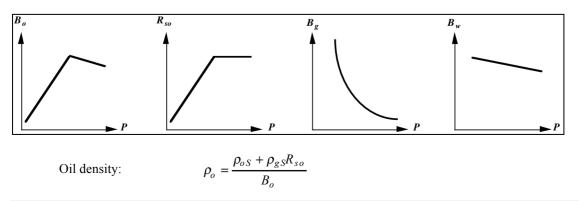## Material Balance Equations

To illustrate the simplest possible model we can have for analysis of reservoir behavior, we will start with derivation of so-called *Material Balance Equations*. This type of model excludes fluid flow inside the reservoir, and considers fluid and rock expansion/compression effects only, in addition, of course, to fluid injection and production. First, let us define the symbols used in the material balance equations:

Symbols used in material balance equations

B_g	Formation volume factor for gas (res.vol./st.vol.)
B_o	Formation volume factor for oil (res.vol./st.vol.)
B_w	Formation volume factor for water (res.vol./st.vol.)
C_r	Pore compressibility (pressure^{-1})
C_w	Water compressibility (pressure^{-1})
ΔP	$P_2 - P_1$
G_i	Cumulative gas injected (st.vol.)
G_p	Cumulative gas produced (st.vol.)
m	Initial gas cap size (res.vol. of gas cap)/(res.vol. of oil zone)
N	Original oil in place (st.vol.)
N_p	Cumulative oil produced (st.vol.)
P	Pressure
R_p	Cumulative producing gas-oil ratio (st.vol./st.vol) = G_p / N_p
R_{so}	Solution gas-oil ratio (st.vol. gas/st.vol. oil)
S_g	Gas saturation
S_o	Oil saturation
S_w	Water saturation
T	Temperature
V_b	Bulk volume (res.vol.)
V_p	Pore volume (res.vol.)
W_e	Cumulative aquifer influx (st.vol.)
W_i	Cumulative water injected (st.vol.)
W_p	Cumulative water produced (st.vol.)
ρ	Density (mass/vol.)
ϕ	Porosity

Then, the Black Oil fluid phase behavior is illustrated by the following figures:

Fluid phase behavior parameters *(Black Oil)*

Oil density:
$$\rho_o = \frac{\rho_{oS} + \rho_{gS}R_{so}}{B_o}$$

Water compressibility: $C_w = -(\dfrac{1}{V_w})(\dfrac{\partial V_w}{\partial P})_T$

Water volume change: $B_{w2} = B_{w1}e^{-c_w \Delta P} \approx B_{w1}(1 - c_w \Delta P)$

Finally, we need to quantify the behavior of the pores during pressure change in the reservoir. The rock compressibility used in the following is the pore compressibility, and assumes that the bulk volume of the rock itself does not change.

Pore volume behavior

Rock compressibility: $C_r = (\dfrac{1}{\phi})(\dfrac{\partial \phi}{\partial P})_T$

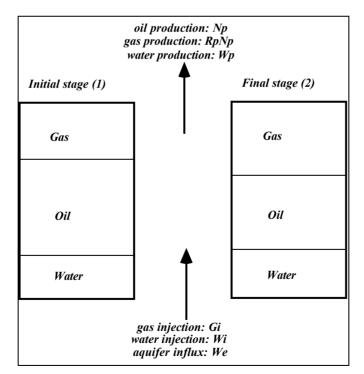Porosity change: $\phi_{w2} = \phi_{w1}e^{c_r \Delta P} \approx \phi_{w1}(1 + c_r \Delta P)$

The material balance equations are based on simple mass balances of the fluids in the reservoir, and may in words be formulated as follows:

Principle of material conservation

$$\left\{ \begin{array}{c} \text{Amount of fluids present} \\ \text{in the reservoir initially} \\ \text{(st. vol.)} \end{array} \right\} - \left\{ \begin{array}{c} \text{Amount of} \\ \text{fluids produced} \\ \text{(st. vol.)} \end{array} \right\} = \left\{ \begin{array}{c} \text{Amount of fluids remaining} \\ \text{in the reservoir finally} \\ \text{(st. vol.)} \end{array} \right\}$$
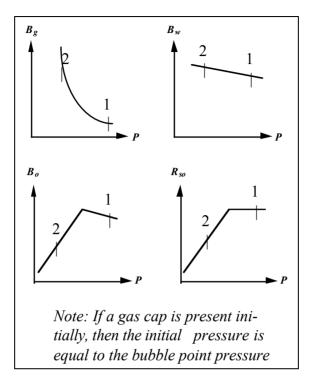
We will define our reservoir system in terms of a simple block diagram, with an initial reservoir stage before production/injection starts, and a final stage at which time we would like to determine pressure and/or production.

Block diagram of reservoir

The two stages on the block diagram are reflected in the fluid phase behavior plots as follows:

Initial and final fluid conditions

Note: If a gas cap is present initially, then the initial pressure is equal to the bubble point pressure

Now, we will apply the above material balance equation to the three fluids involved, oil, gas and water:

Equation 1: Oil material balance

$$\begin{Bmatrix} Oil\ present \\ in\ the\ reservoir \\ initially \\ (st.\ vol.) \end{Bmatrix} - \begin{Bmatrix} Oil \\ produced \\ (st.\ vol.) \end{Bmatrix} = \begin{Bmatrix} Oil\ remaining \\ in\ the\ reservoir \\ finally \\ (st.\ vol.) \end{Bmatrix}$$

or

$$N - N_p = V_{p2}S_{o2} / B_{o2}$$

yielding

$$S_{o2} = \frac{(N - N_p)B_{o2}}{V_{p2}}$$

Equation 2: Water material balance

$$\begin{Bmatrix} Water\ present \\ in\ the\ reservoir \\ initially \\ (st.\ vol.) \end{Bmatrix} - \begin{Bmatrix} Water \\ produced \\ (st.\ vol.) \end{Bmatrix} + \begin{Bmatrix} Water \\ injected \\ (st.\ vol.) \end{Bmatrix} + \begin{Bmatrix} Aquifer \\ influx \\ (st.\ vol.) \end{Bmatrix} = \begin{Bmatrix} Water\ remaining \\ in\ the\ reservoir \\ finally \\ (st.\ vol.) \end{Bmatrix}$$

or

$$V_{p1}S_{w1}/B_{w1} - W_p + W_i + W_e = V_{p2}S_{w2}/B_{w2}$$

yielding

$$S_{w2} = \left[(1+m)NB_{o1}\left(\frac{S_{w1}}{1-S_{w1}}\right)\left(\frac{1}{B_{w1}}\right) + \left(W_i + W_e - W_p\right) \right]\frac{B_{w2}}{V_{p2}}$$

Equation 3: Gas material balance

$$\left\{\begin{array}{c} \textit{Solution gas} \\ \textit{present in the} \\ \textit{reservoir initially} \\ \textit{(st. vol.)} \end{array}\right\} + \left\{\begin{array}{c} \textit{Free gas} \\ \textit{present in the} \\ \textit{reservoir initially} \\ \textit{(st. vol.)} \end{array}\right\} - \left\{\begin{array}{c} \textit{Gas} \\ \textit{produced} \\ \textit{(st. vol.)} \end{array}\right\} + \left\{\begin{array}{c} \textit{Gas} \\ \textit{injected} \\ \textit{(st. vol.)} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \textit{Solution gas} \\ \textit{present in the} \\ \textit{reservoir finally} \\ \textit{(st. vol.)} \end{array}\right\} + \left\{\begin{array}{c} \textit{Free gas} \\ \textit{present in the} \\ \textit{reservoir finally} \\ \textit{(st. vol.)} \end{array}\right\}$$

or

$$NR_{so1} + mNB_{o1} / B_{g1} - R_p N_p + G_i = (N - N_p)R_{so2} + V_{p2}S_{g2} / B_{g2}$$

yielding

$$S_{g2} = \left\{ N\left[(R_{so1} - R_{so2}) + m(\frac{B_{o1}}{B_{g1}}) \right] - N_p(R_p - R_{so2}) + G_i \right\}(\frac{B_{g2}}{V_{p2}})$$

In addition to these three fluid balances, we have the following relationships for fluid saturations and pore volume change:

Equation 4: Sum of saturations

$$S_o + S_w + S_g = 1.0$$

Equation 5: Pore volume change

$$V_{p2} = V_{p1}(1 + c_r \Delta P)$$

By combining the 5 equations above, and grouping terms, we obtain the material balance relationships, as shown below:

THE COMPLETE BLACK OIL MATERIAL BALANCE EQUATION:

$$F = N\left(E_o + mE_g + E_{f,w}\right) + \left(W_i + W_e\right)B_{w2} + G_i B_{g2}$$

where

    production terms are

$$F = N_p\left[B_{o2} + \left(R_p - R_{so2}\right)B_{g2}\right] + W_p B_{w2}$$

    oil and solution gas expansion terms are

$$E_o = \left(B_{o2} - B_{o1}\right) + \left(R_{so1} - R_{so2}\right)B_{g2}$$

    gas cap expansion terms are

$$E_g = B_{o1}\left(\frac{B_{g2}}{B_{g1}} - 1\right)$$

    and rock and water compression/expansion terms are

$$E_{f,w} = -\left(1 + m\right)B_{o1}\frac{C_r + C_w S_{w1}}{1 - S_{w1}}\Delta P$$

MATERIAL BALANCE EQUATION FOR A CLOSED GAS RESERVOIR

The material balance equation for a closed gas reservoir is very simple. Applying the mass balance principle to a closed reservoir with 100% gas, we may derive the general eguation

$$GB_{g1} = (G - G_p)B_{g2}$$

where G is gas initially in place, G_p is cumulative gas production, and B_g is the formation-volume-factor for gas. Since B_g is given by the real gas law
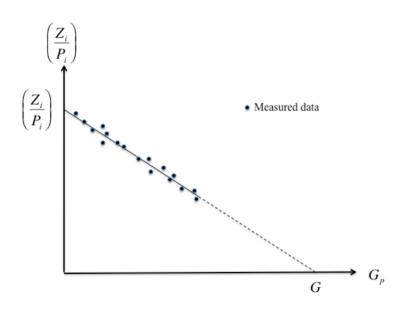
$$B_g = (\text{constant})\frac{Z}{P} \qquad \text{(here temperature is assumed to be constant)}$$

the above material balance equation may be rewritten as

$$G\left(\frac{Z_1}{P_1}\right) = (G - G_p)\left(\frac{Z_2}{P_2}\right)$$

or

$$\left(\frac{P_2}{Z_2}\right) = (1 - \frac{G_p}{G})\left(\frac{P_1}{Z_1}\right)$$

This equation represents a straight line relationship on a $\left(\dfrac{P_2}{Z_2}\right)$ vs. G_p. plot. The line passes through $\left(\dfrac{P_1}{Z_1}\right)$ at $G_p = 0$, and through G at $\left(\dfrac{P_1}{Z_1}\right) = 0$. By making a best-fit straight line to measured data, and extrapolate, we may get an estimate of G.

The straight-line relationship is very useful in estimating the initial volume of gas-in-place (G) from limited production history.