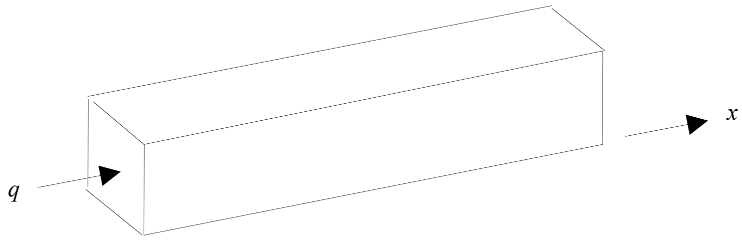
INTRODUCTION TO RESERVOIR SIMULATION

Analytical and numerical solutions of simple one-dimensional, one-phase flow equations

As an introduction to reservoir simulation, we will review the simplest one-dimensional flow equations for horizontal flow of one fluid, and look at analytical and numerical solutions of pressure as function of position and time. These equations are derived using the continuity equation, Darcy's equation, and compressibility definitions for rock and fluid, assuming constant permeability and viscosity. They are the simplest equations we can have, which involve transient fluid flow inside the reservoir.

Linear flow

Consider a simple horizontal slab of porous material, where initially the pressure everywhere is P_0, and then at time zero, the left side pressure (at $x = 0$) is raised to P_L while the right side pressure (at $x = L$) is kept at $P_R = P_0$. The system is shown on the next figure:



Partial differential equation (PDE)

The linear, one dimensional, horizontal, one phase, partial differential flow equation for a liquid, assuming constant permeability, viscosity and compressibility is:

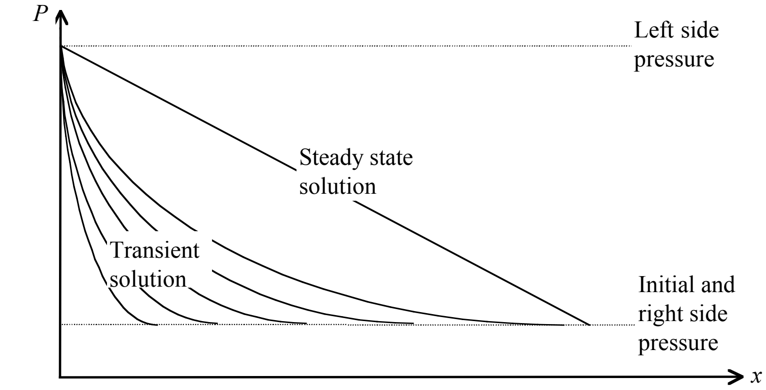
$$\frac{\partial^2 P}{\partial x^2} = (\frac{\phi \mu c}{k}) \frac{\partial P}{\partial t}$$

Transient vs. steady state flow

The equation above includes time dependency through the right hand side term. Thus, it can describe transient, or time dependent flow. If the flow reaches a state where it is no longer time dependent, we denote the flow as steady state. The equation then simplifies to:

$$\frac{d^2 P}{dx^2} = 0$$

Transient and steady state pressure distributions are illustrated graphically in the figure below for a system where initial and right hand pressures are equal. As can be observed, for some period of time, depending on the properties of the system, the pressure will increase in all parts of the system (transient solution), for then to approach a final distribution (steady state), described by a straight line between the two end pressures.



Analytical solution to the linear PDE

The analytical solution of the transient pressure development in the slab is then given by:

$$P(x,t) = P_L + (P_R - P_L)\left[\frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-\frac{n^2\pi^2}{L^2}\frac{k}{\phi\mu c}t)\sin(\frac{n\pi x}{L})\right]$$

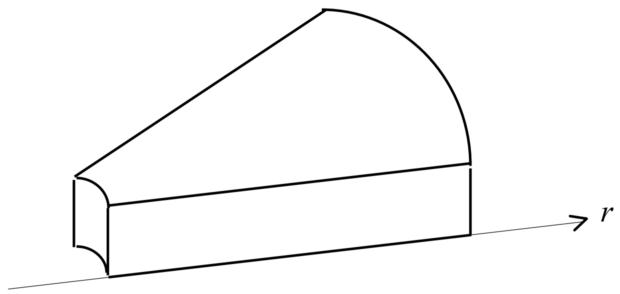
It may be seen from the solution that as time becomes large, the exponential term approaches zero, and the solution becomes:

$$P(x,t) = P_L + (P_R - P_L)\frac{x}{L}.$$

This is, of course, the solution to the steady state equation above.

Radial flow (Well test equation)

An alternative form of the simple one dimensional, horizontal flow equation for a liquid, is the radial equation that frequently is used for well test interpretation. In this case the flow area is proportional to r^2, as shown in the following figure:



The one-dimensional (radial) flow equation in this coordinate system becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial P}{\partial r}) = \frac{\phi\mu c}{k}\frac{\partial P}{\partial t}$$

For an infinite reservoir with $P(r \to \infty) = P_i$ and well rate q from a well in the center *(at r=r_w)* the analytical solution s

$$P = P_i + \frac{q\mu}{4\pi kh}Ei\left(-\frac{\phi\mu cr^2}{4kt}\right),$$

where $Ei(-x) = -\int_x^\infty \frac{e^{-u}}{u} du$ is the exponential integral. A steady state solution does not exist for an infinite system, since the pressure will continue to decrease as long as we produce from the center. However, if we use a different set of boundary conditions, so that $P(r = r_w) = P_w$ and $P(r = r_e) = P_e$, we can solve the steady state form of the equation

$$\frac{1}{r}\frac{d}{dr}(r\frac{dP}{dr}) = 0$$

by integration twice, so that the steady state solution becomes

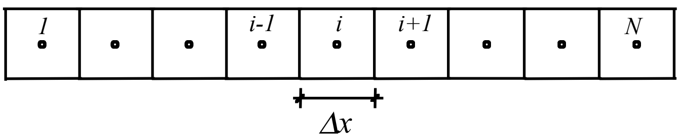
$$P = P_w + \frac{(P_e - P_w)}{\ln(r_e / r_w)} \ln(r / r_w) .$$

Numerical solution

Generally speaking, analytical solutions to reservoir flow equations are only obtainable after making simplifying assumptions in regard to geometry, properties and boundary conditions that severely restrict the applicability of the solution. For most real reservoir fluid flow problems, such simplifications are not valid. Hence, we need to solve the equations numerically.

Discretization

In the following we will, as a simple example, solve the linear flow equation above numerically by using standard finite difference approximations for the two derivative terms $\frac{\partial^2 P}{\partial x^2}$ and $\frac{\partial P}{\partial t}$. First, the *x*-coordinate must be subdivided into a number of discrete grid blocks, and the time coordinate must be divided into discrete time steps. Then, the pressure in each block can be solved for numerically for each time step. For our simple one dimensional, horizontal porous slab, we thus define the following grid block system with *N* grid blocks, each of length Δx :



This is called a *block-centered grid*, and the grid blocks are assigned indices, i, referring to the mid-point of each block, representing the average property of the block.

Taylor series approximations

A so-called Taylor series approximation of a function $f(x+h)$ expressed in terms of $f(x)$ and its derivatives $f'(x)$ may be written:

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots .$$

Applying Taylor series to our pressure function, we may write expansions in a variety of ways in order to obtain approximations to the derivatives in the linear flow equation.

Approximation of the second order space derivative

At constant time, *t*, the pressure function may be expanded forward and backwards:

$$P(x+\Delta x,t) = P(x,t) + \frac{\Delta x}{1!}P'(x,t) + \frac{(\Delta x)^2}{2!}P''(x,t) + \frac{(\Delta x)^3}{3!}P'''(x,t) + \ldots..$$

$$P(x-\Delta x,t) = P(x,t) + \frac{(-\Delta x)}{1!}P'(x,t) + \frac{(-\Delta x)^2}{2!}P''(x,t) + \frac{(-\Delta x)^3}{3!}P'''(x,t) + \ldots..$$

By adding these two expressions, and solving for the second derivative, we get the following approximation:

$$P''(x,t) = \frac{P(x+\Delta x,t) - 2P(x,t) + P(x+\Delta x,t)}{(\Delta x)^2} + \frac{(\Delta x)^2}{12}P''''(x,t) + \ldots.$$

or, by employing the grid index system, and using superscript to indicate time level:

$$(\frac{\partial^2 P}{\partial x^2})_i^t = \frac{P_{i+1}^t - 2P_i^t + P_{i-1}^t}{(\Delta x)^2} + O(\Delta x^2)\,.$$

This is called a *central approximation* of the second derivative. Here, the rest of the terms from the Taylor series expansion are collectively denoted $O(\Delta x^2)$, thus denoting that they are in order of, or proportional in size to Δx^2. This error term, sometimes called discretization error, which in this case is of second order, is neglected in the numerical solution. The smaller the grid blocks used, the smaller will be the error involved. Any time level could be used in the expansions above. Thus, we may for instance write the following approximations at time levels $t+\Delta t$ and $t+\frac{\Delta t}{2}$:

$$(\frac{\partial^2 P}{\partial x^2})_i^{t+\Delta t} = \frac{P_{i+1}^{t+\Delta t} - 2P_i^{t+\Delta t} + P_{i-1}^{t+\Delta t}}{(\Delta x)^2} + O(\Delta x^2)$$

$$(\frac{\partial^2 P}{\partial x^2})_i^{t+\frac{\Delta t}{2}} = \frac{P_{i+1}^{t+\frac{\Delta t}{2}} - 2P_i^{t+\frac{\Delta t}{2}} + P_{i-1}^{t+\frac{\Delta t}{2}}}{(\Delta x)^2} + O(\Delta x^2)$$

Approximation of the time derivative

At constant position, *x*, the pressure function may be expanded in forward direction in regard to time:

$$P(x,t+\Delta t) = P(x,t) + \frac{\Delta t}{1!}P'(x,t) + \frac{(\Delta t)^2}{2!}P''(x,t) + \frac{(\Delta t)^3}{3!}P'''(x,t) + \ldots..$$

By solving for the first derivative, we get the following approximation:

$$P'(x,t) = \frac{P(x,t+\Delta t) - P(x,t)}{\Delta t} + \frac{(\Delta t)}{2}P''(x,t) + \ldots..$$

or, employing the index system:

$$(\frac{\partial P}{\partial t})_i^t = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t)\,.$$

Here, the error term is proportional to Δt, or of the first order. The error therefore approaches zero slower in this case than for the second order term above. This approximation is called a forward approximation. By expanding backwards in time, we may write:

$$P(x,t) = P(x,t+\Delta t) + \frac{-\Delta t}{1!}P'(x,t+\Delta t) + \frac{(-\Delta t)^2}{2!}P''(x,t+\Delta t) + \frac{(-\Delta t)^3}{3!}P'''(x,t+\Delta t) + \ldots..$$

Solving for the time derivative, we get:

$$(\frac{\partial P}{\partial t})_i^{t+\Delta t} = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t) .$$

This expression is identical to the expression above. However, this is now a *backward approximation*. Another alternative for a time derivative approximation may be obtained from forward and backward expansions over an interval of $\frac{\Delta t}{2}$:

$$P(x,t+\Delta t) = P(x,t+\tfrac{\Delta t}{2}) + \frac{\frac{\Delta t}{2}}{1!} P'(x,t+\tfrac{\Delta t}{2}) + \frac{(\frac{\Delta t}{2})^2}{2!} P''(x,t+\tfrac{\Delta t}{2}) + \frac{(\frac{\Delta t}{2})^3}{3!} P'''(x,t+\tfrac{\Delta t}{2}) + \ldots.$$

$$P(x,t) = P(x,t+\tfrac{\Delta t}{2}) + \frac{-\frac{\Delta t}{2}}{1!} P'(x,t+\tfrac{\Delta t}{2}) + \frac{(-\frac{\Delta t}{2})^2}{2!} P''(x,t+\tfrac{\Delta t}{2}) + \frac{(-\frac{\Delta t}{2})^3}{3!} P'''(x,t+\tfrac{\Delta t}{2}) + \ldots.$$

By combination, we obtain the following *central approximation* of the time derivative, with a second order error term:

$$(\frac{\partial P}{\partial t})_i^{t+\frac{\Delta t}{2}} = \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} + O(\Delta t)$$

Explicit difference equation

First, we will use the approximations above at time level t and substitute them into the linear flow equation. The following difference equation is obtained:

$$\frac{P_{i+1}^t - 2P_i^t + P_{i-1}^t}{\Delta x^2} \approx (\frac{\phi\mu c}{k}) \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t}, \quad i = 1,\ldots,N$$

For convenience, the error terms are dropped in the equation above, and the equality sign is replaced by an approximation sign. It is important to keep in mind, however, that the errors involved in this numerical form of the flow equation, are proportional to Δt and Δx^2, respectively.

Boundary conditions (BC's)

The driving force for flow arises from the BC's. Basically, we have two types of BC's, the pressure condition (Dirichlet condition), and the flow rate condition (Neumann condition).

<u>Pressure BC</u>

When pressure boundaries are to be specified, we normally, specify the pressure at the end faces of the system in question. Applied to the simple linear system described above, we may have the following two BC's:

$$P(x=0, t>0) = P_L$$

$$P(x=L, t>0) = P_R$$

or, using the index system:

$$P_{i=1/2}^{t>0} = P_L$$

$$P_{N+1/2}^{t>0} = P_R$$

The reason we here use indices $i=\frac{1}{2}$ and $N+\frac{1}{2}$ is that the BC's are applied to the ends of the first and the last blocks, respectively. Thus, the BC's cannot directly be substituted into the difference equation. However, Taylor series may again be used to derive special formulas for the end blocks. For block 1 we may write:

$$P(x_2,t)=P(x_1,t)+\frac{\Delta x}{1!}P'(x_1,t)+\frac{(\Delta x)^2}{2!}P''(x_1,t)+\frac{(\Delta x)^3}{3!}P'''(x_1,t)+\ldots.$$

$$P(x=0,t)=P(x_1,t)+\frac{(-\frac{\Delta x}{2})}{1!}P'(x_1,t)+\frac{(-\frac{\Delta x}{2})^2}{2!}P''(x_1,t)+\frac{(-\frac{\Delta x}{2})^3}{3!}P'''(x_1,t)+\ldots.$$

By combination of the two expressions, we obtain the following approximation of the second derivative in block 1:

$$(\frac{\partial^2 P}{\partial x^2})_1^t=\frac{P_2^t-3P_1^t+2P_L}{\frac{3}{4}(\Delta x)^2}+O(\Delta x)$$

A disadvantage of this formulation is that the error term is only first order, i.e. proportional to Δx. A similar expression may be obtained for the right hand side:

$$(\frac{\partial^2 P}{\partial x^2})_N^t=\frac{2P_R-3P_N^t+P_{N-1}^t}{\frac{3}{4}(\Delta x)^2}+O(\Delta x).$$

In a real reservoir case, pressure boundary conditions would normally represent bottom hole, or well head, pressures in production or injection wells.

Flow rate BC

Alternatively, we would specify the flow rate, Q, into or out of an end face of the system in question, for instance into the left end of the system above. Making use of the fact that the flow rate may be expressed by Darcy's law, as follows:

$$Q_L=-\frac{kA}{\mu}\left(\frac{\partial P}{\partial x}\right)_{x=0}.$$

We will again apply Taylor series expansion to block 1, but this time we will let the derivative of the pressure be the function:

$$P'(x_1+\tfrac{\Delta x}{2},t)=P'(x_1,t)+\frac{(\frac{\Delta x}{2})}{1!}P''(x_1,t)+\frac{(\frac{\Delta x}{2})^2}{2!}P'''(x_1,t)+\ldots..$$

$$P'(x=0,t)=P'(x_1,t)+\frac{(-\frac{\Delta x}{2})}{1!}P''(x_1,t)+\frac{(-\frac{\Delta x}{2})^2}{2!}P'''(x_1,t)+\ldots..$$

Subtracting the second expression from the first and solving for the second derivative, we obtain the following approximation for grid block 1:

$$P''(x_1,t)=\frac{P'(x_1+\frac{\Delta x}{2},t)-P'(x=0,t)}{\Delta x}+O(\Delta x^2)$$

Now we replace the derivative at the end face by the expression given by the boundary condition:

$$P''(x_1,t)=\frac{P'(x_1+\frac{\Delta x}{2},t)+Q_L\dfrac{\mu}{kA}}{\Delta x}+O(\Delta x^2)$$

The other in the expression derivative may be replaced by a central formula:

$$P'(x_1 + \tfrac{\Delta x}{2}, t) = \frac{P(x_2, t) - P(x_1, t)}{\Delta x} + O(\Delta x^2),$$

so that the final formula for the second derivative in block 1 for this boundary condition becomes:

$$(\frac{\partial^2 P}{\partial x^2})_1^t = \frac{P_2^t - P_1^t}{(\Delta x)^2} + Q_L \frac{\mu}{\Delta x A k} + O(\Delta x)$$

Similarly, a constant rate at the right hand side, Q_R, would result in the following expression:

$$(\frac{\partial^2 P}{\partial x^2})_N^t = \frac{P_N^t - P_{N-1}^t}{(\Delta x)^2} - Q_R \frac{\mu}{\Delta x A k} + O(\Delta x)$$

In a real reservoir case, flow rate conditions would normally represent production or injection rates for wells. A special case is the no-flow boundary, where $Q = 0$. This condition is specified at all outer limits of the reservoir, between non-communicating layers, and across sealing faults in the reservoir.

Initial condition (IC)

The initial condition (initial pressures) for our horizontal system may be specified as:

$$P_i^{t=0} = P_0, i = 1, \ldots, N.$$

For non-horizontal systems, hydrostatic pressures are normally computed based on a reference pressure and fluid densities.

Solution of the difference equation

Having derived the difference equation above, and specified the grid system, the BC's and the IC, we can solve for pressures. However, one issue of importance needs to be discussed first. In deriving the difference approximations, we assigned a time level of t to the terms in the Taylor series. Obviously, we could as well assigned a time level of $t + \Delta t$ with equivalent generality. Or we could assign a time level of $t + \frac{\Delta t}{2}$. We will discuss these cases below, starting with the explicit formulation. For convenience, error terms are not included below.

Explicit formulation

This is exactly the case we derived above. By approximation of all the terms at time t, we obtain a set of difference equation that can be solved explicitly for average pressures in the grid blocks (*i=1,...,N*) for each time step, as follows (below we give the expressions for the case of constant pressure BC's; if rate conditions are used, the expressions should be modified accordingly):

$$P_1^{t+\Delta t} = P_1^t + \frac{4}{3}(\frac{\Delta t}{\Delta x^2})(\frac{k}{\phi \mu c})(P_2^t - 3P_1^t + 2P_L)$$

$$P_i^{t+\Delta t} = P_i^t + (\frac{\Delta t}{\Delta x^2})(\frac{k}{\phi \mu c})(P_{i+1}^t - 2P_i^t + P_{i-1}^t), \quad i = 2, \ldots, N-1$$

$$P_N^{t+\Delta t} = P_N^t + \frac{4}{3}(\frac{\Delta t}{\Delta x^2})(\frac{k}{\phi \mu c})(2P_R^t - 3P_N^t + P_{N-1})$$

Implicit formulation

In this case, all time levels in the approximations are changed to $t+\Delta t$, except for in the time derivative approximation, which now will be of the backward type.

$$\frac{P_2^{t+\Delta t} - 3P_1^{t+\Delta t} + 2P_L}{\frac{3}{4}\Delta x^2} = (\frac{\phi\mu c}{k})\frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} \quad (i=1)$$

$$\frac{P_{i+1}^{t+\Delta t} - 2P_i^{t+\Delta t} + P_{i-1}^{t+\Delta t}}{\Delta x^2} = (\frac{\phi\mu c}{k})\frac{P_i^{t+\Delta t} - P_i^t}{\Delta t}, \quad i=2,\ldots,N-1$$

$$\frac{2P_R^{t+\Delta t} - 3P_N^{t+\Delta t} + P_{N-1}^{t+\Delta t}}{\frac{3}{4}\Delta x^2} = (\frac{\phi\mu c}{k})\frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} \quad (i=N)$$

Now we have a set of N equations with N unknowns, which must be solved simultaneously. For simplicity, the set of equations may be written on the form:

$$a_i P_{i-1}^{t+\Delta t} + b_i P_i^{t+\Delta t} + c_i P_{i+1}^{t+\Delta t} = d_i, \quad i=1,..N$$

where

$$\alpha = (\frac{\phi\mu c}{k})(\frac{\Delta x^2}{\Delta t})$$

and

$$a_1 = 0$$

$$a_i = 1, \quad i=2,\ldots,N$$

$$b_1 = b_N = -3 - \tfrac{3}{4}\alpha$$

$$b_i = -2 - \alpha, \quad i=2,\ldots,N-1$$

$$c_N = 0$$

$$c_i = 1, \quad i=1,\ldots,N-1$$

$$d_1 = -\tfrac{3}{4}\alpha P_1^t - 2P_L$$

$$d_i = -\alpha P_i^t, \quad i=2,\ldots,N-1$$

$$d_N = -\tfrac{3}{4}\alpha P_N^t - 2P_R$$

This linear set of equations may be solved for average block pressures using for instance the Gaussian elimination method.

Crank-Nicholson formulation

As mentioned above, we also have the possibility of writing the equation at a time level between t and $t+\Delta t$ (Crank-Nicholson's method). For $t+\frac{\Delta t}{2}$, we may write the difference equation for block i as:

$$\frac{P_{i+1}^{t+\frac{\Delta t}{2}} - 2P_i^{t+\frac{\Delta t}{2}} + P_{i-1}^{t+\frac{\Delta t}{2}}}{\Delta x^2} = (\frac{\phi\mu c}{k})\frac{P_i^{t+\frac{\Delta t}{2}} - P_i^t}{\Delta t},$$

Since the pressures are defined at time levels t and $t+\Delta t$, and not at $t+\frac{\Delta t}{2}$, we cannot solve this equation as it is. Therefore, we rewrite the left side as the average of explicit and implicit formulations:

$$\frac{1}{2}\left[\frac{P_{i+1}^{t}-2P_{i}^{t}+P_{i-1}^{t}}{\Delta x^2}+\frac{P_{i+1}^{t+\Delta t}-2P_{i}^{t+\Delta t}+P_{i-1}^{t+\Delta t}}{\Delta x^2}\right]=(\frac{\phi\mu c}{k})\frac{P_i^{t+\Delta t}-P_i^t}{\Delta t}$$

The resulting set of linear equations may be solved simultaneously just as in the implicit case. All the coefficients may be deducted from the explicit and implicit cases above.

Discussion of the formulations

Obviously, the explicit formulation is simpler to use than the implicit formulation, as explicit expressions for pressures are obtained directly. Discretization errors are the same for the two formulations. The amount of work involved is less for the explicit case. In one-dimensional solutions, this may not have any importance, however, in two and three dimensional cases with large numbers of grid blocks, the difference in computational time per time step will become large.

However, the explicit formulation is seldom used. As it turns out, it becomes unstable for large time steps. It will be shown below, using von Neumann stability analysis, that the explicit formulation has the following stability requirement:

$$\Delta t \le \frac{1}{2}(\frac{\phi\mu c}{k})\Delta x^2,$$

This requirement has the consequence that time step size is limited by both grid block size and properties of the rock and fluid. This limitation may be severe, as it is the grid block with the smallest value of $(\frac{\phi\mu c}{k})\Delta x^2$ that determines the limiting time step size.

Application of von Neumann stability analysis to the implicit formulation, shows that it is unconditionally stable for all time step sizes. Practice shows that the additional computational work per time step involved in the implicit method, generally is compensated for by permitting much larger time step. Larger time steps lead to larger numerical errors, so it is important in any numerical solution application to check that the errors are within acceptable limits.

The Crank-Nicholson formulation has less discretization error than the two others, since the central approximation of the time derivative has a second order error term. The solution of the set of equations is similar to the implicit case. However, the Crank-Nicholson method often results in oscillations in the solved pressures, and is therefore seldom used.

<u>Stability analysis for explicit formulation</u>

The explicit difference equation may be written

$$\frac{P(x+\Delta x,t)-2P(x,t)+P(x-\Delta x,t)}{(\Delta x)^2}=\alpha\frac{P(x,t+\Delta t)-P(x,t)}{\Delta t},$$

where

$$\alpha=\frac{\phi\mu c}{k}.$$

In von Neumann stability analysis, we assume that if $P(x,t)$ is a solution to the equation above, and that its perturbation $P(x,t)+\varepsilon(x,t)$ also is a solution. Thus, we may obtain the following equation:

$$\frac{\varepsilon(x+\Delta x,t)-2\varepsilon(x,t)+\varepsilon(x-\Delta x,t)}{(\Delta x)^2}=\alpha\frac{\varepsilon(x,t+\Delta t)-\varepsilon(x,t)}{\Delta t}.$$

We now assume that the error introduced is of the form:

$$\varepsilon(x,t)=\psi(t)e^{i\beta x},$$

where

$$i=\sqrt{-1}.$$

Thus,

$$\varepsilon(x+\Delta x,t)=\psi(t)e^{i\beta(x+\Delta x)}$$

$$\varepsilon(x-\Delta x,t)=\psi(t)e^{i\beta(x-\Delta x)}$$

$$\varepsilon(x,t+\Delta t)=\psi(t+\Delta t)e^{i\beta x}$$

By substitution and simplification, and making use of the fact that

$$e^{i\beta\Delta x}+e^{-i\beta\Delta x}-2=-4\sin^2(\frac{\beta\Delta x}{2}),$$

we get the following expression:

$$\frac{\psi(t+\Delta t)}{\psi(t)}=\left[1-\frac{4\Delta t}{\alpha\Delta x^2}\sin^2(\frac{\beta\Delta x}{2})\right].$$

The ratio $\frac{\psi(t+\Delta t)}{\psi(t)}$ may be interpreted as the ratio of increase in error during the time interval Δt. Obviously, if this ratio is larger than one, the solution becomes unstable. Thus, we may formulate the following criterion for stability:

$$\left|\frac{\psi(t+\Delta t)}{\psi(t)}\right|\leq 1,$$

or

$$\left|1-\frac{4\Delta t}{\alpha\Delta x^2}\sin^2(\frac{\beta\Delta x}{2})\right|\leq 1.$$

Since

$$1\geq\sin^2(\frac{\beta\Delta x}{2})\geq 0,$$

the condition for stability becomes:

$$\frac{4\Delta t}{\alpha\Delta x^2}-1\leq 1,$$

or

$$\Delta t\leq\frac{1}{2}(\frac{\phi\mu c}{k})\Delta x^2.$$

<u>Stability analysis for implicit formulation</u>

The implicit form of the difference equation is

$$\frac{P(x+\Delta x,t+\Delta t)-2P(x,t+\Delta t)+P(x-\Delta x,t+\Delta t)}{(\Delta x)^2}=\alpha\frac{P(x,t+\Delta t)-P(x,t)}{\Delta t}.$$

Following a similar procedure as above, we obtain the following equation for the error term:

$$\frac{\varepsilon(x+\Delta x,t+\Delta t)-2\varepsilon(x,t+\Delta t)+\varepsilon(x-\Delta x,t+\Delta t)}{(\Delta x)^2}=\alpha\frac{\varepsilon(x,t+\Delta t)-\varepsilon(x,t)}{\Delta t}.$$

Again assuming that

$$\varepsilon(x,t)=\psi(t)e^{i\beta x},$$

we get the following expression for the error ratio:

$$\frac{\psi(t+\Delta t)}{\psi(t)}=\frac{1}{1+\frac{4\Delta t}{\alpha\Delta x^2}\sin^2(\frac{\beta\Delta x}{2})}.$$

The condition for stability now becomes:

$$\left|\frac{1}{1+\frac{4\Delta t}{\alpha\Delta x^2}\sin^2(\frac{\beta\Delta x}{2})}\right|\leq 1,$$

which is always true, since the denominator is greater than 1. Thus, the stability criterion simply becomes:

$$\Delta t\leq\infty.$$

<u>Stability analysis for Crank-Nicholson formulation</u>

Application of the von Neumann stability analysis to the Crank-Nicholson formulation, shows that it also is unconditionally stable, just as the implicit case.