{2} \begin{bmatrix}
\Delta \alpha & \Delta \rho \\
\alpha & \rho
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
\Delta \alpha & \Delta \beta & \frac{\beta^2}{\alpha^2} & \Delta \rho \\
\alpha & -4 \frac{\beta^2}{\alpha^2} & 2 \frac{\beta^2}{\rho} & \sin^2 \theta
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
\Delta \beta \\
\beta & \rho
\end{bmatrix} \left( \frac{\tan^2 \theta - \sin^2 \theta}{\alpha^2} \right)
\]

Equation (2) is interesting in two regards. It was rearranged by Shuey in terms of Poisson’s ratio rather than S-wave velocity to give his well-known approximation, and was also rearranged by Wiggens at Mobil, and published by Gelfand and Larner (SEG Expanded Abstracts, 1986, p. 335) as an approximation based on S and P-wave reflectivity. First, set the S-wave to P-wave velocity ratio:

\[\beta/\alpha = 0.5\]

and then ignore the third term in equation (2), which leads to:

\[
R(\theta) = \frac{1}{2} \begin{bmatrix}
\Delta \alpha & \Delta \rho \\
\alpha & \rho
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
\Delta \alpha & \Delta \beta & 1 & \Delta \rho \\
2 \alpha & -\beta & 2 & \rho
\end{bmatrix} \sin^2 \theta
\]

Notice that, if we let

\[
R_p = \frac{1}{2} \begin{bmatrix}
\Delta \alpha & \Delta \rho \\
\alpha & \rho
\end{bmatrix}
\]

and

\[
R_s = \frac{1}{2} \begin{bmatrix}
\Delta \beta & \Delta \rho \\
\beta & \rho
\end{bmatrix}
\]
we can rewrite equation (3) as:

\[
R(\theta) = R_p + (R_p - 2R_s)\sin^2\theta
\]  

(4)

From this, we see that:

\[
R_s = (R_p - G)/2
\]

Shuey’s Approximation

Whereas the approximation in equation (1) involved \(\alpha, \beta\) and \(\rho\), Shuey (Geophysics 50, 609-614, 1985), published a closed form approximation of the Zoeppritz equations which involved \(\alpha, \rho\) and \(\sigma\), or Poisson’s ratio. The equation is given by:

\[
R(\theta) = R_p + \left[R_pA_0 + \frac{\Delta\sigma}{(1 - \sigma)^2}\sin^2\theta + \frac{\Delta\alpha}{2\alpha}(\tan^2\theta - \sin^2\theta)\right]
\]  

(5)

where:

\[
\sigma = (\sigma_1 + \sigma_2)/2
\]

\[
\Delta\sigma = \sigma_2 - \sigma_1
\]

\[
A_0 = B - 2(1 + B) \frac{1 - 2\sigma}{1 - \sigma}
\]

\[
B = \frac{\Delta\alpha/\alpha}{\Delta\alpha/\alpha + \Delta\rho/\rho}
\]

and the other variables are as in equation (1).

Hilterman (unpublished notes) simplified Shuey’s equation even further by making the following assumptions:

1. Use only the first two terms
2. Set \(s = 1/3\), which means that \(A_\rho = -1\)

Then, equation (5) simplifies to:

\[
R(\theta) = R_p[1 - \sin^2\theta] + 9/4 \Delta\sigma\sin^2\theta
\]  

(6)
With further simplification, we can get either:

\[ R(\theta) = R_p \cos^2 \theta + \frac{9}{4} \Delta \sigma \sin^2 \theta \]  \hspace{1cm} (7)

\[ = R_p + \left[ \frac{9}{4} \Delta \sigma - R_p \right] \sin^2 \theta \]  \hspace{1cm} (8)

\[ = R_p + G \sin^2 \theta \]

Notice that equation (8) suggests that from an estimate of \( R_p \) and \( G \), the change in Poisson’s ratio can be estimated using the rearranged equation:

\[ \Delta \sigma = \frac{4}{9} \left( R_p + G \right) \]  \hspace{1cm} (9)

Figure 10 shows a comparison of results obtained for a simple geological model (see page 13) at the top and bottom interfaces, using the full Zoeppritz calculations, and the approximations of the earlier equations. Figure 11 shows the error distribution of the top layer (bottom curve, since the reflection coefficient is negative) of Figure 10. Notice that all the fits are within 2\% to 20°. Gelfand’s approximation is best to 35°, and Shuey’s approximation is best overall.

Figure 10. A comparison of the Zoeppritz equations and its approximations for a simple gas sand model.
Zoeppritz’ Equation and Approximations

Figure 11. The error for the negative reflector shown in Figure 10.

Notice that both the Aki/Richards/Gelfand approximation and the Shuey approximation can be expressed by the following simple equation:

\[ R(\theta) = R_p + G\sin^2\theta \]

This equation is linear if we plot \( R \) as a function of \( \sin^2\theta \). We could then perform a linear regression analysis on the seismic amplitudes to come up with estimates of both intercept \( R_p \), and gradient \( G \). But, first, we must transform our data from constant offset form to constant angle form.
The Smith and Gidlow Approximation

Yet another approximation based on the Aki and Richards equation was given by Smith and Gidlow (Geophysical Prospecting 35, 993-1014, 1987). They used this approximation to perform a weighted stack on the corrected seismic gathers to produce information about the rock properties of reservoirs.

Smith and Gidlow started by rearranging equation (1) to give:

\[
R(\theta) = \frac{1}{2} \left( \begin{array}{cc} \Delta \alpha & \Delta \rho \\ \alpha & \rho \end{array} \right) - 2 \frac{\beta^2}{\alpha^2} \left( \begin{array}{cc} \Delta \beta & \Delta \rho \\ \beta & \rho \end{array} \right) \sin^2 \theta
\]

\[
\quad + \frac{1}{2} \Delta \alpha \tan^2 \theta
\]

(10)

Next, they simplified equation (10) by removing the dependence on density, using Gardner’s equation:

\[
\rho = a \alpha^{1/4}
\]

(11)

which can be differentiated to give:

\[
\frac{\Delta \rho}{\rho} = \frac{1}{4} \frac{\Delta \alpha}{\alpha}
\]

(12)

Substituting equation (12) into equation (10) gives:

\[
R(\theta) = \frac{\Delta \alpha}{\alpha} + d \frac{\Delta \beta}{\beta}
\]

(13)

where:

\[
c = \frac{5}{8} - \frac{1}{2} \frac{\beta^2}{\alpha^2} \sin^2 \theta + \tan^2 \theta
\]

\[
d = -4 \frac{\beta^2}{\alpha^2} \sin^2 \theta
\]
Equation (13) was solved by least squares to derive weights that can be applied to the seismic gather to produce estimates of both $\Delta \alpha / \alpha$ and $\Delta \beta / \beta$. Smith and Gidlow also derived two other types of weighted stacks, the “Pseudo-Poisson’s ratio reflectivity”, defined as:

$$\frac{\Delta \sigma}{\sigma} = \frac{\Delta \alpha}{\alpha} - \frac{\Delta \beta}{\beta}$$ (14)

and the “fluid factor” stack, which requires a little more discussion. To derive the fluid factor, Smith and Gidlow used the ARCO mudrock equation, which is the straight-line fit that appears to hold for water-bearing clastics around the world. This equation is written as:

$$\alpha = 1360 + 1.16 \beta \text{ (velocities in m/s)}$$ (15)

Equation (15) can be differentiated to give:

$$\Delta \alpha = 1.16 \Delta \beta$$ (16)

Equation (16) can be expressed in ratio form as:

$$\frac{\Delta \alpha}{\alpha} = 1.16 \frac{\beta}{\alpha} \frac{\Delta \beta}{\beta}$$ (17)

However, equation (17) only holds for the wet case. In a hydrocarbon reservoir, it should not hold, and we can define a “fluid factor” residual:

$$\Delta F = \frac{\Delta \alpha}{\alpha} - 1.16 \frac{\beta}{\alpha} \frac{\Delta \beta}{\beta}$$ (18)
Equating the Gradient/Intercept and Smith/Gidlow Methods

In the first two sections, we derived two approximations to the Zoeppritz equations which could both be expressed as:

\[
R(\theta) = R_p + G \sin^2 \theta
\]  

(19)

where: \( R_p = \) P-wave intercept

\[
\begin{bmatrix}
1

\Delta \alpha & \Delta \rho \\
\hline
2 & \alpha & \rho
\end{bmatrix}
\]

\[
G = \text{Gradient} = R_p - 2R_s
\]

\[
\begin{bmatrix}
1

\Delta \beta & \Delta \rho \\
\hline
2 & \beta & \rho
\end{bmatrix}
\]

In the Smith and Gidlow approximations, the actual physical parameters, \( \Delta \alpha/\alpha \) and \( \Delta \beta/\beta \), were estimated. However, recalling the Gardner approximation that was used by Smith and Gidlow (equation 12), we can show how both methods can be equated. First, substitute equation 12 into the relationship for \( R_p \), to get:

\[
R_p = \frac{1}{2} \left[ \frac{\Delta \alpha}{\alpha} + \frac{1}{4} \frac{\Delta \alpha}{\alpha} \right] = \frac{5}{8} \frac{\Delta \alpha}{\alpha}
\]

(20)

which implies that:

\[
\frac{\Delta \alpha}{\alpha} = 1.6R_p
\]

(21)
Next, substitute equation (12) into the relationship for $R_s$, to get:

$$
R_s = \begin{bmatrix}
1 & \Delta \beta & 1 & \Delta \alpha \\
2 & \beta & 4 & \alpha
\end{bmatrix}
\begin{bmatrix}
\Delta \beta \\
\Delta \alpha
\end{bmatrix}
+ \begin{bmatrix}
\Delta \beta \\
\Delta \alpha
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

(22)

or:

$$
\frac{\Delta \beta}{\beta} = 2R_s - \frac{1}{4} \frac{\Delta \alpha}{\alpha} = 2R_s - \frac{2}{5} R_p
$$

(23)

Thus, $\Delta \alpha/\alpha$ and $\Delta \beta/\beta$ are simply linear combinations of $R_p$ and $G$.

Next, we can derive Pseudo-Poisson’s ratio reflectivity from $R_p$ and $G$. Note that:

$$
\frac{\Delta \sigma}{\sigma} = \frac{\Delta \alpha}{\alpha} - \frac{\Delta \beta}{\beta}
$$

$$
\begin{bmatrix}
8 & 3 \\
5 & 5
\end{bmatrix}
= \begin{bmatrix}
R_p \\
R_p - G
\end{bmatrix}
$$

(24)
Finally, let us derive the “fluid factor”, $F$.

$$\Delta F = \frac{\Delta \alpha}{\alpha} - 0.58 \frac{\Delta \beta}{\beta}$$  \hspace{1cm} \text{(if we assume } \beta/\alpha = 1/2)$$

$$= \frac{8}{5} R_p - 0.58 \left[ \frac{3}{5} R_p - G \right]$$

$$= 1.252 R_p + 0.58G \quad \text{(25)}$$

The past few pages have presented a bewildering array of approximations. However, each one has been reduced to the form:

$$X = aR_p + bG$$

The following table is a summary of all these approximations:

<table>
<thead>
<tr>
<th>Term</th>
<th>$a$</th>
<th>$b$</th>
<th>Approximation to Aki/Richards</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_p$</td>
<td>1</td>
<td>0</td>
<td>Drop 3rd term</td>
</tr>
<tr>
<td>$G$</td>
<td>0</td>
<td>1</td>
<td>Drop 3rd term</td>
</tr>
<tr>
<td>$R_s$</td>
<td>0.5</td>
<td>-0.5</td>
<td>Assume $\beta/\alpha = 1/2$</td>
</tr>
<tr>
<td>$\Delta \sigma$</td>
<td>4/9</td>
<td>4/9</td>
<td>Assume $\sigma = 1/3$</td>
</tr>
<tr>
<td>$\Delta \alpha/\alpha$</td>
<td>1.6</td>
<td>0</td>
<td>$\rho = a\alpha^{1/4}$</td>
</tr>
<tr>
<td>$\Delta \beta/\beta$</td>
<td>0.6</td>
<td>-1</td>
<td>$\rho = a\alpha^{1/4}$</td>
</tr>
<tr>
<td>$\Delta \sigma/\sigma$</td>
<td>1</td>
<td>1</td>
<td>$\rho = a\alpha^{1/4}$</td>
</tr>
<tr>
<td>$\Delta F$</td>
<td>1.252</td>
<td>0.58</td>
<td>$\rho = a\alpha^{1/4}$, $\beta/\alpha = 1/2$</td>
</tr>
</tbody>
</table>
Transforming From the Offset to the Angle Domain

As we have discussed, both Zoeppritz equations and Shuey’s equation (which is an approximation to the Zoeppritz equation) are dependent upon the angle of incidence at which the seismic ray strikes the horizon of interest.

However, we record seismic data as a function of offset. While offset and angle are roughly similar, there is a nonlinear relationship between them, which must first be accounted for in processing and analysis schemes which require that angle be used instead of offset. We term this type of analysis AVA (amplitude versus angle) rather than AVO. An example of such a transform is shown in Figure 12.

![Figure 12. (a) shows AVO response and (b) shows transform of (a) in AVA (amplitude versus angle) response.](Western Geophysical)
An offset gather is shown in Figure 12(a), and the equivalent angle gather is shown in Figure 12(b). At the top of each gather is shown a schematic of the raypath geometry assumed for the reflected events in a particular trace of each gather. Notice that the angle of incidence for a constant offset trace decreases with depth, whereas the angle remains constant with depth for a constant angle trace. The operation for computing will transform is quite straightforward and will be discussed in the next paragraph.

To transform from constant offset to constant angle, we need to know the relationship between $X$ and $\theta$. For a complete solution, a full ray tracing must be done. However, a good approximation is to use straight rays. In this case we find that:

$$\tan \theta = \frac{X}{2Z}$$

(26)

where:

$\theta$ = angle of incidence

$X$ = offset

$Z$ = depth

If we know the velocity down to the layer of interest, we can write:

$$Z = \frac{V_{t_0}}{2}$$

(27)

where:

$V$ = velocity (RMS or average)

$t_0$ = total zero-offset traveltme

Substituting equation (27) into (26) gives:

$$\tan \theta = \frac{X}{V_{t_0}}$$

(28)

which gives us the mapping from offset to angle. By inverting equation (28), we get the mapping from angle to offset:

$$X = V_{t_0} \tan \theta$$

(29)
Equation (29) thus allows us to map the amplitudes on an offset gather to amplitudes on an angle gather. Figure 13 shows a theoretical set of constant angle curves superimposed on an offset versus time plot, where the following velocity relationship has been used:

\[
V = V_0 + kt
\]  

where:  
\[
V_0 = 1000 \text{ m/s} \\
k = 100 \text{ m/s}
\]

Figure 13 shows curves for four different angles: 5, 10, 20 and 30 degrees. Notice that these curves increase to larger offsets at deeper times. This means that a constant angle seismic trace would contain amplitudes collected from longer offset on the AVO gather as time increases.

![Figure 13. A plot of constant angle curves superimposed on constant offset traces.](image)

The preceding equations are strictly valid only for a single layer. An approximation that can be used for the multi-layer case involves using the ray parameter \( p \) and total traveltime \( t \), where:

\[
p = \frac{\sin \theta}{V_{\text{INT}}}
\]  

\[
t^2 = t_0^2 + \frac{X^2}{V_{\text{RMS}}^2}
\]
In the above equations,
\[ V_{\text{INT}} = \text{interval velocity for a particular layer} \]
\[ V_{\text{RMS}} = \text{RMS velocity down to the layer} \]

Note also that \( p \) and \( t \) are related by the equation:
\[
\frac{dt}{dx} = p \tag{33}
\]

By substituting equation (33) into (32), we get:
\[
X = \frac{p}{t V_{\text{RMS}}^2} \tag{34}
\]

By substituting equation (34) into (31), we get:
\[
sin\theta = \frac{X V_{\text{INT}}}{t V_{\text{RMS}}^2} \tag{35}
\]

To see that equation (35) reduces to equation (28) for the single-layer case, refer to Figure 14. Notice that:
\[
t_0 = t \cos \theta \tag{36}
\]

Thus, by substituting equation (36) into (35), and noting that \( V_{\text{RMS}} = V_{\text{INT}} = V \) for a single layer, we see that:
\[
\frac{\sin \theta}{\cos \theta} = \tan \theta = \frac{X}{V t_0} \tag{37}
\]

![Figure 14. Raypath geometry for a single shot-receiver pair in a constant velocity medium.](image-url)
Detailed Implementation of Gradient Analysis

Given an input CDP gather, $R(t, x)$, we assume that for each sample, $t$, the data is given by:

$$R(t, x) = R_p(t) + G(t)\sin^2\theta(t, x)$$

(38)

In equation (38), which is Shuey’s approximation to the Zoeppritz equations, $\theta(t, x)$ is the incident angle corresponding to the data sample recorded at $(t, x)$.

Expressing equation (34) in terms of $t_0$, we get:

$$\sin\theta = \frac{x}{\sqrt{x^2 + \frac{t_0^2}{V_{RMS}^2}}}$$

(39)

Now that we have the relationship between $x$ and $\theta$ in the general case, we can consider how to derive this information to form a CDP gather.

For any given value of zero-offset time, $t_0$, assume that $R$ is measured at $N$ offsets.

$$x_i, \quad i = 1, N$$

From equation (38), we can write the defining equation for this time as:

$$\begin{bmatrix}
R(x_1) \\
R(x_2) \\
\vdots \\
R(x_N)
\end{bmatrix} = \begin{bmatrix}
1 & \sin^2\theta(t, x_1) \\
1 & \sin^2\theta(t, x_2) \\
\vdots & \vdots \\
1 & \sin^2\theta(t, x_N)
\end{bmatrix} \begin{bmatrix}
R_p(t) \\
G(t)
\end{bmatrix}$$

(40)

This matrix equation ($B = AC$) represents $N$ equations in two unknowns. The least-squares solution to equation (40) is:

$$C = (A^T A)^{-1}(A^T B)$$

(41)
where: 

$$A^TA = \begin{bmatrix} N & \sum_{i=1}^{N} \sin^2 \theta(t, x_i) \\ \sum_{i=1}^{N} \sin^2 \theta(t, x_i) & \sum_{i=1}^{N} \sin^4 \theta(t, x_i) \end{bmatrix}$$

$$A^TB = \begin{bmatrix} N \\ \sum_{i=1}^{N} R(x_i) \\ \sum_{i=1}^{N} R(x_i) \sin^2 \theta(t, x_i) \end{bmatrix}$$

If we define the following constants:

$$S2 = \sum_{i=1}^{N} \sin^2 \theta(t, x_i)$$

$$S4 = \sum_{i=1}^{N} \sin^4 \theta(t, x_i)$$

$$SD = \sum_{i=1}^{N} R(x_i)$$

$$S2D = \sum_{i=1}^{N} R(x_i) \sin^2 \theta(t, x_i)$$

We have to solve:

$$\begin{bmatrix} N & S2 \\ S2 & S4 \end{bmatrix} \begin{bmatrix} R_P \end{bmatrix} = \begin{bmatrix} SD \\ S2D \end{bmatrix}$$

(42)
The solution is:

\[
G = \frac{S^2SD - NS^2D}{(S^2)^2 - NS^4} \quad (43)
\]

\[
R_p = \frac{S^2S2D - S^4SD}{(S^2)^2 - NS^4}
\]

Also, we can write \( G \) in terms of \( R_p \) as:

\[
G = \frac{(S2D - S^2R_p)/S^4}{(S^2)^2 - NS^4} \quad (44)
\]

To add a pre-whitening term to the AVO gradient analysis just discussed, modify the original defining equation to be:

\[
\begin{bmatrix}
R(x_1) \\
R(x_2) \\
\vdots \\
R(x_N)
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\begin{bmatrix}
sin^2 \theta (x_1, t) \\
sin^2 \theta (x_2, t) \\
\vdots \\
sin^2 \theta (x_N, t)
\end{bmatrix}
= \begin{bmatrix}
R_p \\
G \\
\vdots \\
0
\end{bmatrix}
\quad (45)
\]

We have added \( M \) equations, all of which state that \( G = 0 \). The only change to the solution is:

\[
S^4 = \sum_{i=1}^{N} \sin^2 \theta (x_i, t) + M \quad (46)
\]
Cross Plotting AVO Attributes

Cross Plotting Well Log Attributes

The idea of cross plotting AVO attributes, such as intercept and gradient, has been discussed by a number of authors in the literature.

In Windhofer (1993), the attribute cross plot technique is applied to the Campbell-2 well from the Campbell field near Barrow Island, Australia. The approach was to build a synthetic model using the well logs from the Campbell-2 well and then derive attributes from the model using the equation:

\[
R(\theta) = R_P + G \sin^2 \theta
\]  

where:  
\(R_P\) = Intercept or true P-wave reflectivity  
\(G\) = gradient

Figure 15 shows their results from the well, where the small squares indicate the wet trend in the well, and the asterisks indicate the hydrocarbon column values. Also, notice that the “wet trend” appears to fall on a straight line.

Figure 16 shows the results of a model study that was done to “calibrate” the well values. On the cross plot, the models are indicated by:

- TW – Top water sand
- TG – Top gas sand
- BG – Base gas sand

Notice that the water sands fall on the original wet trend line quite well. Surprisingly, the gas sands from the Campbell well are even more anomalous than the models would indicate.
Figure 15. Campbell-2 Cross plot of intercept vs. gradient, 1500 ms to 1700 ms with points within the hydrocarbon column shown by an asterisk and the interpreted “wet trend” shown by dashed line.

(Windhofer et al)

Figure 16. Campbell-2 models. Cross plot of intercept vs. gradient for the top gas sandstone models (TG1-TG4), the base gas sandstone models (BG1-BG4) and the top water sandstone models (TW1-TW4). The top gas sandstone and oil-water contact points and the “wet trend” for Campbell-2 are replotted for comparative purposes.

(Windhofer et al)
Cross Plotting Seismic and Log Attributes

The next logical step is to add real data to the plot, and this is described in Foster (1993). They also derive an equation that shows why a linear fit should be observed over the wet trend. In their derivation, they made use of Castagna’s mud-rock equation (shown here after differentiation):

\[ \Delta V \rho = \alpha \Delta V_S \]  \hspace{1cm} (2)

and also of Gardner’s equation (again shown after differentiation):

\[ \frac{\Delta \rho}{\rho} = \beta \frac{\Delta V_P}{V_P} \]  \hspace{1cm} (3)

This leads to the following relationship (starting with the Aki-Richards approximation):

\[
G = R_P \left[ 1 + \beta \frac{V_P}{\rho} \right] \left[ 1 - \left( \frac{4\beta V_P}{\rho} \right) + \frac{8V_P}{\alpha V_S} \left( \frac{V_S}{V_P} \right)^2 \right]
\]  \hspace{1cm} (4)

(Note that this paper uses A for \( R_P \) and B for G.)

Thus, in the absence of hydrocarbons, a linear trend is predicted by the Aki-Richards equation and is observed on well logs. Figure 17 shows this fit on a set of blocked values taken from a well log. Also shown are anomalous values from a gas sand in the well, and a straight line fit to the wet trend.

![Figure 17: The result is similar to that shown in Figure 16, but the values are derived using blocked logs.](Foster et al)
Foster et al then plotted values taken from field seismograms, as shown in Figure 18. Notice that the trend here is not nearly as obvious. However, the anomalous points, shown as triangles, are obvious in the top right quadrant of the cross plot.

![Figure 18](image)

**Figure 18.** The values of A and B derived from field seismograms are shown. These data come from a seismic line that crosses over the well used in deriving Figures 16 and 17. *(Foster et al)*

### Cross-Sections Derived From Cross Plots

The final analysis step was taken by Verm and Hilterman (1994). One of the problems with the cross plots shown previously was that it is hard to know where the points are coming from on the cross plot. Of course, you could color code them. This works for a single well trace but is difficult for seismic data in which the window is defined in both CDP and time. Verm and Hilterman’s idea was to identify anomalous points on the cross plot and then redisplay these points in seismic cross-section form. This is best illustrated using an example from their paper. Keep in mind that they use a slightly modified form of equation (1):

\[
R(\theta) = R_p \cos^2 \theta + \frac{\Delta \sigma}{(1-\sigma)^2} \sin^2 \theta
\]

where:
- \( R_p \) = Normal Incidence Reflectivity
- \( \Delta \sigma \) = Poisson Reflectivity
- \( (1-\sigma)^2 \) = Poisson Reflectivity

AVO September 2004
Figure 19 shows a cross plot of these two attributes.

One final use of the cross plot is discussed by Verm and Hilterman. They show that they can turn a class 2 anomaly into a class 3 anomaly. Before showing this, let us review Rutherford and Williams’ classification scheme:

<table>
<thead>
<tr>
<th>Anomaly Type</th>
<th>Acoustic Impedance Change</th>
<th>Poisson’s Ratio Change</th>
<th>Product Section $R_p^*G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1</td>
<td>Large Increase</td>
<td>Large Decrease</td>
<td>Negative top and base</td>
</tr>
<tr>
<td>Class 2</td>
<td>Small Change</td>
<td>Large Decrease</td>
<td>Indeterminate</td>
</tr>
<tr>
<td>Class 3</td>
<td>Large Decrease</td>
<td>Large Decrease</td>
<td>Positive top and base</td>
</tr>
</tbody>
</table>
Notice that a class 2 anomaly cannot be displayed using a product ($R_p^*G$) display. However, if a rotation of the cross plot is done, the result is to convert it into a class 3 type anomaly.

Figure 20.  
(Verma and Hilterman)
The Common Offset Stack

One of the main ways in which the Hampson-Russell AVO program differs from other AVO programs is in its ability to perform inversion. Inversion involves creating a forward model and comparing this forward model to an observed seismic gather. The model is then updated to create a better fit with the real data. Although the observed seismic gather could be a single CMP gather, it is more normal to use a common offset stack for the analysis. The theory behind the common offset stack is given in the next paragraph.

The common offset stack is also given other names, such as the super gather, the Ostrander gather, and the COS (which stands for Common Offset Stack). The basic concept of the common offset stack is to stack traces within a box that is defined by an offset range, and a CMP range. To see this, refer to Figure 21, which shows a standard stacking diagram with the geometry for a common offset stack superimposed. In this particular stacking diagram, we have shown a 24-trace split-spread profile, in which we are rolling into and rolling out of the spread. Notice that the common offset stack is defined by a CMP window that is 11 traces wide and 6 individual offset ranges. Traces on both sides of the split are stacked together. In this simple example, only a 6-trace common offset stack will result. However, with more realistic geometries, any number of traces can be reproduced in the common offset stack.

The big advantages of the common offset stack are that signal to noise ratio is improved, while, at the same time, the offset dimension is preserved. Therefore, any amplitude versus offset anomalies can be seen more clearly. The dangers involved in the common offset stack are two-fold:

1. We may use too many traces in the CMP range, and therefore smear over structure.

2. We may use too many traces with the offset range, and therefore distort the amplitude response.

It is therefore important to try various common offset stacks before finalizing the geometry over which to model.
Figure 21.

Note: Offsets with same number are stacked together.
**AVO MODELLING**

**Introduction**

One of the main tasks in the use of AVO is to generate a synthetic seismogram from a given earth model. That seismogram may then be compared with a real dataset and conclusions may be drawn from the correspondence (or lack of correspondence) between the two. Forward modelling always involves a trade-off between the complexity of the modelling system and the computer time required to calculate the synthetic. The model complexity is a function of both the detail in the earth model and the algorithm used. The following sections describe the modelling algorithm in AVO.

**One-Dimensional Earth Model**

The earth model used in AVO is a one-dimensional earth model. This means that the earth is assumed to consist of a series of flat layers as shown in Figure 22:

![Figure 22](image_url)
Each layer, \( i \), is characterized by four parameters:

\[
D_i = \text{the thickness of layer } i \\
V_{Pi} = \text{P-wave velocity of layer } i \\
V_{Si} = \text{S-wave velocity of layer } i \\
\rho_i = \text{density of layer } i
\]

Note that if there are \( N \) layers, there must be \( N \) thicknesses and \( N+1 \) values of P-wave velocity, S-wave velocity and density.

A synthetic seismogram consists of a series of traces that represent the effect of recording seismic data over the one-dimensional earth model. Each trace is calculated independently, assuming that both the source and receiver are at the surface of the earth.

The convolutional model may be used to describe the behavior of each trace:

\[
T(t) = \sum r(i)W(t-\tau_i) \quad (1)
\]

where:

- \( T(t) \) = the recorded seismic trace at receiver \( R \) as a function of recording time
- \( W(t) \) = the source wavelet or impulse imparted into the earth at \( S \)
- \( \Sigma \) = the sum of all possible reflection paths from \( S \) to \( R \) through the earth model. For example, path 1 in Figure 23 shows a primary reflection from the fourth interface, while path 2 shows a peg-leg multiple path. In theory, there are an infinite number of combinations for any number of layers, if multiples are taken into account. If multiples are ignored, as is the case in AVO, the number of paths is exactly equal to the number of layers \( N \).
- \( r(i) \) = the effective reflection coefficient for path \( i \). This may include the effect of transmission losses as well as the actual reflection coefficient at the deepest interface. This will be dependent on the angle of incidence at the reflecting interface.
- \( \tau_i \) = the total travel-time for this path. This will be a function of the thicknesses of the layers traversed by the path as well as the P-wave velocities in those layers.
In order to handle the possibility that the travel-time, \( t_i \), may be a non-integral number of time samples, the frequency domain representation of (1) is used:

\[
T(f) = \sum r(i) w(f) e^{j2\pi f t_i}
\]  

(2)

Thus, the forward modelling of a trace may be summarized as follows:

1) Determine all acceptable paths from source to receiver.

2) For each path, calculate the effective reflection coefficient \( r(i) \) and the travel time \( t_i \).

3) Form the complex sum shown in equation (2).

4) Do the inverse Fourier transform to get the final trace.

The path calculation requires ray tracing, and this is described in the next section.
Ray-Tracing

The ray-tracing problem in AVO may be stated as follows: for a given source and receiver location, and a given reflection layer, find the raypath which connects the source and receiver, while obeying Snell’s Law in each layer.

The following derivation follows that of Dahl and Ursin (1991). Define the ray-parameter, \( p \), as:

\[
p = \frac{\sin \theta_1}{V_{P1}} \quad (3)
\]

where \( \theta_1 \) is the emergence angle for energy from the source and \( V_{P1} \) is the P-wave velocity of the first layer. Snell’s Law tells us that for any layer

\[
p = \frac{\sin \theta_i}{V_{P_i}} \quad (4)
\]

If we know the emergence angle, we can calculate the offset, \( y \), at which the reflected energy will come back to the surface:

\[
y(p) = \sum_k \frac{D_k V_k p}{(1 - V_k^2 p^2)^{1/2}} \quad (5)
\]

In this equation, \( D_k \) is the thickness of the \( k \)th layer, and the sum is over all layers down to the reflecting interface.

In the ray-tracing problem, we know that the desired offset is \( y_d \), the distance between the source and receiver for the trace in question. The problem is to find the value of \( p \) that gives the correct value of \( y \) in equation (5):

\[
y_d = \sum_k \frac{D_k V_k p}{(1 - V_k^2 p^2)^{1/2}} \quad (6)
\]

Unfortunately, there is no explicit solution to this problem. The approach used in AVO is an iterative solution, sometimes called a “shooting method”. The first step is to make a guess at the correct value of \( p \), say \( p_0 \). With this value of \( p \), we can calculate the initial value of \( y \), from equation (5):

\[
y_0 = \sum_k \frac{D_k V_k p_0}{(1 - V_k^2 p_0^2)^{1/2}} \quad (7)
\]
Usually, $y_0$ will not be the same as the desired value $y_d$. The error is:

$$\Delta y = y_d - y_0$$  \hspace{1cm} (8)

An improved value for $p$ is then given by:

$$p' = p_0 + \frac{\partial p}{\partial y} \Delta y$$  \hspace{1cm} (9)

where $\partial p/\partial y$ is calculated from equation (5) as:

$$\frac{\partial p}{\partial y} = \begin{bmatrix} \frac{-1}{D_k V_k} \\ \sum_k \frac{(1 - V_k^2 p_0^2)^{3/2}}{k} \end{bmatrix}^{-1}$$  \hspace{1cm} (10)

The new value, $p'$, when used in equation (5) should give a value of $y$ closer to the desired $y_d$ than the initial guess. Ray tracing consists of continually applying equations (7) to (10) until a “convergence condition” is reached.

The convergence condition in AVO is reached when either of the following is true:

1) The maximum number of iterations is reached. For synthetic modelling in AVO, the maximum number of iterations is set to 100. This value should never be reached under normal circumstances. If it is, an error message will be printed on the console.

2) The offset error in equation (8) becomes “small enough”. In AVO, the tolerance is set to the greater of 1% of the desired offset or 1 distance unit. If, for example, the desired offset is 1000 m, the iterations will stop when the error is less than 10 m. If the desired offset is 10 m, the iterations will stop when the error is less than 1 m.

The algorithm described by equations (7) to (10) is so efficient that for the vast majority of cases, convergence is reached in two or three iterations. Unfortunately, certain pathological conditions can occur from time to time, which cause the algorithm to “spin its wheels”. The worst case occurs when a single layer between the surface and the target interface has a P-wave velocity much higher than average. In this case, the propagation angle $\theta_i$ in equation (4) is close to $90^\circ$. This also means that the ray-path is close to critical but not quite. Instability can then arise, which causes the error, $\Delta y$, to converge very slowly. The AVO program will then appear to take an excessively long time to produce a synthetic. In extreme cases, the maximum number of iterations will be exceeded and an error message printed out.
A typical cause of anomalously high velocity layers is, in fact, errors in the log. In an attempt to reduce the impact of such errors from depths above the target zone, AVO applies a uniform blocking for those layers above the target zone before ray tracing. For more detail on this, see the section entitled “Target Zone”.

Zoeppritz Equations

After the ray path has been determined for reflection from a particular interface, the amplitude of the reflected energy is calculated using the Zoeppritz equations. These equations determine the transformation of an incident plane wave upon striking a plane reflector. As shown in Figure 24, an incident p-wave produces four resulting waves, consisting of two reflected waves and two transmitted waves.

In the AVO program, only the reflected p-wave amplitude is used in the calculated synthetic. This means that the source is assumed to emit p-wave energy, the receivers detect only p-wave energy, and all converted waves are negligible in their effect.

Numerous versions of the Zoeppritz equations have been published in the literature. The AVO program uses two formulations. The formulation of Dahl and Ursin (1991) gives explicit values for the reflected p-wave and transmitted p-wave under the assumption that the angle of incidence is less than the critical angle, i.e., \( \sin \theta_i < \frac{V_{p1}}{V_{p2}} \).
The Dahl and Ursin algorithm is reproduced below:

The PP reflection coefficient is given by:

\[ r_{PP} = \frac{N_r}{D} \]

where:

\[ N_r = q^2p^2P_1P_2P_3P_4 + \rho_1\rho_2(V_{s1}V_{p2}P_1P_4 - V_{p1}V_{s2}P_2P_3) \]
\[ - V_{p1}V_{s1}P_3P_4Y^2 + V_{p2}V_{s2}P_2X^2 - V_{p1}V_{p2}V_{s1}V_{s2}p^2Z^2 \]

\[ D = q^2p^2P_1P_2P_3P_4 + \rho_1\rho_2(V_{s1}V_{p2}P_1P_4 + V_{p1}V_{s2}P_2P_3) \]
\[ + V_{p1}V_{s1}P_3P_4Y^2 + V_{p2}V_{s2}P_2X^2 + V_{p1}V_{p2}V_{s1}V_{s2}p^2Z^2 \]

while the PP transmission coefficient is given by:

\[ t_{PP} = \frac{N_t}{D} \]

where:

\[ N_t = 2V_{p1}\rho_1P_1(V_{s2}P_2X + V_{s1}P_4Y) \]

and where the denominator \( D \) is the same as for the reflection coefficient.

The expressions \( q, X, Y, Z, P_1, P_2, P_3, \) and \( P_4 \) are given by:

\[ q = 2(\rho_2V_{s2}^2 - \rho_1V_{s1}^2) \]
\[ X = \rho_2 - qp^2 \]
\[ Y = \rho_1 + qp^2 \]
\[ Z = \rho_2 - \rho_1 - qp^2 \]
\[ P_1 = (1 - V_{p1}^2p^2)^{1/2} \]
\[ P_2 = (1 - V_{s1}^2p^2)^{1/2} \]
\[ P_3 = (1 - V_{p2}^2p^2)^{1/2} \]
\[ P_4 = (1 - V_{s2}^2p^2)^{1/2} \]

The indices 1 and 2 for the model parameters correspond to the layers including the incident wave and the transmitted wave for a given interface, respectively.
Note that the previous equations provide only two of the four possible components, i.e. reflected and transmitted p-wave. Where four components are required, (e.g. to produce the AVO Curves plot) or if the incidence angle, $\theta_i$, exceeds the critical angle, the formulation of Schoenberg and Protazio (1992) is used.

The Schoenberg and Protazio algorithm is summarized below:

Let:
\[
\begin{align*}
\rho &= \text{density of upper layer} \\
\alpha &= \text{P-wave velocity of upper layer} \\
\beta &= \text{S-wave velocity of upper layer} \\
\rho' &= \text{density of lower layer} \\
\alpha' &= \text{P-wave velocity of lower layer} \\
\beta' &= \text{S-wave velocity of lower layer} \\
S_1 &= \text{slowness} = \frac{\sin \theta}{\alpha}
\end{align*}
\]

Calculate:
\[
\begin{align*}
S_{3P} &= \left[(\alpha^2 - S_1^2)\right]^{1/2} \\
S_{3S} &= \left[(\beta^2 - S_1^2)\right]^{1/2} \\
\Gamma &= 1 - 2\beta S_1^2
\end{align*}
\]

Define the 2 x 2 matrices:
\[
X = \begin{bmatrix}
\alpha S_1 & \alpha S_{3S} \\
-\rho \alpha + 2\rho \beta S_1 S_{3S} & S_1 S_{3S}
\end{bmatrix}
\]
\[
Y = \begin{bmatrix}
-2\rho \alpha \beta S_1 S_{3P} & -\rho \beta \Gamma \\
\alpha S_{3P} & -\beta S_1
\end{bmatrix}
\]

Similarly, define the analogous quantities $S_1'$, $S_{3P}'$, $S_{3S}'$, $\Gamma'$, $X'$, $Y'$ by replacing $\rho$, $\alpha$, $\beta$ by the corresponding primed values.

From these definitions, it can be shown that the reflection and transmission matrices can be derived as:
\[
\begin{align*}
T &= 2Y^{-1}X(X^{-1}X'Y^{-1}Y + \Gamma)^{-1} \\
R &= (X^{-1}X'Y^{-1}Y + \Gamma)^{-1}(X^{-1}X'Y^{-1}Y - \Gamma)
\end{align*}
\]
As described in the original paper, special handling is required when the term \((\phantom{0})^{-1}\) is singular.

The 2 x 2 matrices \(R\) and \(T\) are:

\[
R = \begin{bmatrix}
R_{PP} & R_{PS} \\
R_{SP} & R_{SS}
\end{bmatrix}
\quad T = \begin{bmatrix}
T_{PP} & T_{PS} \\
T_{SP} & T_{SS}
\end{bmatrix}
\]

From these matrices, we can get the four components:

\[
R_{PP} = \text{P-wave reflection coefficient} \\
R_{PS} = \text{S-wave reflection coefficient} \\
T_{PP} = \text{P-wave transmission coefficient} \\
T_{PS} = \text{S-wave transmission coefficient}
\]

The other four components, not used by AVO, describe the transformation of an incident S-wave.

The Schoenberg and Protazio formulation is more time consuming to calculate, but has the advantage that post-critical reflections may be handled. A post-critical reflection occurs when \(S_{3P}\) or \(S_{3S}\) becomes imaginary. In that case, complex arithmetic must be used all through the algorithm. The resulting coefficient contains not only an amplitude, but a phase component. Of course, for pre-critical reflection, both formulations give the same result.

The Zoeppritz equations calculate the effect on the event amplitude of reflection from the target interface. Several other factors may modify the actual measured energy. One of these is transmission losses. This effect is described in the next section.

**Transmission Loss**

Transmission loss refers to the gradual loss of energy by the incident wavelet as it propagates through the series of layers above the target interface. The Zoeppritz equation allows us to calculate these transmission coefficients for each layer and apply them as a product.

In Figure 25, the effective amplitude measured at the receiver is:

\[1 \ast t_1 \ast t_2 \ast r \ast t_3 \ast t_4\]

Notice that the transmission coefficient for propagating down through an interface is generally not the same as for propagating up.

Usually the transmission component \(t_1 \ast t_2 \ast \ldots\) has a far smaller tendency to change with offset than the reflection coefficient itself. This is why it is often reasonable to neglect the effect of transmission losses in AVO.
Figure 25.

Figure 26.
To properly calculate and apply transmission losses could be an enormously time-consuming process. This is because each reflection coefficient contains a product of transmission coefficients from shallower events. In order to keep the calculation time reasonable AVO makes the approximation shown in Figure 26.

When the reflection coefficient, \( r_2 \), is calculated using the Zoeppritz equations, the transmission coefficient, \( t_2 \), is calculated as well. Unfortunately, what is required in order to correct the next reflection coefficient, \( r_3 \), is the transmission coefficient \( t_2' \), calculated at a slightly different angle. The approximation made in AVO is to use \( t_2 \) instead of \( t_2' \). This means that by performing the calculation downward through the layers, the cumulative product \( t_1 \cdot t_2 \cdot \ldots \) can be calculated very efficiently. However, the consequence is that an error is being made. In AVO, we are usually interested in the effect of an offset-dependent transmission loss from one layer on a deeper target layer. If these layers are not too far apart, say less than several hundred milliseconds in travel-time, the AVO approximation will be excellent. As the layers become farther removed in travel-time, the effect of the approximation will be to over-emphasize the shallow transmission loss, since larger angles are being used than necessary.

If a Target Zone is being used in the AVO program, the calculated transmission coefficients for layers outside the target zone will be given by the zero-offset equation: \( t = 1 - r \). For more information on the Target Zone, refer to the next section.

**Target Zone**

Since AVO is performing its synthetic calculation for each depth layer in the model, calculation times could be very large. A typical well log could have thousands of depth layers. While log blocking can be used to reduce the effective number of layers, it is usually inconvenient to do this over the entire log.

The solution to this problem in AVO is to define what is called a “Target Zone”. The Target Zone is a depth range. Any model interfaces within this zone are assumed to be important and these effects are calculated exactly. Any model interfaces outside this zone are assumed to be less important and approximations are used.

Referring to the synthetic defining equation (2), the sum over interfaces is broken into two parts:

\[
T(f) = \sum_{i=\text{inside target}} + \sum_{i=\text{outside target}}
\]
Each term within the summation depends on two parameters which must be calculated: $\tau_i$, the arrival time, and $r(i)$, the effective reflection coefficient. Each of these parameters is handled differently, depending on whether or not the interface is inside the target zone:

**Inside Target Zone:** Calculate $\tau_i$ and $r(i)$ exactly as described in the previous sections. Notice that the calculation of $\tau_i$ requires ray-tracing through all layers above the current layer $i$, not just those within the Target Zone. For this reason, the calculation of $\tau_i$ and $r(i)$ are exact.

**Outside Target Zone:** Calculate $\tau_i$ using the Dix approximation:

$$\tau_i = \left[ \tau_i^{o2} + \left( \frac{OFFSET}{V_{RMS}} \right)^2 \right]^{1/2}$$

where:

- $\tau_i^{o}$ = the zero-offset two-way travel-time to this layer
- $OFFSET$ = source/receiver distance
- $V_{RMS}$ = Root-mean-square velocity to this layer

Calculate $r(i)$ using the zero-offset equation:

$$r(i) = \frac{Impedance(i) - Impedance(i-1)}{Impedance(i) + Impedance(i-1)}$$

Effectively, the portion of the synthetic calculated using layers outside the Target Zone is an extension of the conventional zero-offset synthetic.

The objective in using the Target Zone is to create a synthetic that is exact over the region of interest, and “reasonable” outside. Because of wavelet interference, there will be some tendency for the inaccuracies outside the Target Zone to be smeared into the part of the synthetic within the Target Zone. For this reason, the Target Zone should be chosen long enough that the actual layers of interest are at least one wavelet length away from the edge of the Target Zone.

A secondary use of the Target Zone is to compensate for the effect of shallow logging errors on ray tracing. As explained in the section on ray tracing, the iterative algorithm can become unstable if there are large velocity anomalies in the path. Experience has shown that real velocity variations are not enough to cause instability, but errors or “spikes” in the sonic log can be a problem. In order to provide a robust ray-tracing algorithm, AVO temporarily modifies the model for all layers above the Target Zone by performing uniform blocking to a thickness of five depth units. The effect of this blocking is to average out velocity spikes. The blocking is only done for the calculation of $\tau_i$ and the original log is used for all other calculations.
Geometrical Spreading

Geometrical Spreading refers to the decrease in amplitude of the source wavelet as it propagates away from the source point. In AVO, the option exists to include this effect in the calculation of the effective reflection coefficient $r(i)$ in equation (2).

The decision about whether to use the Geometrical Spreading option is not always simple. The main effect of this option on the synthetic is to produce a very significant decrease in amplitude with two-way travel-time and a much less significant decrease with offset. The argument against using this option is that real seismic data has usually been corrected during processing for geometrical spreading. Since the main purpose of creating a synthetic is to compare it with real data, the effect should not be present in the synthetic in that case. On the other hand, it can be informative to see the offset-dependence of geometrical spreading since this is a key issue in AVO.

Ursin and Dahl (1991) give an exact expression for geometrical spreading:

$$ g = \frac{\cos \theta_1}{V_{P1}} \left[ \sum \frac{D_k V_{Pk}}{\cos \theta_k} \sum \frac{D_n V_{Pn}}{\cos^3 \theta_n} \right]^{1/2} \quad (11) $$

The calculated value, $g$, is a correction or multiplier which is applied to the reflection coefficient, $r(i)$. Note that expression (11) is spreading for one-way travel only, and the calculated $g$ must be applied twice to accommodate the return trip from the reflector.

Analyzing expression (11), we can see that there are three terms. The first term, $\frac{\cos \theta_1}{V_{P1}}$, is a constant multiplier for the entire synthetic. The second term, $\sum \frac{D_k V_{Pk}}{\cos \theta_k}$ is precisely equal to $\tau_i$, which is the travel-time for the reflector in question. The third term, $\sum \frac{D_n V_{Pn}}{\cos^3 \theta_n}$
is approximately equal to $V_{\text{RMS}}^2$, where $V_{\text{RMS}}$ is the root-mean-square velocity down to the interface in question.

The equation used by AVO is based on the above analysis and is attributed to Newman (1973):

$$g = \frac{1}{V_{p1}} (\tau_i V_{\text{RMS}}^2)$$  \hspace{1cm} (12)

Note that the simplicity of this expression allows very fast calculation times.

**Array Effect**

The **Array Effect** option in AVO allows you to include the effects on the calculated synthetic of a receiver array or a source array or both. In either case, the array is assumed to be a set of $N$ receiver or source elements, placed with an equal spacing, $d$, on the line between source and receiver.

The response of this array is calculated as follows:

Assume that the wavelet is dominated by a single frequency component, $f$. The energy arriving at the receiver will generally be attenuated by the summation of the $N$ receivers. The **Array Effect** multiplier will then be:

$$R = \frac{1}{N} \frac{\sin(\pi fd/V_a)}{\sin(\pi d/V_a)}$$  \hspace{1cm} (13)

where $V_a$ is the apparent velocity at which a particular phase component travels along the surface from one receiver to the next. This apparent velocity is a function of both the first layer velocity, $V_1$, and the propagation angle, $\theta_1$, at which energy propagates through the first layer. In fact:

$$V_a = \frac{V_1}{\sin \theta_1}$$

Note that for the case of vertical incidence, $\theta_1 = 0$, and $V_a$ becomes infinite. This means there is zero time delay between the recording of energy at each of the recording elements in the array. Also, in that limit $R \to 1$, meaning that there is no attenuation. For non-vertical incidence, corresponding to offset traces, equation (13) produces a multiplier less than 1, which attenuates the measured response.

The **Array Effect** multiplier, $R$, changes with both time and offset, since the angle $\theta_1$ depends on the ray-path. While this effect could be calculated by ray tracing, a more efficient procedure is to use the Dix approximation as follows.
Note that the apparent velocity of a particular event on the seismogram can be written as:

\[ V_a = (dt/dx)^{-1} \]  \hspace{1cm} (14)

where: \( t \) = the measured arrival time for this event at offset \( x \)

But the Dix approximation is:

\[ t^2 = t_0^2 + (x^2/V_{RMS}^2) \]  \hspace{1cm} (15)

Combining (14) and (15) gives:

\[ V_a = t V_{RMS}^2/x \]  \hspace{1cm} (16)

By combining (13) and (16), we get an array multiplier for each trace sample of the trace.

**NMO Correction**

The synthetic generation equation (2) produces a synthetic comparable with the seismogram that would be measured in the field. In particular, the synthetic displays the effects of Normal Moveout (NMO). This is because the arrival times, \( \tau_i \), are the actual offset-dependent travel-paths calculated by ray tracing through the model. Tuning effects produced by the wavelet interference between top and bottom of a thin layer are accurately modelled. Because a frequency domain formulation is used, thin layers are handled as accurately as ray-tracing permits, regardless of the time sample rate of the synthetic.

Often, it is desirable to compare the generated synthetic with real data that has been NMO-corrected. For this reason, the option exists in AVO to apply NMO-correction while the synthetic is being generated. Note that the process used in AVO correctly models the real earth process, i.e. it produces a complete synthetic without NMO-correction, then applies NMO-correction to that synthetic. This means, for example, that the true impact of NMO stretch can be seen on the generated synthetic.

The NMO-correction applied in AVO is slightly more accurate than that applied in most processing flows for the following reason. Most NMO-correction algorithms assume that the reflection time is given by the Dix equation:

\[ \tau_i^2 = \tau_0^2 + x^2/V_{RMS}^2 \]
The correction time that is applied is then:

\[ \tau_i - \tau_0 \]

Since AVO has used ray tracing to calculate the original synthetic, the actual travel times are known. For this reason, the NMO-correction in AVO can be thought of as the “ideal” process without the limitation of the hyperbolic Dix equation.

A problem for any NMO-correction algorithm occurs when the travel time for a shallow reflection exceeds that for a deep reflection at a particular offset. This situation can only arise if there is an increase in velocity with depth. In order for NMO-correction to correct this, it would be necessary to reverse the ordering of events on the trace. This cannot be done because the wavelet has already been convolved with the reflection coefficients. For this reason, it is standard for NMO-correction routines to either mute the trace at the time where events cross or handle the problem in a different way. In AVO, the crossing problem is handled by always applying the smaller correction. This means that an event may appear to be under-corrected at the far offsets if it has crossed another event.

In general, the NMO-correction in the AVO Modelling window should be similar to, but not exactly the same as, the process of sending the uncorrected synthetic to the AVO Processing window and applying NMO correction there. The difference is that the AVO Processing option uses the Dix equation.

**Plane Wave Synthetic**

The NMO-corrected synthetic generated by the process described in the previous section contains the effects of NMO stretch and NMO tuning. These effects can be very significant and may, in fact, exceed the amplitude variations produced by the Zoeppritz equations. Of course, this correctly models what should be seen on real data.

Occasionally, it is useful to produce a synthetic that does not have the NMO effects. In AVO, this is called the Plane Wave Synthetic, and it is generated as follows: In the synthetic generation equation (2), replace \( \tau_i \) (the actual event arrival time) by \( \tau_i^o \) (the zero-offset event arrival time). All other parameters are calculated as described previously. In particular, the effective reflection coefficient, \( r(i) \), contains the Zoeppritz amplitude calculated using the real incidence angle.

The Plane Wave Synthetic has the property that it looks like a real NMO-corrected synthetic, except that there is no NMO stretch. Any amplitude variation with offset is due entirely to the Zoeppritz calculation. Of course, the Plane Wave Synthetic is not like any real dataset that could be measured. However, at any particular time, it is similar to the result of taking an uncorrected CDP gather and correcting it by applying time shifts.


**Elastic Wave Algorithm**

The Zoeppritz algorithm, described in a preceding section, uses ray-tracing to determine the angle at which energy is incident on each interface. From knowledge of the incidence angle and the lithologic properties above and below the interface, the reflection amplitude is calculated using the Zoeppritz equations. This algorithm is strictly accurate for plane wave energy incident on a single interface. In reality, a spherical wave is incident on a collection of interfaces. The main effect of this difference is to create a series of inter-bed multiples and model-converted waves. These waves may interfere with the primary reflections and change the resulting waveform. The Zoeppritz algorithm, because it models the primary energy only, may give an inaccurate result, especially for thin layer models with large impedance contrasts. A particularly good example of this effect is presented in Simmons and Backus (1994).

The Elastic Wave Algorithm attempts to solve these problems by modelling all components simultaneously. Because multiples and mode-converted events are modelled, this algorithm is theoretically exact for the one-dimensional case. The problem is that this algorithm is calculated in the frequency-wave number domain, and artifacts may be generated if the model is insufficiently sampled in this domain. This means that very large CPU times may be required for particular models.

The Elastic Wave Algorithm implemented in AVO is the reflectivity-modelling algorithm. This procedure is explained in detail in Appendix D.

**Sensitivity Analysis**

The Sensitivity Analysis process in AVO is a tool for analyzing how sensitive the generated synthetic is to small changes in the parameters. A typical AVO Modelling session consists of producing the synthetic response to a gas-saturated sand layer. The question then arises: how much would the model change if the Poisson’s Ratio of the sand changes from 0.1 to 0.2? One way of answering this question would be to change the Poisson’s Ratio of the model and generate a second synthetic. Sensitivity Analysis streamlines that process.

Equation (2) describes the general synthetic generation process:

\[ T(f) = \sum r(i)w(f)e^{i2\pi f\tau_i} \]

Assume that we wish to analyze the effect of a particular layer, \( k \). We can write exactly:

\[ T(f) = r(k-1)w(f)e^{i2\pi f\tau_{k-1}} + r(k)w(f)e^{i2\pi f\tau_k} + \sum r(i)w(f)e^{i2\pi f\tau_i} \quad (17) \]

where \( r(k) \) is the reflection coefficient at the base of layer \( k \), and now the sum (\( \sum \)) is over all interfaces except \( k-1 \) and \( k \).
Equation (17) would appear to give us a very efficient way of doing Sensitivity Analysis: calculate the summation once and store the result; then each change of the layer parameters can be effected by the addition of the first two terms. Unfortunately, the catch is that the $r(i)$ and the $\tau_i$ within the summation may be indirectly affected by the change in layer $k$. This occurs if the ray-path changes and that will happen if either the P-wave velocity or layer thickness is changed. On the other hand, changing density or S-wave velocity has no impact on the ray-path.

The Sensitivity Analysis option in AVO makes the following approximation: assume the ray-path does not change when the layer parameter is changed. As discussed above, this will be exact for changes of S-wave velocity or density, and will be a good approximation for small changes of P-wave velocity or layer thickness.

**AVO Inversion**

AVO Inversion is the process of automatically deriving a geological model in such a way that the resulting synthetic seismogram matches the real data seismogram to within some tolerance level. The process of AVO Inversion is a topic of current research and there is as yet no general consensus on how the process should be done or even whether it is valid for standard seismic data. For this reason, the AVO Inversion process in the AVO program should be thought of as a tool to be used in conjunction with other tools in the program, and whose results must be evaluated critically in the light of all other information.

The AVO Inversion algorithm has the following general properties:

1. It is iterative. This means that the algorithm starts with an initial guess model supplied by you, and proceeds to modify it in a series of steps. Each step is guaranteed to produce a synthetic match at least as good as the previous step. The initial guess model can be critical in determining the final result – for instance, it fixes the number of layers.

2. The algorithm is linearized. This means that the possibility exists for the algorithm to become “trapped” in a local minimum. The consequence is that the initial guess needs to be reasonably close to the right answer.

3. The problem is non-unique. This is not a function of the particular algorithm, but rather a consequence of the real data we are using to do the inversion – bandlimited P-wave seismic data. This property re-emphasizes the significance of the initial guess model.

4. The algorithm allows fixed constraints. This means that the user can “guide” the flow of the algorithm by restricting the allowable change for any parameter. This reinforces the significance of the initial guess and the expertise of the user.

The following sections describe the details of the AVO Inversion algorithm.
The Objective Function

The objective function in any inversion process is a mathematical function which measures how “good” a particular solution is. An inversion algorithm always consists of defining a suitable objective function and then minimizing it. The following objective function derivation follows from that of Dahl and Ursin (1991).

Equation (2) allows us to calculate a synthetic trace, $T$, for any given model. The model itself is parameterized by using the following vector:

$$m_0 = [V_{P1}, V_{P2}, ..., V_{PN+1}, V_{S1}, V_{S2}, ..., V_{SN+1}, D_1, D_2, ..., D_N, \rho_1, \rho_2, ..., \rho_{N+1}]$$

where:

- $V_{Pi}$ = P-wave velocity of $i^{th}$ layer
- $V_{Si}$ = S-wave velocity of $i^{th}$ layer
- $D_i$ = thickness of $i^{th}$ layer
- $\rho_i$ = density of $i^{th}$ layer

Let us assume that we have a real-data CDP gather, denoted by the vector $d$:

$$d = \begin{bmatrix}
  d_{11} \\
  d_{12} \\
  . \\
  . \\
  d_{1\text{NSAMP}} \\
  d_{21} \\
  . \\
  . \\
  d_{2\text{NSAMP}} \\
  d_{31} \\
  . \\
  . \\
  d_{N\text{TRACE*NSAMP}}
\end{bmatrix}$$

where $d_{ij}$ = data sample $i$ on $j^{th}$ trace

Note that the CDP gather may have any number of traces, and the vector $d$ is simply formed by writing all the traces one after the other in a long string. The number of elements in $d$ is:

$$\text{NSAMP*TRACE}$$
Note, also, that we make no particular assumption about the form of the CDP gather, except that each trace has the same number of samples. For example, the offsets may be increasing uniformly or they may be randomly ordered, the traces may or may not be NMO-corrected, etc.

Given the initial model, \( m_o \), we can generate a synthetic seismogram, \( d_o \), which has the same general properties as \( d \). This means that it has the same offset distribution in exactly the same order as \( d \). If \( d \) is NMO-corrected, then \( d_o \) will be calculated in the same way. Of course, the vector \( d_o \) has exactly the same number of elements as \( d \).

The fundamental assumption in inversion is that if the model, \( m_o \), is correct -- i.e. a good representation of the earth -- then the synthetic \( d_o \) will be almost the same as \( d \). The only difference between them will be the noise in the measurement of \( d \). The further assumption is that if \( d_o \) and \( d \) are significantly different, then we should modify the model until they become the same. This is expressed by defining the objective function:

\[
J = [d - d_o(m)]^T[d - d_o(m)]
\] (18)

In this equation, \( d - d_o(m) \) is itself a vector of length \( NTRACE \times NSAMP \) and the superscript \( T \) means to take the transpose. Multiplying the transpose of a vector by itself is the sum of the squares of the elements. In other words, \( J \) is the sum of the squared differences between the real data and the synthetic data:

\[
J = \sum_{i=1}^{NSAMP} \sum_{j=1}^{NTRACE} (d_{ij} - d_{0ij})^2
\]

Since \( d_o \) is a function of the model vector, \( m \), the inversion problem consists of finding the value of \( m \) that minimizes \( J \). This is also called least-squares inversion and the process of minimizing \( J \) is described in the following sections.

**Calculating Sensitivity Elements**

Finding the minimum value of \( J \) in equation (18) is complicated by the fact that the synthetic, \( d_o \), is a non-linear function of the model parameters, \( m \). The approach used in AVO is the conjugate-gradient algorithm. This algorithm requires us to calculate the gradient of \( J \), a vector whose elements are the derivatives of \( J \) with respect to each of the model parameters:

\[
\partial J/\partial m_k = \text{the derivative of } J \text{ with respect to the } k^{th} \text{ model parameter}
\]

Minimizing \( J \) is equivalent to finding the point where all the derivatives are zero.
From equation (14), we can write:

\[ J = d^T d - d_0^T d - d^T d_0 + d_0^T d_0 \]

\[ \frac{\partial J}{\partial d_0^T} \]

and

\[ \frac{\partial J}{\partial m_k} = -2 \frac{\partial d_0^T}{\partial m_k} E \]  \hspace{1cm} (19)

where: \[ E = d - d_0(m) = \text{the difference between the real data and the synthetic data} \]

Calculating the gradient elements thus reduces to calculating the derivatives of the synthetic trace with respect to each of the model parameters.

In the time domain, the defining equation for the generation of a synthetic trace is:

\[ d_0(m, t) = \sum_i r_i w(t - \tau_i) \]

where:
- \( w(t) \) = the wavelet used in the synthetic
- \( \Sigma \) = a sum over all reflection interfaces
- \( \tau_i \) = arrival time for the \( i^{th} \) interface

From this, we can write:

\[ \frac{\partial d_0}{\partial m_k} = \frac{\partial r_i}{\partial m_k} \frac{\partial \tau_i}{\partial m_k} \frac{\partial T_i}{\partial m_k} \]

\[ \frac{w(t - \tau_i)}{\partial T_i} \]  \hspace{1cm} (20)

The components \( \frac{\partial r_i}{\partial m_k} \) and \( \frac{\partial \tau_i}{\partial m_k} \) measure the changes in a particular interface reflection coefficient and travel-time as a result of a small change in any particular model parameter. These may be calculated analytically. The details are given in Dahl and Ursin (1991) and are not reproduced here. Note that the calculations are complicated – for example, if \( m_k \) represents the P-wave velocity of a particular layer, then the reflection coefficient change \( \Delta r_i \) must include the effect of a changing ray-path as well as a modification in the Zoeppritz equation. The equations of Dahl and Ursin are exact and they are used in the AVO program.
Returning now to equation (19), remember that the vector $d_0$ was the synthetic for all traces, not just a single trace. Combining (19) and (20), we can write:

\[
\frac{\partial J}{\partial m_k} = \sum_j \left( \sum_i \text{CORR}(\tau_i) - r_i \right) \left( \frac{\partial \text{CORRP}(\tau_i)}{\partial m_k} \right) \tag{21}
\]

where:
- $\Sigma = $ sum over all traces in the CDP gather
- $\Sigma = $ sum over all interfaces in the model
- $\text{CORR}(\tau_i) = \sum_t w(t)E(t + \tau_i)$
- $= $ the cross-correlation between the wavelet and the residual error for a particular trace
- $\text{CORRP}(\tau_i) = \sum_t \frac{\partial w(t)}{\partial t}E(t + \tau_i)$
- $= $ the cross-correlation between the derivative of the wavelet and the residual error for a particular trace

The final effect of all these calculations is to produce the gradient vector:

\[
g = \begin{bmatrix}
\frac{\partial J}{\partial m_1} \\
\frac{\partial J}{\partial m_2} \\
\vdots \\
\frac{\partial J}{\partial m_n}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial J}{\partial V_{P1}} \\
\frac{\partial J}{\partial V_{P2}} \\
\vdots \\
\frac{\partial J}{\partial V_{S1}} \\
\frac{\partial J}{\partial V_{S2}} \\
\vdots
\end{bmatrix}
\]

The number of elements in $g$ is equal to the number of parameters in the model vector, $m_o$, which is equal to four times the number of layers in the model.
It is also possible to calculate $g$ by the finite difference approximation. This consists of starting with a particular model, $m_0$, and calculating its objective function value, $J_0$. Now change one of the parameters, say $V_{S2}$, by a small amount, $\Delta V$, and calculate the new objective function, $J'$. The gradient element, $\frac{\partial J}{\partial V_{S2}}$ is approximately given by:

$$\frac{\partial J}{\partial V_{S2}} \approx \frac{J' - J_0}{\Delta V} \quad (22)$$

In the limit where $\Delta V$ approaches zero, equation (22) should give the same result as the exact expression (21). This procedure has been used in AVO to check the correct programming of the complicated steps used in equation (21).

**The Conjugate-Gradient Algorithm**

Equations (18) and (21) allow us to calculate the two fundamental quantities required to perform inversion, namely, the objective function and its gradient for any particular value of the model vector, $m$. From this point on, the inversion problem is a standard function minimization problem for a non-linear function. There are a number of options for performing that minimization.

The minimization algorithm used in the AVO program is the CONJUGATE-GRADIENT minimization described in detail in Numerical Recipes in C, by Press et al (1988). The particular routines used are:

- **FRPRMN** – Conjugate gradient algorithm using Polak-Ribiere minimization
- **MNBRAK** – Bracket the minimum value. This has been modified to incorporate fixed constraints.
- **LINMIN** – Minimize along a chosen line direction.
- **BRENT** – Minimize along a line direction.

The conjugate-gradient was chosen for the following reasons:

1. It is very efficient for large systems. It requires only the calculation of the gradient and does not require calculating the so-called Hessian matrix, nor does it require solving a large linear system of equations.
2. It is stable. This is particularly important for a non-unique problem like **AVO**. Note that the objective function in equation (18) does not contain any stabilization terms, tending to limit the deviation of the final model from the initial guess. This means that the movement is totally unconstrained. In fact, the constraints are imposed in two ways: explicitly, as described in the next section, and implicitly, by limiting the number of iterations in the conjugate-gradient.

3. It allows the explicit use of fixed constraints, as described below.

The conjugate-gradient algorithm is summarized as follows:

<table>
<thead>
<tr>
<th>Initialize the following vectors:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = m_0 ) Initial model value</td>
</tr>
<tr>
<td>( \nabla I = g ) Gradient at initial model point</td>
</tr>
<tr>
<td>( h = g )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Loop on iterations:</th>
</tr>
</thead>
<tbody>
<tr>
<td>{ Find optimum value along the direction, ( \nabla I ), measured by the distance ( \alpha ) from the current point.</td>
</tr>
<tr>
<td>Update the model</td>
</tr>
<tr>
<td>( x = x + \alpha \nabla I )</td>
</tr>
<tr>
<td>Calculate the new gradient, ( \nabla I )</td>
</tr>
<tr>
<td>Calculate the multipliers</td>
</tr>
<tr>
<td>( \Delta g g = (\nabla I + g) \cdot \nabla I )</td>
</tr>
<tr>
<td>( g g = g \cdot g )</td>
</tr>
<tr>
<td>Calculate the new search direction</td>
</tr>
<tr>
<td>( a = \Delta g g / g g )</td>
</tr>
<tr>
<td>( g = -\nabla I )</td>
</tr>
<tr>
<td>( \nabla I = h = g + \alpha h )</td>
</tr>
<tr>
<td>Go back to the top of the loop</td>
</tr>
<tr>
<td>}</td>
</tr>
</tbody>
</table>

Note that the new search direction is always a linear combination of the current gradient and the previous search direction. In this way, the conjugate-gradient algorithm differs from the pure gradient-descent algorithm.
The algorithm thus consists of continuously finding a new search direction, and then optimizing along that direction. The process is terminated when either of these conditions is reached:

a) The gradient becomes very small, i.e. $g g$ is very small. This effectively means the model fit is as close as it is going to get.

b) The maximum number of iterations is reached. This is the condition actually used in the **AVO** program.

### Use of Fixed Constraints

The conjugate-gradient algorithm described in the previous section does not contain any limitations on the movement of the model vector, $m$, from the initial guess value, $m_o$. This can cause two problems: the non-uniqueness problem may cause erratic or unpredictable results, and the rate of convergence may be very slow. The **AVO** program alleviates these conditions by allowing fixed constraints to limit the movement of the model vector.

The use of fixed constraints means that the conjugate-gradient algorithm is actually initialized by three vectors:

- $m_o = \text{initial guess model vector}$
- $m_L = \text{lower-bound model vector}$
- $m_U = \text{upper-bound model vector}$

Note that each of these vectors is of dimension $4 \times \text{(number of layers)}$. Considering the $i^{\text{th}}$ component of each vector, the final model answer has the property that

$$m_{Li} \leq m_i \leq m_{Ui}$$

Of course, it is assumed that the initial guess value is within the allowed range:

$$m_{Li} \leq m_{oi} \leq m_{Ui}$$

This formulation implies that if any model component has a fixed constraint, then all components must have them, but the constraints for the various components are independent. For instance, the density components may have a very narrow allowed range, while the S-wave velocity components have a very large range. Or, the S-wave component range for one layer may be very different from that of another layer.
The fixed constraints modify the conjugate-gradient algorithm in two ways. First, they are used to scale the parameters:

\[ m_i' = \frac{2m_i}{m_{Ui} - m_{Li}} \]

This is an important step that renders the m-vector dimensionless and means that all parameters vary approximately within the range \( m_{oi} +/- 1.0 \).

The second use of the constraints is to limit the line search along the search direction, \( X_i \), in the conjugate-gradient to the allowable region. A problem may occur if the model vector at some point in the iterations falls exactly on the boundary of the allowed region and the gradient vector is such as to point outside the region. In that case, no further movement would occur. To alleviate this problem, the following rule is applied: If the above condition is encountered, zero out that component within the gradient vector which is normal to the allowable surface.

In summary, the use of fixed constraints has been found very effective in directing convergence along a desired direction.
REFERENCES


APPENDIX A

Seismic Wave Propagation in a Stratified Elastic Half Space
Seismic Wave Propagation in a Stratified Elastic Half Space

Qing Li

Introduction

Elastic wave modeling is designed to simulate the AVO effect of the P-wave reflectivity for a model with complex reflection layering. As demonstrated by Simmons and Backus (1994), mode conversions and multiples through those layers may affect the AVO response at the target, and cause the P-wave reflection to display a completely different AVO response from that of a single elastic interface. For these reasons it is important to model the pre-stack seismic amplitudes and the waveforms using the elastic wave modeling.

It is generally adequate for AVO analysis to simulate the excitation and propagation of seismic waves in a one dimensional, stratified elastic half space. The elastic parameter distribution is only a function of the depth coordinate, and is homogeneous in the lateral direction. This type of earth model can be expressed as a series of homogeneous isotropic layers. The elastic equations of motion may be expressed in a cylindrical system, where the separable solution of the wave equation may be expressed in terms of P-SV and SH components. The solution for the P-SV component may be further reduced by transform techniques to a set of first-order equations. The propagator matrix, which gives the solution of these equations, relates the stresses and displacements at the top and bottom of the layers. The calculation is done in the frequency domain and includes the effects of all possible wave propagation modes. This method allows for the decomposition of the solution by ray type. This ray decomposition is particularly useful in the case of reflection seismology, for it allows the separation of different propagation modes relating multiple reflections and mode conversions. This approach was introduced by Thompson (1950), and extended by many others (see Kennett 1985 for details).

The elastic wave modeling algorithm in AVO follows a series of publications by several authors in the 1970’s, especially the work of Frasier (1970), Kennett and Kerry (1979), and Kennett (1979, 1980). Kennett’s method provides a complete solution for wave propagation in stratified media. The accuracy is limited only by how well a stratified model resembles the real geology. Despite all the theoretical achievements, there are still many difficulties in the numerical implementation. The major difficulty is the numerical instability associated with the presence of evanescent waves within the elastic medium, and the poles that correspond to surface-wave modes in the double transformed solution. These poles lie on the real k-axis, and hence are on the path of integration. Many numerical techniques have been introduced to overcome those difficulties. Interested readers are referred to the references for details.
Description of the layered model

The elastic wave modeling algorithm implemented in AVO assumes a model consisting of \( n \) plane, homogeneous and isotropic layers over a half space. Each layer is characterized by its thickness, density, and the \( P \) and \( S \) wave velocities. In the current implementation, we neglect the effect of the free surface, that is, we assume the first layer extends to the upper half of the space (Figure 1). The seismic source and receivers are buried in the first layer at the location where the surface is assumed to be. There are no surface waves or surface related multiples in the solution. The direct wave is also excluded from the solution because it is of no interest in AVO analysis. We also only consider the situation where shot and receivers are located at the surface.

The model is divided into two zones: the target zone and the overburden. The reason for specifying a target zone is to reduce unwanted reflections from overburden layers and to speed up the computation. For the first purpose, we artificially set all the reflection and transmission coefficients in the overburden to zero except for the \( P-P \) transmission coefficient. This transmission coefficient is kept at the true value if the transmission loss option is selected, and set to one otherwise. For the second purpose we block the overburden layers into approximately 10 blocks. We assume that by using 10 layers in the overburden, the curved ray path will be approximated fairly well. The blocking method can be either Wyllie’s time average or Backus averaging, as specified by the user. These two methods of blocking the model are discussed further in the next section.

The solution of the elastic wave modeling algorithm incorporates the effect of geometrical spreading. If the user wishes to see the synthetic that would have been generated without this attenuation, then the amplitudes must be recalculated to restore the energy that was lost as a result of geometrical spreading. Otherwise, when the synthetic is displayed with geometrical spreading losses shown, the synthetic is derived directly from the solution of the algorithm.
Blocking of the model

As mentioned above, there are two methods used to block the model: Wyllie’s time averaging and Backus averaging. The blocking is applied to both the overburden and the target zone of the model. The overburden will be blocked into ten layers of uniform thickness while the blocking within the target zone is controlled by a minimum layer thickness parameter.

When a layered medium is blocked using Wyllie’s time averaging, the P-wave velocity of the resulting composite layer will be defined so as to maintain the same transit time through that layer as through all of the individual layers taken together. The S-wave velocity and density of the blocked layer will be a P-wave transit-time weighted average of the values for each of the component layers. This method is the same as the one that is used for blocking when a synthetic is calculated from conventional ray-tracing using the Zoeppritz equations.

In contrast, Backus averaging derives the effective elastic constants of the composite layer from a weighted average of the elastic constants of each of the included layers. The weighting in this case is the thickness of a given layer relative to the thickness of all the layers. From the effective elastic constants, we can calculate the P and S wave velocities and density for the blocked layer. This method, then, allows us to replace a number of thin layers with a single layer that has similar elastic properties, although the transit time through the aggregate may not be preserved. For a detailed description of these two methods see Mavko et al (1998).

Within the target zone, the blocking is set by the definition of a minimum layer thickness. When no minimum is specified, the elastic wave modeling algorithm is applied directly to the initial model, which may be either the raw, unblocked logs or externally blocked logs. If the minimum layer thickness is set to a non-zero value, an additional blocking will be applied to layers with a thickness less than this minimum value. When a layer is encountered whose thickness is less than the minimum value, subsequent layers will be accumulated until the total thickness exceeds the minimum, and then a single layer will be created using the selected blocking method. The use of the minimum layer thickness will permit significant reduction in the calculation time for a model. Typically, there will not be a noticeable degradation in the quality of the derived synthetic, provided that the minimum thickness is small relative to the length of the wavelet. A real data example showing the effect of this parameter is included later in this paper.
Source and receivers

We assume a point compressional wave source traveling down from the surface. The receivers at the surface record both the vertical and horizontal components of displacement. Since there is a 90 degree phase difference between pressure and displacement, we apply a 90 degree phase rotation to the displacement in order to match the phase of the wavelet at the receivers to the phase of the wavelet at the source.

While the signs of the vertical and horizontal components of the displacement are well defined according to the coordinate system used, the sign for total displacement is not defined. We define the sign of the total displacement to be the same as the vertical component if the option to calculate the $P-P$ wave response is selected. We define the sign of the total displacement to be the same as the horizontal component if the option to calculate the $P-S$ wave response is selected. Figure 2 displays the sign convention.

The user may also include the effects of source and geophone arrays that attenuate the upcoming waveform. The effect of the array is simulated as a linear filter applied to the synthetic traces.

In the current implementation, the seismic source and receivers are assumed to be located at the surface.

Figure 2. Sign convention for P-P reflection and P-S reflection.
Various wave propagation modes

The program provides the user with various options for selecting different wave propagation modes. It has options for primaries with or without interbed multiples, and mode conversions between $P$ and $SV$ waves. The panel of conversion modes consists of 16 check buttons that describe various reflections and transmissions. For example, $PdPu$ refers to $P$ wave down and $P$ wave up, i.e. both incidence and reflection are $P$ waves; $PdSu$ refers to $P$ wave down and $S$ wave up, i.e. we have $P$ wave incidence and $S$ wave reflection; and so on. This check button panel corresponds to the reflection and transmission matrix described by Frasier (1970) and Aki and Richards (1980). It can be written as

$$
\begin{bmatrix}
R_D & T_U \\
T_D & R_U
\end{bmatrix} =
\begin{bmatrix}
R_{PP} & R_{Sp} & t_{pp} & t_{sp} \\
R_{Ps} & R_{SS} & t_{ps} & t_{ss} \\
t_{pp} & t_{sp} & R_{PP} & R_{SP} \\
t_{ps} & t_{ss} & R_{Ps} & R_{SS}
\end{bmatrix}
$$

with capital letters indicating the down-going wave, and lower case letters indicating the up-going wave. These reflection and transmission coefficients are calculated using the Zoeppritz equations. The total reflection response for the interface $k$ and all layers below can be written by the recursive construction scheme (Frasier, 1970; Kennett, 1985). Using the notation of Kennett, we write

$$
R_D(z^-_k) = R_D^k + T_U R_D(z^+_k) \left( I - R_U^k R_D(z^+_k) \right)^{-1} T_D^k
$$

where $R_D(z^-_k)$ indicates the total reflectivity just above the $k$th interface, and $R_D(z^+_k)$ indicates the total reflectivity just below the $k$th interface. $R_D^k$, $R_U^k$, $T_D^k$ and $T_U^k$ are the reflection and transmission matrices at the $k$th interface.

The total reflectivity at the $k$th interface relates to the $k+1$th interface through a wave propagator. For a uniform layer, we have

$$
R_D(z^-_k) = E R_D(z^-_{k+1}) E
$$

where the matrix $E$ is a diagonal matrix which gives the vertical phase delay in the frequency domain for both $P$ and $S$ waves:

$$
E = \begin{bmatrix}
\exp(i\alpha \xi z) & 0 \\
0 & \exp(i\beta \eta z)
\end{bmatrix}
$$

where $\xi = \sqrt{\alpha^2 - \beta^2}$, and $\eta = \sqrt{\beta^2 - \beta^2}$.
Since there is no reflection below the last interface, the iteration can start at the bottom of the model and set

\[ R_D(z_{D-}) = R_D^a \]

This reflection response includes all the interbed multiples within the model. Note that the matrix expression inside the parentheses describes the multiples between the layer \( k \) and the layer \( k+1 \) and below. We could obtain the primary reflection without multiples by removing this term.

\[ R_D(z_{D-}) = R_D^k + T^k U^k R_D(z_{k-}) T^k_D \]

If we require only first order multiples, the term inside the parentheses is expanded in a series keeping only the first two terms

\[ R_D(z_{D-}) = R_D^k + T^k U^k R_D(z_{k-})(I + R_D^k R_D(z_{k-})) T^k_D \]

It should be noted that the truncation of the series is artificial. The idea of separating the total wave field into primary, the first order multiple, and the higher order multiples has its clear physical interpretation for the pre-critical reflections. However, it becomes vague in the post-critical range. The truncation of evanescent waves generated in the post-critical range could cause instability to the solution. For the simple model with thick layers, the evanescent wave will decay to insignificance before reaching another interface. The solution will be stable under this situation. However for a complex model with many thin layers, the solution may become unstable.

When the option to include only the primary or the first order multiples is selected, we need to stabilize the solution by artificially damping part of energy due to the evanescent wave before it starts to increase exponentially. Nevertheless, there is still some unwanted energy left in the solution, which may make the solution appear noisy, especially in the post critical range.
Formulation of reflected waves

We follow the derivation described in Aki and Richards (1980), and assume a homogeneous, isotropic, elastic body with $P$ wave velocity $\alpha$, $S$ wave velocity $\beta$, and density $\rho$. We use a cylindrical coordinate system to describe the problem, and assume a point source located at the origin. In this case $P-SV$ components and $SH$ component become decoupled. The problem becomes two dimensional, with solutions dependent only on $r$ and $z$. For the AVO modeling we are only interested in $P-SV$ components.

By using potential functions $\phi$ and $\psi$, the displacement is represented as

$$\vec{u} = \nabla \phi + \nabla \times \nabla \times (0,0,\psi)$$

with the $P$ wave potential $\phi$ satisfying

$$\frac{\partial^2 \Phi}{\partial t^2} - \alpha^2 \nabla^2 \phi = \frac{\Phi}{\rho}$$

and $SV$ wave potential $\psi$ satisfying

$$\frac{\partial^2 \Psi}{\partial t^2} - \beta^2 \nabla^2 \psi = \frac{\Psi}{\rho}$$

where $\Phi$ and $\Psi$ are potentials for the body force.

A point compressional wave source sends out a spherical wave. We can write its potential function $\phi_s$ as

$$\phi_s(x,t) = A \frac{1}{R} \exp(i\omega(\frac{R}{\alpha} - t))$$

with $R = \sqrt{x^2 + y^2 + z^2}$. This wave is then incident on the first reflection interface at $z = z_{i1}$. The potential for generalized $P-P$ reflection is

$$\phi = A\omega \exp(-i\omega t) \int_0^\infty R_{\phi p}(z_{i1}) \frac{P}{\xi} J_0(\omega r) \exp(i2\omega \xi z) \, dp$$

and for generalized $P-S$ reflection is

$$\psi = A\omega \exp(-i\omega t) \int_0^\infty R_{\phi s}(z_{i1}) \frac{1}{\omega \xi} \frac{\beta}{\alpha} J_0(\omega r) \exp(i\omega (\xi z + \eta z)) \, dp$$
where $J_0$ and $J_1$ are the Bessel functions of the first and second kind. The vertical and radial components of the displacement can be obtained with the previous formulas

$$U_z = \frac{\partial \phi}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right)$$

$$= A \omega^2 \exp(-i\omega t) \int_0^\infty J_0(\omega pr) \left[ p \Gamma_{pp}(z_i^+) \exp(i2\omega \xi z) + p^2 \frac{\beta}{\alpha \xi} \Gamma_{pp}(z_i^+) \exp[i\omega(\xi z + \eta z)] \right] dp$$

$$U_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z}$$

$$= -A i \omega^2 \exp(-i\omega t) \int_0^\infty J_1(\omega pr) \left[ \frac{p^2}{\xi} \Gamma_{pp}(z_i^+) \exp(i2\omega \xi z) - p \frac{\beta \eta}{\alpha \xi} \Gamma_{pp}(z_i^+) \exp[i\omega(\xi z + \eta z)] \right] dp$$

The calculated displacements $u_z$ and $u_r$ are given in the frequency domain as an impulse response. The final shot record is obtained by transforming the solution back to the time domain, and convolving with a wavelet.
Example 1: Ostrander’s Model

As the first example we create the Ostrander model shown in Table 1. Ostrander’s case (Ostrander, 1984) was an early example showing that the AVO effect could be observed in real pre-stack data. The sudden decrease in Poisson’s ratio simulates a gas sand layer.

Table 1. Ostrander’s model

<table>
<thead>
<tr>
<th>Depth (ft)</th>
<th>Vp (ft/s)</th>
<th>σ</th>
<th>ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1565</td>
<td>10,000</td>
<td>0.4</td>
<td>2.4</td>
</tr>
<tr>
<td>1615</td>
<td>8,000</td>
<td>0.2</td>
<td>2.14</td>
</tr>
<tr>
<td>2100</td>
<td>10,000</td>
<td>0.4</td>
<td>2.4</td>
</tr>
<tr>
<td>2500</td>
<td>12,000</td>
<td>0.4</td>
<td>2.42</td>
</tr>
</tbody>
</table>

Figure 3 shows the result from the elastic wave modeling. It can be clearly seen that the amplitude increases with offset at the depth where the gas sand is located.

Figure 3. The result of the elastic wave modeling for the Ostrander model.
Example 2: Simmons and Backus Model

The second example is a model from Simmons and Backus (1994). Table 2 shows the input rock properties for the forward modeling. This model contains many thin layers. It is a good example to show how thin layers can affect the AVO response.

Table 2. A model for thin layer assemblage

<table>
<thead>
<tr>
<th>Depth (m)</th>
<th>Vp (m/s)</th>
<th>Vs (m/s)</th>
<th>ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>2909.3</td>
<td>2771</td>
<td>1385</td>
<td>2.42</td>
</tr>
<tr>
<td>2909.9</td>
<td>2903</td>
<td>1508</td>
<td>2.32</td>
</tr>
<tr>
<td>2913.6</td>
<td>2771</td>
<td>1451</td>
<td>2.42</td>
</tr>
<tr>
<td>2916.6</td>
<td>2903</td>
<td>1508</td>
<td>2.42</td>
</tr>
<tr>
<td>2920.9</td>
<td>2651</td>
<td>1325</td>
<td>2.40</td>
</tr>
<tr>
<td>2922.1</td>
<td>3586</td>
<td>2398</td>
<td>2.55</td>
</tr>
<tr>
<td>2913.9</td>
<td>2903</td>
<td>1451</td>
<td>2.30</td>
</tr>
<tr>
<td>2915.2</td>
<td>3586</td>
<td>2398</td>
<td>2.42</td>
</tr>
<tr>
<td>3000.0</td>
<td>2903</td>
<td>1451</td>
<td>2.43</td>
</tr>
</tbody>
</table>

Figure 4 shows the models calculated using Elastic Wave Modeling and with Zoeppritz primaries-only modeling. As pointed out by Simmons and Backus (1994), the difference is due to the effect of the converted waves generated within the thin layers.

Figure 4. A comparison between the two types of modeling, as applied to the Simmons and Backus model. As discussed by the authors, for this thin-layer model, ignoring mode-converted events causes the Zoeppritz primaries-only model to predict an erroneous AVO increase.
Example 3: Real Data Example

The third example is a real data example from Western Canada. The log is about 800 meters long, containing about 3200 layers. A synthetic shot record is modeled, and shown in the left panel of Figure 5. It takes about 10 hours to generate this result with a Sun Sparc 20 workstation. In order to reduce run time, we set the minimum thickness to 2 meters. This reduces the number of layers in the model to about 400. The right panel of Figure 5 shows the result. There is no visual difference between the two synthetics; however, the run time was reduced from 10 hours to 1 hour.

<table>
<thead>
<tr>
<th>Complete Log (3200 Layers)</th>
<th>2 meter uniform blocking</th>
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<tr>
<td>Complete Log (3200 Layers)</td>
<td>2 meter uniform blocking</td>
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Figure 5. A synthetic shot record calculated with a real log. The number of layers in the model is about 3200. The left panel shows the calculation with all 3200 layers. The right panel shows the calculation using a minimum block size of 2 meters.
To evaluate the differences more accurately, Figure 6 shows the difference plot which is calculated by subtracting the 2 meter synthetic from the full synthetic. Note that differences occur mainly on the first arrivals and at the far offsets for the event around 600 ms.

Figure 6. A synthetic shot record calculated with a real log. The number of layers in the model is about 3200. The left panel shows the calculation with all 3200 layers. The right panel shows the difference between the synthetic calculated with 2 meter blocking and the full synthetic.
These figures suggest that a good strategy for minimizing the computation time is to set the minimum block size carefully. Figure 7 shows the calculation of this synthetic with a range of block sizes. Obviously there is a continuous tradeoff between accuracy and computer run-time.

Figure 7. A comparison of synthetics calculated with varying minimum block sizes. In each case, the synthetic is shown at the top and the difference plot is shown below. The difference is calculated by subtracting the synthetic from that calculated using all 3200 layers.
References


Kennett, B., 1985, Seismic wave propagation in stratified media: Cambridge University Press.


