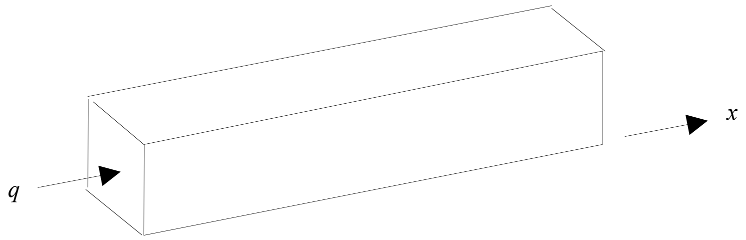
REVIEW OF BASIC STEPS IN DERIVATION OF FLOW EQUATIONS

Generally speaking, flow equations for flow in porous materials are based on a set of mass, momentum and energy conservation equations, and constitutive equations for fluids and the porous material. For simplicity, we will in the following assume isothermal conditions, so that we not have to involve an energy conservation equation. However, in cases of changing reservoir temperature, such as in the case of cold water injection into a warmer reservoir, this may be of importance.

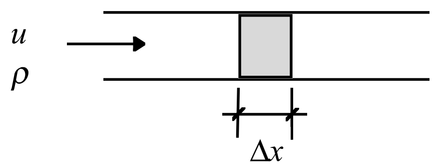
Below, equations are described for linear, one-dimensional systems, but can easily be extended to two and three dimensions, and to other coordinate systems.

Conservation of mass

Again we will consider the following one dimensional slab of porous material:



Mass conservation may be formulated across a control element of the slab, with one fluid of density ρ is flowing through it at a velocity *u*:



The mass balance for the control element is then written as:

$$\begin{Bmatrix} Mass\ into\ the \\ element\ at\ x \end{Bmatrix} - \begin{Bmatrix} Mass\ out\ of\ the \\ element\ at\ x+Dx \end{Bmatrix} = \begin{Bmatrix} Rate\ of\ change\ of\ mass \\ inside\ the\ element \end{Bmatrix},$$

or

$$\{u\rho A\}_x - \{u\rho A\}_{x+\Delta x} = \frac{\partial}{\partial t}\{\phi A \Delta x \rho\}.$$

Dividing by Δx, and taking the limit as Δx goes to zero, we get the conservation of mass, or continuity equation:

$$-\frac{\partial}{\partial x}(A\rho u) = A\frac{\partial}{\partial t}(\phi\rho).$$

For constant cross sectional area, the continuity equation simplifies to:

$$-\frac{\partial}{\partial x}(\rho u) = \frac{\partial}{\partial t}(\phi\rho).$$

Conservation of momentum

Conservation of momentum is governed by the Navier-Stokes equations, but is normally simplified for low velocity flow in porous materials to be described by the semi-empirical Darcy's equation, which for one dimensional, horizontal flow is:

$$u = -\frac{k}{\mu}\frac{\partial P}{\partial x}.$$

Alternative equations are the Forchheimer equation, for high velocity flow:

$$-\frac{\partial P}{\partial x} = u\frac{\mu}{k} + \beta u^n,$$

where n is proposed by Muscat to be 2, and the Brinkman equation, which applies to both porous and non-porous flow:

$$-\frac{\partial P}{\partial x} = u\frac{\mu}{k} - \mu\frac{\partial^2 u}{\partial x^2}.$$

Brinkman's equation reverts to Darcy's equation for flow in porous media, since the last term then normally is negligible, and to Stoke's equation for channel flow because the Darcy part of the equation then may be neglected.

In the following, we assume that Darcy's equation is valid for flow in porous media.

Constitutive equation for porous materials

To include pressure dependency in the porosity, we use the definition of rock compressibility:

$$c_r = (\frac{1}{\phi})(\frac{\partial \phi}{\partial P})_T.$$

Keeping the temperature constant, the expression may be written:

$$\frac{d\phi}{dP} = \phi c_r$$

Normally, we may assume that the bulk volume of the porous material is constant, i.e. the bulk compressibility is zero. This is not always true, as witnessed by the subsidence in the Ekofisk area.

Constitutive equation for fluids

Recall the familiar fluid compressibility definition, which applies to any fluid at constant temperature:

$$c_f = -(\frac{1}{V})(\frac{\partial V}{\partial P})_T.$$

Equally familiar is the gas equation, which for an ideal gas is:

$$PV = nRT,$$

and for a real gas includes the deviation factor, Z:

$$PV = nZRT.$$

The gas density may be expressed as:

$$\rho_g = \rho_{gS}\frac{P}{Z}\frac{Z_S}{P_S}$$

where the subscript S denotes surface (standard) conditions. These equations are frequently used in reservoir engineering applications. However, for reservoir simulation purposes, we normally use either so-called *Black Oil* fluid description, or *compositional* fluid description. For now, we will consider the Black Oil model, and get back to compositional models later on.

The standard Black Oil model includes *Formation Volume Factor*, B, for each fluid, and *Solution Gas-Oil Ratio*, R_{so}, for the gas dissolved in oil, in addition to viscosity and density for each fluid. A modified model may also include oil dispersed in gas, r_s, and gas dissolved in water, R_{sw}. The definitions of formation volume factors and solution gas-oil ratio are:

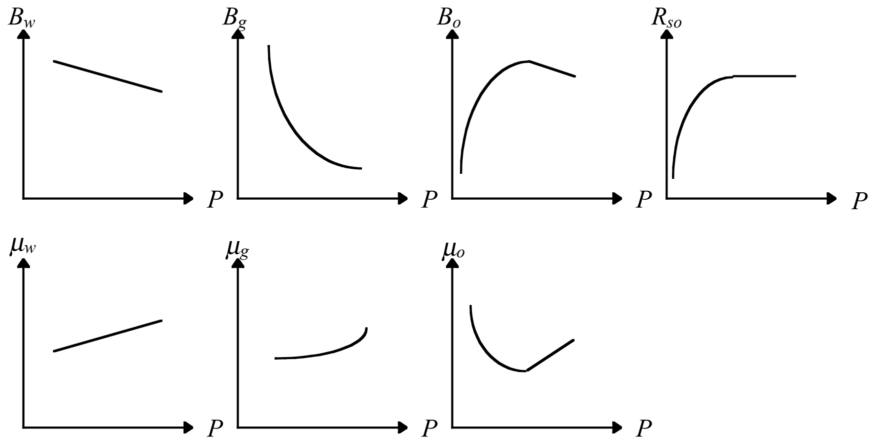
$$B = \frac{\text{volume at reservoir conditions}}{\text{volume at standard conditions}}$$

$$R_{so} = \frac{\text{volume of gas evolved from oil at standard conditions}}{\text{volume of oil at standard conditions}}$$

The density of oil at reservoir conditions is then, in terms of these parameters and the densities of oil and gas, defined as:

$$\rho_o = \frac{\rho_{oS} + \rho_{gs} R_{so}}{B_o}.$$

Typical pressure dependencies of the standard Black Oil parameters are:



Flow equation

For single phase flow, in a one-dimensional, horizontal system, assuming Darcy's equation to be applicable and that the cross sectional area is constant, the flow equation becomes:

$$\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B}\right)$$

Boundary conditions

As discussed previously, we basically have two types of BC's; pressure conditions (Dirichlet conditions) and rate conditions (Neumann conditions). The most common boundary conditions in reservoirs, including sources/sinks, are discussed in the following.

Dirichlet conditions

When pressure conditions are specified, we normally would specify the pressures at the end faces of the system in question. Applied to the simple linear system described above, we may have the following two pressure BC's at the ends:

$$P(x=0,t>0)=P_L$$

$$P(x=L,t>0)=P_R$$

For reservoir flow, a pressure condition will normally be specified as a bottom-hole pressure of a production or injection well, at some position of the reservoir. Strictly speaking, this is not a boundary condition, but the treatment of this type of condition is similar to the treatment of a boundary pressure condition.

Neumann condition

Alternatively, we would specify the flow rates at the end faces of the system in question. Using Darcy's equation at the ends of the simple system above, the conditions become:

$$Q_L=-\frac{kA}{\mu}\left(\frac{\partial P}{\partial x}\right)_{x=0}$$

$$Q_R=-\frac{kA}{\mu}\left(\frac{\partial P}{\partial x}\right)_{x=L}$$

For reservoir flow, a rate condition may be specified as a production or injection rate of a well, at some position of the reservoir, or it is specified as a zero-rate across a sealed boundary or fault, or between non-communicating layers.

Initial condition (IC)

The initial condition specifies the initial state of the primary variables of the system. For the simple case above, a constant initial pressure may be specified as:

$$P(x,t=0)=P_0$$

The initial pressure may be a function of postition. For non-horizontal systems, hydrostatic pressure equilibrium is normally computed based on a reference pressure and fluid densities:

$$P(z,t=0)=P_{ref}+(z-z_{ref})\rho g.$$

Multiphase flow

A continuity equation may be written for each fluid phase flowing:

$$-\frac{\partial}{\partial x}\left(\rho_l u_l\right)=\frac{\partial}{\partial t}\left(\phi\rho_l S_l\right),\quad l=o,w,g,$$

and the corresponding Darcy equations for each phase are:

$$u_l = -\frac{k k_{rl}}{\mu_l}\frac{\partial P_l}{\partial x}, \quad l = o, w, g.$$

However, the continuity equation for gas has to be modified to include solution gas as well as free gas, so that the oil equation only includes the part of the oil remaining liquid at the surface:

$$\rho_o = \frac{\rho_{oS} + \rho_{gS} R_{so}}{B_o} = \rho_{oL} + \rho_{oG}$$

where ρ_{oL} represents the part of the oil remaining liquid at the surface (in the stock tank), and ρ_{oG} the part that is gas at the surface. Thus, the oil and gas continuity equations become:

$$-\frac{\partial}{\partial x}\left(\rho_{oL} u_o\right) = \frac{\partial}{\partial t}\left(\phi \rho_{oL} S_o\right)$$

$$-\frac{\partial}{\partial x}\left(\rho_g u_g + \rho_{oG} u_o\right) = \frac{\partial}{\partial t}\left(\phi \rho_g S_g + \phi \rho_{oG} S_o\right)$$

After substitution for Darcy's equations and Black Oil fluid properties, and including well rate terms, the flow equations become:

$$\frac{\partial}{\partial x}\left(\frac{k k_{ro}}{\mu_o B_o}\frac{\partial P_o}{\partial x}\right) - q'_o = \frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{k k_{rg}}{\mu_g B_g}\frac{\partial P_g}{\partial x} + R_{so}\frac{k k_{ro}}{\mu o B_o}\frac{\partial P_o}{\partial x}\right) - q'_g - R_{so} q'_o = \frac{\partial}{\partial t}\left(\frac{\phi S_g}{B_g} + R_{so}\frac{\phi S_o}{B_o}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{k k_{rw}}{\mu_w B_w}\frac{\partial P_w}{\partial x}\right) - q'_w = \frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right)$$

where

$$P_{cow} = P_o - P_w$$

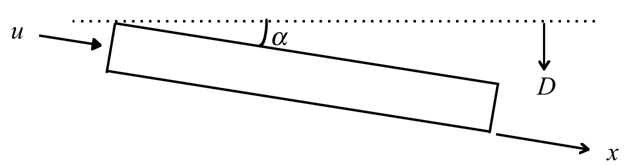
$$P_{cog} = P_g - P_o$$

$$\sum_{l=o,w,g} S_l = 1$$

The oil equation could be further modified to include dispersed oil in the gas, if any, similarly to the inclusion of solution gas in the oil equation.

Non-horizontal flow

For one-dimensional, inclined flow,



the Darcy equation becomes:

$$u = -\frac{k}{\mu}\left(\frac{\partial P}{\partial x} - \rho g \frac{dD}{dx}\right),$$

or, in terms of dip angle, α, and hydrostatic gradient:

$$u = -\frac{k}{\mu}\left(\frac{\partial P}{\partial x} - \gamma \sin(\alpha)\right),$$

where $\gamma = \rho g$ is the hydrostatic gradient of the fluid.

Multidimensional flow

The continuity equation for one-phase, three-dimensional flow in Cartesian coordinates, is:

$$-\frac{\partial}{\partial x}(\rho u_x) - \frac{\partial}{\partial y}(\rho u_y) - \frac{\partial}{\partial z}(\rho u_z) = \frac{\partial}{\partial t}(\phi\rho),$$

and the corresponding Darcy equations are:

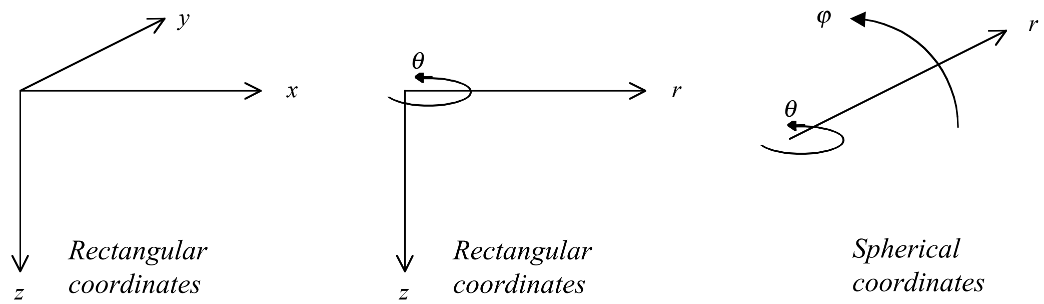
$$u_x = -\frac{k_x}{\mu}\left(\frac{\partial P}{\partial x} - \gamma \frac{\partial D}{\partial x}\right)$$

$$u_y = -\frac{k_y}{\mu}\left(\frac{\partial P}{\partial y} - \gamma \frac{\partial D}{\partial y}\right)$$

$$u_z = -\frac{k_z}{\mu}\left(\frac{\partial P}{\partial z} - \gamma \frac{\partial D}{\partial z}\right)$$

Coordinate systems

Normally, we use either a rectangular coordinate system or a cylindrical coordinate system in reservoir simulation.



In operator form, the continuity and the Darcy equations for one-phase flow may be written:

$$-\nabla \cdot (\rho \vec{u}) = \frac{\partial}{\partial t}(\phi\rho)$$

$$\vec{u} = -\frac{K}{\mu}(\nabla P - \gamma \nabla D),$$

where the operators for rectangular coordinates (x, y, z) are defined as:

$$\nabla \cdot (\ ) = \frac{\partial}{\partial x}(\ ) + \frac{\partial}{\partial y}(\ ) + \frac{\partial}{\partial z}(\ ) \qquad \text{(divergence)}$$

$$\nabla(\ ) = \hat{i}\frac{\partial}{\partial x}(\ ) + \hat{j}\frac{\partial}{\partial y}(\ ) + \hat{k}\frac{\partial}{\partial z}(\ ) \quad \text{(gradient)}$$

for cylindrical coordinates (r, θ, z):

$$\nabla \cdot (\ ) = \frac{1}{r}\frac{\partial}{\partial r}\big(r(\ )\big) + \frac{1}{r}\frac{\partial}{\partial \theta}(\ ) + \frac{\partial}{\partial z}(\ )$$

$$\nabla(\ ) = \hat{i}\frac{\partial}{\partial r}(\ ) + \hat{j}\frac{\partial}{\partial \theta}(\ ) + \hat{k}\frac{\partial}{\partial z}(\ )$$

and for spherical coordinates (r, θ, φ).

$$\nabla \cdot (\ ) = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2(\ )\big) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}(\ ) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \varphi}(\ )$$

$$\nabla(\ ) = \hat{i}\frac{\partial}{\partial r}(\ ) + \hat{j}\frac{\partial}{\partial \theta}(\ ) + \hat{k}\frac{\partial}{\partial \varphi}(\ )$$

Boundary conditions of multiphase systems

The pressure and rate BC's discussed above apply to multiphase systems. However, for a production well in a reservoir, we normally specify either an oil production rate at the surface, or a total liquid rate at the surface. Thus, the rate(s) not specified must be computed from Darcy's equation. The production is subjected to maximum allowed GOR or WC, or both. We will discuss these conditions later.

Initial conditions of multiphase systems

In addition to specification of initial pressures, we also need to specify initial saturations in a multiphase system. This requires knowledge of water-oil contact (WOC) and gas-oil contact (GOC). Assuming that the reservoir is in equilibrium, we may compute initial phase pressures based on contact levels and densities. Then, equilibrium saturations may be interpolated from the capillary pressure curves. Alternatively, the initial saturations are based on measured logging data.